CENTER MANIFOLDS FOR INVARIANT SETS

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ABSTRACT. We derive a general center manifolds theory for a class of compact invariant sets of flows generated by a smooth vector fields in $\mathbb{R}^n$. By applying the Hadamard graph transform technique, it is shown that, associated to certain dynamical characteristics of the linearized flow along the invariant set, there exists an invariant manifold (called a *center manifold*) of the invariant set which contains every locally bounded solution (in particular, contains the invariant set) and is persistent under small perturbations.

1. INTRODUCTION

Consider the following ordinary differential equation

$$z' = f(z), \ z \in \mathbb{R}^n, \tag{1}$$

where f is of class C^r for $r \geq 3$.

If Y is a normally hyperbolic invariant manifold of (1), then the theory of normally hyperbolic invariant manifolds ([10] [16] [22] [26]) assures that not only Y will persist under a C^1 perturbation, but the hyperbolic structure of Y (stable and unstable manifolds and their invariant foliations) will also be preserved.

Center manifolds theory deals with the case when an invariant set becomes non-hyperbolic. As the persistence of such set is generally not expected, a fundamental problem in this context is to find a smooth, normally hyperbolic, invariant manifold, called *center manifold*, containing the invariant set with the smallest possible dimension. The classical center manifolds theory was initiated by Pliss ([24]) and Kelley ([20]) for a non-hyperbolic equilibrium x_0 of (1). It says that there exists a center manifold in a neighborhood of x_0 which is tangent to the generalized eigenspace of $Jf(x_0)$ associated to the non-hyperbolic eigenvalues (i.e. those having zero real parts). A comprehensive and complete version of the classical center manifolds theorem was later given in [31] and [33] which, among other things, particularly showed the same degree of smoothness of a center manifold as the vector field. The theory has found a tremendous amount of applications in problems of bifurcation, stability, and perturbation etc, and, it has been generalized to maps (include periodic orbits of a flow as a special case) and infinite dimensional dynamical systems. For further references on the development and application of the classical center manifolds theory, we refer the readers to [1] [2] [3] [7] [13] [15] [25] [32].

For higher dimensional invariant sets, it was already known that center manifolds exist for an invariant manifold consisting of equilibria (see [11] [19] [21]), for an invariant torus having special structures (see [4] [8]), for skew-product flows (see [9]), and for any small piece of trajectory of maps or flows (see [16] [30]). Recently, motivated by global bifurcation problems, Homburg ([17]) and Sandstede ([28])

Key words and phrases. Center manifold, graph transform, overflowing.
Partially supported by NSF grant DMS9803581.

have succeeded in constructing center manifolds for certain homoclinic orbits. In [6], the authors derived a center manifolds theorem for any smooth, compact invariant manifold of flows, which generalized both the classical center manifolds theorem and the persistence theorem for normally hyperbolic invariant manifolds.

In this paper, continuing our previous work [6] on smooth invariant manifolds, we shall prove a parallel center manifolds theorem for the case of general compact invariant sets.

As in [6], a modification of the vector field is unavoidable in constructing such a center manifold. This creates essential difficulties as the modified vector field is no longer C^1 close to the original one. Of importance is the behavior of the Jacobian of the unit normal vectors at the boundary of a neighborhood of the invariant set Y. If Y is a smooth invariant manifold, a tubular neighborhood of Y then provides the properties on the unit normal vectors which enable the construction of a center manifold. Concerning general invariant sets, as tubular neighborhoods do not exist, one must impose conditions on the existence of certain neighborhoods of Y so that a fairly low dimensional center manifold can exist. We identify a class of invariant sets, called *admissible* ones (see Definition 2), and show the existence of center manifolds for these sets. The class of admissible invariant sets is fairly large. It includes Lipschitz manifolds, smooth manifolds with corners (such as homoclinic orbits and heteroclinic cycles), unions of compact manifolds with different dimensions, etc.

The paper is organized as follows. In Section 2, we introduce the concept of generalized tangent bundles for a subset of $\mathbb{R}^n$ and show how to use them to construct submanifolds containing the set. The center manifolds theorem is stated in Section 3 followed by a corollary in term of the Sacker-Sell spectral theory. In Section 4, to set up the problem into the framework in which the graph transform applies, we first construct an approximate center-unstable manifold and a bundle structure of its neighborhood. The modification of the vector field is then made and the relevant properties are studied. In Section 5, we define the graph transform and outline the proofs of our main results. Applications of our theory to various perturbation and bifurcation problems will be discussed in Section 6.

This paper is dedicated to Professor Jack K. Hale on the occasion of his 70th birthday. The authors like to thank Jack for his great inspirations and supports on their academic careers.

The work is completed while the second and the third authors were visiting the National University of Singapore.

2. Generalized tangent bundle of subsets of $\mathbb{R}^n$

We shall apply the graph transform technique ([12]) to show the existence of a center-unstable manifold. The existence of a center-stable manifold can be obtained similarly after reversing time, and the intersection of the center-stable and the center-unstable manifolds in a neighborhood of Y gives a desired center manifold. To show the existence of a center-unstable manifold, a primary step is the construction of an approximate center-unstable manifold by taking both topological and dynamical issues into considerations. The topological issue concerns that, given a topological subset Y of $\mathbb{R}^n$, how to construct a smooth submanifold of $\mathbb{R}^n$ which contains Y, while the dynamical issue concerns the dynamical natures of the linearization along Y. We study the topological issue in this section.

Definition 1. Let Y be a connected and compact subset of $\mathbb{R}^n$.

(a) Given $y \in Y$. A subspace $V(y)$ of $T_y\mathbb{R}^n$ is called a *generalized tangent space* of Y at y if

$$d\left(\frac{y_2 - y_1}{|y_2 - y_1|}, V(y)\right) \to 0 \text{ as } (y_2, y_1) \to (y, y), \tag{2}$$

where d denotes the Euclidean distance from a point to a subspace.

(b) A continuous subbundle $V(Y) = \cup\{V(y) \subset T_y\mathbb{R}^n; y \in Y\}$ of the tangent bundle $T_Y\mathbb{R}^n$ of $\mathbb{R}^n$ over Y is called a *generalized tangent bundle* (GTB) of Y if, for each $y \in Y$, $V(y)$ is a generalized tangent space of Y at y.

The above concepts are motivated by the following simple observation: if M is a smooth submanifold of $\mathbb{R}^n$ containing Y, then $T_Y M$ is a GTB of Y. Thus, to construct a smooth submanifold of $\mathbb{R}^n$ containing Y, it is necessary to have a GTB of Y to begin with. Conversely, as shown in Proposition 1 below, if $V(Y)$ is a GTB of Y, then there exists a submanifold M of $\mathbb{R}^n$ containing Y such that $T_Y M = V(Y)$.

Remark 1. (a) For a given subset Y of $\mathbb{R}^n$, there exists a GTB of the least dimension, whereas the GTBs realizing the smallest dimension need not be unique. However, if Y itself is a smooth submanifold of $\mathbb{R}^n$, then the tangent bundle TY of Y is the unique GTB with the smallest dimension.

(b) Suppose Y admits cusp singularities. Although the tangent bundle over regular points of Y might be continuously extended to the singularities, the convergence property (2) excludes it as a GTB. In other word, for such a set, the dimension of a GTB must be bigger than the dimension of the tangent bundle over its regular part.

(c) Any continuous subbundle of $T_Y\mathbb{R}^n$ containing a GTB of Y is necessarily a GTB of Y.

To construct a submanifold of $\mathbb{R}^n$ containing a given subset Y of $\mathbb{R}^n$, we first describe a local result, as an easy consequence of the Whitney's Extension Theorem ([34]).

Lemma 1. *For a given compact set $K \subset \mathbb{R}^m$, let $h : K \to \mathbb{R}$ be a continuous function and $Y := \{(x, h(x)); x \in K\}$ denote the graph of f over K. Suppose that $V(Y) = \cup\{V(y) : y \in Y\}$ is a GTB of Y with the fiber dimension* $\dim V(y) = m$ *such that $|\langle n(y), n_0\rangle| \geq \lambda$ for all $y \in Y$, where $\lambda > 0$ is a constant, $n(y)$ is the upward unit normal vector to $V(y)$ and $n_0 = (0, 0, \cdots, 0, 1) \in \mathbb{R}^{m+1}$. Then there exists an open neighborhood U of K in $\mathbb{R}^m$ and a C^1 function H defined on U such that $H|_K = h$, and, for each $y \in Y$, $V(y)$ is the tangent space of the graph of H at y. In other word, h admits a C^1 extension in U.*

Proof. Since $|\langle n(y), n_0\rangle| \geq \lambda > 0$, there exist $\alpha_1, \alpha_2, \cdots, \alpha_m$ such that $V(y)$ is spanned by

$$v_1(y) = (1, 0, \cdots, 0, \alpha_1), \ \ldots, \ v_m(y) = (0, \cdots, 0, 1, \alpha_m).$$

It follows that $n(y) = \frac{1}{\Delta}(-\alpha_1, \cdots, -\alpha_m, 1)$, where $\Delta = \sqrt{1 + \alpha_1^2 + \cdots \alpha_m^2}$ clearly satisfies $|\Delta| \leq \lambda^{-1}$.

In order to apply the Whitney's Extension Theorem, we need to show that h is of class C^1 in K in the sense of Whitney; that is, there exist continuous functions l_i on K for $i = 1, 2, \cdots, m$ such that, for any $x = (x_1, \cdots, x_m) \in K$ and $x' = (x'_1, \cdots, x'_m) \in K$, if

$$h(x') = h(x) + \sum_{i=1}^{m} l_i(x)(x'_i - x_i) + R(x', x),$$

then R satisfies the following property: for any $\epsilon > 0$ small, there exists $\delta > 0$ such that whenever $x^i \in K$ and $|x^i - x| \le \delta$, $i = 1, 2$, then $|R(x^2, x^1)| \le \epsilon$.

Define $l_i(x) = \alpha_i(y)$ where $y = (x, h(x))$. Then,

$$R(x^2, x^1) = h(x^2) - h(x^1) - \sum_{i=1}^{m} \alpha_i(x_i^2 - x_i^1).$$

Since $V(Y)$ is a GTB, for any $\epsilon > 0$, there exists a $\delta > 0$ such that if $|x^i - x| \le \delta$, $i = 1, 2$, then

$$|\langle \left(x^2 - x^1, h(x^2) - h(x^1)\right), n(y)\rangle| \le \epsilon \left(|x^2 - x^1| + |h(x^2) - h(x^1)|\right).$$

This implies that

$$\begin{aligned}|h(x^2) - h(x^1) - \sum_{i=1}^{m} \alpha_i(x_i^2 - x_i^1)| &\le \epsilon\Delta \left(|x^2 - x^1| + |h(x^2) - h(x^1)|\right) \\ &\le \epsilon(\Delta + \Delta^2)|x^2 - x^1| + \epsilon\Delta|h(x^2) - h(x^1) - \sum_{i=1}^{m} \alpha_i(x_i^2 - x_i^1)|,\end{aligned}$$

and hence

$$|R(x^2, x^1)| = |h(x^2) - h(x^1) - \sum_{i=1}^{m} \alpha_i(x_i^2 - x_i^1)| \le \frac{(\Delta + \Delta^2)\delta}{1 - \epsilon\Delta}\epsilon,$$

i.e., $R(x^2, x^1)$ satisfies the desired property.

Therefore, $h : K \to \mathbb{R}$ is of class C^1 in the sense of Whitney. The lemma immediately follows from the Whitney's Extension Theorem (Theorem I in [34]). □

To accomplish the global construction, we utilize the standard technique of Partition of Unity to 'glue' the local ones together.

Proposition 1. *Let Y be a connected compact subset of $\mathbb{R}^n$ and let $V(Y)$ be a GTB of Y. Then there exists a smooth manifold $M(Y)$ such that* (i) $Y \subset M(Y)$, *and* (ii) $T_Y M(Y) = V(Y)$.

Proof. We proceed in two steps.

Step 1. We first show that, for any $y_0 \in Y$, there exist a neighborhood $N(y_0)$ of y_0 and a C^1 manifold $M(y_0)$ such that $Y \cap N(y_0) \subset M(y_0)$ and $T_y M(y_0) = V(y)$ for all $y \in Y \cap M(y_0)$.

By choosing an appropriate local coordinates system, we may assume without loss of generality that $y_0 = 0$, $V(y_0) = \mathbb{R}^l$ and $V^\perp(y_0) = \mathbb{R}^{n-l}$, where $V^\perp$ is a complement of V in $T_Y\mathbb{R}^n$. Hence, there exists a neighborhood $N(y_0)$ of y_0 such that for any $x \in N(y_0)$, $x = (p, q)$ for some $p \in V(y_0)$ and $q \in V^\perp(y_0)$. Define the projection $P_{y_0} : Y \cap N(y_0) \to V(y_0)$ by $P_{y_0}(x) = p$. We claim that P_{y_0} is injective if $N(y_0)$ is sufficiently small. To see this, let $y_i = (p_i, q_i) \in Y \cap N(y_0)$, $i = 1, 2$. If $p_2 = p_1$, then

$$d\left(\frac{y_2 - y_1}{|y_2 - y_1|}, V(y_0)\right) = d\left(\frac{(0, q_2 - q_1)}{|q_2 - q_1|}, V(y_0)\right) = 1,$$

a contradiction to the fact that V is a GTB.

Now let $K = P_{y_0}(Y \cap N(y_0))$ and define $h : K \to V^{\perp}(y_0)$ by $h(p) = y$ if $P_{y_0}(y) = p$. By applying Lemma 1 to each component of h, we obtain a smooth extension of h whose graph $M(y_0)$ satisfies the desired properties (i) and (ii).

Step 2. We now construct a global manifold $M(Y)$ from the family of local manifolds $M(y)$ obtained in Step 1.

The observation is that if two graphs overlap on a common domain of Y, then the two generating functions of the graphs, together with their partial derivatives, will agree on the same domain, so does any convex interpolation of these two functions. We show this in details below.

Let $M(y_0) = \text{graph}(H_{y_0})$, $y_0 \in Y$, be the smooth manifolds obtained in Step 1 of dimension m_0. Since $T_y M(y_0) = V(y)$ for all $y \in M(y_0) \cap Y$, we may assume that the Lipschitz constant of H_{y_0} is small by shrinking the domain of H_{y_0} if necessary. Furthermore, since $V(y)$ depends continuously on y, we may also assume that, for any y' and y'' in Y, if $M(y') \cap M(y'') \neq \emptyset$, then $M(y')$ is also a graph of a function from $P_{y''} M(y') \subset V(y'')$ to $V^{\perp}(y'')$. For simplicity, we still use $H_{y'}$ to denote this function.

Since Y is compact, there exist $y_1, \cdots, y_l \in Y$ such that $Y \subset \cup_{i=1}^{l} M(y_i)$. Assume that $Y \not\subset \cup_{i=1}^{l} M(y_i) \backslash M(y_j)$ for any $j = 1, 2, \cdots, l$. We denote B_i as the domains of H_{y_i} respectively.

We will glue $M(y_i)$ for $i = 1, 2, \cdots, l$ together inductively from y_1 to y_l. Starting from y_1 to y_2, there are two cases.

Case 1. $M(y_1) \cap M(y_2) = \emptyset$. In this case, we do nothing.

Case 2. $M(y_1) \cap M(y_2) \neq \emptyset$. Let $B = P_{y_2} M(y_1) \cap M(y_2) \subset V(y_2)$ and, as remarked above, denote H_{y_1} the function on B whose graph is $M(y_1)$. Choose an open set $\hat{B} \subset B$ such that $\text{cl}(\hat{B}) \subset B$, $\hat{B} \cap P_{y_2} M(y_1) \neq \emptyset$ and $Y \subset \cup_{i=2}^{l} M(y_i) \cup M(y_1)|_{\hat{B}}$.

By Uryson's lemma, there exists a C^r cut-off function χ on $V(y_2)$ such that $0 \leq \chi \leq 1$, $\chi(p) = 1$ for $p \in \hat{B}$ and $\chi(p) = 0$ for $p \notin B$.

Define $H_{12} = \chi H_{y_1} + (1 - \chi) H_{y_2}$ on $B \cup B_2 \subset V(y_2)$, then H_{12} is of class C^{r-1}. For any $p \in B \cap P_{y_2} Y$, say $p = P_{y_2} y$ for some $y \in Y$, we have $H_{y_1}(p) = H_{y_2}(p) = y$, and hence, $H_{12}(p) = y$. Moreover, for all $|k| \leq r - 1$,

$$\begin{aligned} D^k H_{12}(p) =& D^k \chi(p) H_{y_1}(p) + \chi(p) D^k H_{y_1}(p) \\ & - D^k \chi(p) H_{y_2}(p) + (1 - \chi(p)) D^k H_{y_2}(p). \end{aligned}$$

Since $D^k H_{y_1}(p) = D^k H_{y_2}(p)$ for all $|k| \leq r - 1$, we have

$$D^k H_{12}(p) = \chi(p) D^k H_{y_1}(p) + (1 - \chi(p)) D^k H_{y_2}(p) = D^k H_{y_1}(p).$$

Thus, graph(H_{12}) satisfies the properties (i) and (ii) locally and coincides with $M(y_1)$ and $M(y_2)$ on the corresponding portion of Y. We denote this graph by M_{12}.

Next, consider y_3. By the way that M_{12} is constructed, if $M_{12} \cap M(y_3) \neq \emptyset$, then it is a graph of a function on $P_{y_3}(M_{12} \cap M(y_3)) \subset V(y_3)$. We can then apply the same argument to obtain a manifold, say M_{123}, replacing M_{12} and $M(y_3)$.

Continuing this process, we gradually reduce the number of overlapped pieces and finally obtain a global C^r manifold containing Y with the desired properties (i) and (ii). ◻

Due to the presence of nontrivial center directions, the construction of a center manifold requires a modification of the vector field in a neighborhood of the invariant set. To be able to control the behavior of the modified vector field, the nature of the embedding of the invariant set into the phase space becomes evidently crucial and has to be somewhat restricted.

Definition 2. A connected compact subset Y of $\mathbb{R}^n$ is said to be *admissible* if there exist constants $C_1, C_2, C_3 > 0$ such that, for any $\epsilon > 0$, one can find a smooth neighborhood N of Y on which the following holds for all $z \in \partial N$:

(i) $C_1\epsilon \le d(z, Y) \le \epsilon$;
(ii) $|\langle f(z), v(z)\rangle| \le C_2\epsilon$;
(iii) In a neighborhood U of z, if $\partial N \cap U$ is the graph of a function $\phi : T_z\partial N \to T_z^{\perp}\partial N$, then $|u^{\top}\text{Hess}(\phi)(z)u| \le C_3\epsilon^{-1}$ for any unit vector $u \in T_z\partial N$.

Remark 2. (a) The function ϕ above is unique up to an orthogonal change of variables on $T_z\partial N$ and thus the property (iii) is independent of choice of ϕ.
(b) The admissible class includes most typical compact invariant sets of flows, for example, smooth or Lipschitz manifolds, homoclinic loops and heteroclinic cycles with corners, etc. In fact, if $Y \in R^n$ and if there exists a neighborhood U of Y and a diffeomorphism $\phi : U \to \mathbb{R}^n$ such that $\phi(Y)$ is a CR-complex, then Y is admissible for any vector field which leaves Y invariant.

3. Statements of main results

Let Y be an invariant set of (1) and let $y \cdot t$ denote the flow on Y for $y \in Y$ and $t \in \mathbb{R}$. The linearization of (1) along $y \cdot t$ reads

$$z' = A(y \cdot t)z, \tag{3}$$

where $A(y) = Jf(y)$ is the Jacobian matrix of f at $y \in Y$. Denote $\Phi(y, t)$ as the principal matrix solution of (1). We make the following hypotheses:

(H1) System (3) admits a continuous invariant splitting of $T_Y\mathbb{R}^n$:

$$T_Y\mathbb{R}^n = V_s(Y) \oplus V_c(Y) \oplus V_u(Y),$$

i.e., $V_i(Y) = \cup\{V_i(y) \subset T_y\mathbb{R}^n : y \in Y\}$ and $\Phi(y,t)V_i(y) = V_i(y \cdot t)$, $i = s, c, u$, for all $t \in \mathbb{R}$, $y \in Y$.

(H2) $V_c(Y)$ is a GTB of Y.

(H3) There exist constants α, β, K and a positive integer d with $\alpha < 0$, $0 \le \beta < \frac{1}{d}$, $d \le r$, $K > 0$, such that for all $y \in Y$

$$||\Phi_s(y,t)|| \le Ke^{\alpha t},\ t \ge 0; \quad ||\Phi_s(y,t)||^{\beta} \le Km(\Phi_c(y,t)),\ t \ge 0;$$

$$||\Phi_u(y,t)|| \le Ke^{-\alpha t},\ t \le 0; \quad ||\Phi_u(y,t)||^{\beta} \le Km(\Phi_c(y,t)),\ t \le 0;$$

where, $\Phi_i(y,t) = \Phi(y,t)P_i(y)$, $P_i(y)$ is the projections of $\mathbb{R}^n$ to $V_i(y)$ for $i = s, c, u,$, respectively, and, for a linear operator L, $||L||$ stands for the operator norm of L and $m(L) = \min\{|Lz| : |z| = 1\}$ denotes the co-norm.

Definition 3. (a) A submanifold M of $\mathbb{R}^n$ with boundary ∂M is called *locally invariant* under (1) if, for any point $p \in M \backslash \partial M$, there exists an $\epsilon > 0$ such that $z(t,p) \in M$ for $t \in (-\epsilon, \epsilon)$, where $z(t,p)$ is the solution of (1) with $z(0,p) = p$.

(b) Let f be a C^r vector field on $\mathbb{R}^n$, a locally invariant C^k ($k \leq r$) manifold M of f is said to be C^k *persistent* if there exists a neighborhood $\mathcal{U}(f)$ in the space of C^k vector fields, such that for any $g \in \mathcal{U}(f)$, there exists a locally invariant C^k manifold $M(g)$ (not necessarily unique) of g which is C^k close to M in the Hausdorff metric.

Our main result states as follows.

Theorem 1. *Assume that* (H1),(H2),(H3) *hold for an admissible invariant set* Y *of* (1). *Then there exists a sub-manifold* $M_c(Y)$ *of* $\mathbb{R}^n$ *with the following properties:*

(i) $M_c(Y)$ *is of class* C^d *and is locally invariant;*

(ii) $T_y M_c(Y) = V_c(y)$ *for all* $y \in Y$*;*

(iii) $M_c(Y)$ *is* C^d *persistent;*

(iv) *There exists an* $\epsilon > 0$ *such that* $M_c(Y)$ *contains all solutions bounded in* $N_\epsilon(Y)$. *In particular,* $M_c(Y)$ *contains* Y.

We refer to a manifold satisfying the above properties (i) – (iv) as a C^d *center manifold* of Y.

Remark 3. (a) As in the classical center manifolds case ([5]), center manifolds of a compact invariant set are not unique in general.

(b) If Y is smooth and $V_c(Y) = TY$, then Y must be normally hyperbolic and (iii) generalize the theory of normally hyperbolic invariant manifolds.

(c) The above result not only extends our previous result [6] to the non-smooth cases, but also it improves our previous result even for the smooth case alone, in the sense that an arbitrary constant K is allowed in the above exponential rate condition (H3) while in [6] only $K = 1$ was allowed.

The applications of the theorem depends on a careful choice of a linear invariant splitting. A basic strategy would be to first choose a smallest possible $V_c(Y)$ in (H1) then try to verify (H2) and (H3).

A natural (but not necessarily optimal) linear invariant splitting satisfying (H1) – (H3) above can be obtained from the well known Sacker-Sell ([27]) spectral theory of exponential dichotomy (see also [29]).

Let Y be an admissible invariant set of (1) and $\Sigma(Y) = [a_1, b_1] \cup \cdots \cup [a_k, b_k]$ be its Sacker-Sell spectrum, where $k \leq n$ and $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k$. For $1 \leq i \leq k$, let $V_i = V_i(Y)$ denote the invariant subbundle (called the *spectral subbundle* associated to the spectral interval $[a_i, b_i]$) of $T_Y\mathbb{R}^n$. Then

$$T_Y\mathbb{R}^n = V_1(Y) \oplus \cdots \oplus V_k(Y).$$

Consider also, for $i \leq j$, the union of spectral intervals of form $\Sigma_{i,j} = \cup_{p=i}^{j}[a_p, b_p]$ and denote the corresponding spectral subbundle by $V_{i,j} = V_i \oplus \cdots \oplus V_j$. Let

$$i_0 = \max\{i : \ V_{i,k} \text{ is a GTB}\}, \qquad j_0 = \min\{j : \ V_{1,j} \text{ is a GTB}\}.$$

Then, V_{i_0,j_0} is the smallest spectral subbundle which remains as a GTB of Y. Clearly, i_0 and j_0 are uniquely defined. We refer to $\Sigma_c = \Sigma_{i_0,j_0}$, $V_c = V_{i_0,j_0}$ as the *generalized center spectrum* and the *generalized center subbundle* of Y, respectively. To unify the notation, we let $a_{j_0+1} = +\infty$ if $j_0 = k$ and let $b_{i_0-1} = -\infty$ if $i_0 = 1$.

Theorem 2. *Assume the following spectrum gap condition: there is a positive integer $d \le r$ such that*

$$-a_{i_0} d < -b_{i_0 - 1}, \quad b_{j_0} d < a_{j_0+1}.$$

Then, with $V_c(Y)$ as the generalized center subbundle of Y, there exists a center manifold $M_c(Y)$ of Y satisfying all properties of Theorem 1. □

4. Modified vector field

The proof of our theorem follows the framework of our previous work [6]. It involves the same crucial steps such as the constructions of an approximate center-unstable manifold and a fiber bundle structure of a neighborhood, modification of the vector field, estimates dealing with large perturbations, and the application of graph transformation.

Traditionally, analysis carried on a manifold and its fiber bundles makes use of local coordinates of the manifold and local trivializations of the fiber bundles, which generates certain complexities into the analysis. We avoid some of these complexities by taking advantage of the fact that the manifold and its fiber bundles which we work with are submanifolds of the phase space $\mathbb{R}^n$. Therefore, instead of using local coordinates, we shall only use the global coordinate inherited from $\mathbb{R}^n$.

4.1. **An approximate center-unstable manifold.** The first step of the construction is to construct an approximate center-unstable manifold. Such an approximate center-unstable manifold should be tangent to $V_{cu}(y)$ at all the points $y \in Y$.

Lemma 2. *There exists a smooth manifold $\hat{M}_{cu}(Y)$ such that $Y \subset \hat{M}_{cu}(Y)$ and $T_Y \hat{M}_{cu}(Y) = V_{cu}(Y)$.*

Proof. The lemma follows immediately from (H2) and Proposition 1. □

4.2. **A bundle structure in a neighborhood of Y.** The next step is to construct a bundle structure over $\hat{M}_{cu}(Y)$ in a neighborhood of Y. At the end, a center-unstable manifold will be obtained as the graph of a function over $\hat{M}_{cu}(Y)$ with values on fibers of a such bundle.

With the Nash's Embedding Theorem [23], we assume without loss of generality that $V_{cu}(y) \perp V_s(y)$ for $y \in Y$ (see [6]). By the tubular neighborhood theorem, there exists a neighborhood N of $\hat{M}_{cu}(Y)$ such that for any $z \in N$, one finds a unique $p_z \in \hat{M}_{cu}(Y)$ such that $d(z, \hat{M}_{cu}(Y)) = d(z, p_z)$. Define $\pi_{cu} : N \to \hat{M}_{cu}(Y)$ by $\pi_{cu}(z) = p_z$, $\pi_s : N \to N$ by $\pi_s(z) = z - \pi_{cu}(z) := q$; and fix a projection $Q : \hat{M}_{cu}(Y) \to Y$ by $Q(p) = y_p$, where $y_p \in Y$ is a point with $d(p, Y) = d(p, y_p)$. Note that Q is not necessarily continuous. By Whitney's Embedding Theorem ([35]), we also assume without loss of generality that the bundle and the maps (except Q) defined above are C^r by jiggling the bundle slightly if necessary (see also [10]). Denote the differential of π_{cu}, π_s at z by $D\pi_{cu}(z)$ and $D\pi_s(z)$. We then have $D\pi_{cu}(y) = P_{cu}(y)$, $D\pi_s(y) = P_s(y)$.

Let $\epsilon_0 > 0$ be small. Since Y is admissible, we further assume without loss of generality that the above neighborhood N also satisfies the properties of Definition 2 with respect to ϵ_0. For $0 \le \epsilon < C_1 \epsilon_0$ and $C_0 > 0$ where C_1 is as in Definition 2 and

C_0 is specified in Lemma 8 , denote

$$\begin{aligned} U_\epsilon =&\{z \in \hat{M}_{cu}(Y) \cap N_0 : d(z, \partial N_0) \geq \epsilon\}, \\ U_\epsilon(z_0) =&\{z \in N_0 :\ d(z, \hat{M}_{cu}(Y)) = d(z_0, \hat{M}_{cu}(Y))\}, \quad z_0 \in N_0, \\ S(z_0) =&\{z \in N_0 :\ d(z, \hat{M}_{cu}(Y)) = d(z, z_0)\}, \quad z_0 \in \hat{M}_{cu}(Y), \\ S_\epsilon(z_0) =&\{z \in S(z_0) :\ d(z, z_0) < C_0\epsilon_0^2\}, \quad z_0 \in \hat{M}_{cu}(Y), \\ N_\epsilon =& \cup_{z_0 \in U_\epsilon} S_{\epsilon_0}(z_0), \quad \partial^* N_\epsilon = \cup_{z_0 \in \partial U_\epsilon} S_{\epsilon_0}(z_0). \end{aligned}$$

We note that $\partial^* N_\epsilon$ is the portion of ∂N_ϵ over ∂U_ϵ, $\dim \partial^* N_\epsilon = n - 1$, and, $\dim S(z_0) = \dim V_s(y)$ for any $z_0 \in U_0$.

4.3. **Overflowing and modification of vector field.** To define a graph transform induced by the time T-map φ^T of the flow (1), the vector field f has to be modified to satisfy the so called *overflowing* property (see [6] [10] [16] [30], etc.).

Definition 4. Let $N \subset \mathbb{R}^n$ be a submanifold of dimension n with smooth boundary ∂N and let $A \subset \partial N$ be a subset of ∂N. A vector field f is said to satisfy the *overflowing property with respect to* (N, A) if at each point $z \in A$, f is either tangent to or points outward to ∂N.

Basic ideas of the modification are the following. In a neighborhood of $\partial^* N_0$, we first add a vector field along the normal direction to $\partial^* N_0$ to meet the overflowing property. As remarked before, this will generally introduce a large C^1 error between the new vector field and the original one. We then add a strong contraction component along the stable direction to balance any possible contractions along the center-unstable direction caused by the first modification.

Let $\chi :\ \mathbb{R} \to \mathbb{R}$ be the cut-off function defined by

$$\chi(x) = \begin{cases} 0, & x \geq \frac{1}{2}C_1\epsilon_0 \\ \exp\{\frac{3\epsilon_0 x}{3x - C_1\epsilon_0}\}, & x \leq \frac{1}{3}C_1\epsilon_0 \end{cases}$$

where C_1 is as in Definition 2. In particular,

$$\epsilon_0|\chi'(x)| \leq C_1^{-1}\chi(x) + \epsilon_0^2. \tag{4}$$

Let $v(z)$ be the inward unit normal vector to $T_z\partial^* N_\epsilon(z)$ for $z \in \partial^* N_\epsilon$ and define $\eta(z) = d(\pi_{cu}z, \partial U_0)$ for $z \in N_0$. We modify the original vector field (1) to the following:

$$z' = \hat{f}(z) = f(z) - C_2\epsilon_0\chi(\eta(z))v(z) - d_0\chi(\eta(z))(z - \pi_{cu}(z)), \tag{5}$$

where $d_0 > 0$ is chosen to be of order $O(\epsilon^{-1})$ and such that $d_0 C_0 \epsilon_0$ is bounded, and C_2 is as in Definition 2.

It turns out that the new vector field (5) satisfies the overflowing property with respect to $(N_0, \partial^* N_0)$. Indeed, for $z \in \partial^* N_0$, $\chi(\eta(z)) = 1$, and hence,

$$\langle \hat{f}(z), v(z)\rangle = \langle f(z), v(z)\rangle - C_2\epsilon_0 < 0.$$

4.4. **Estimates.** Analysis on the modified vector field depends crucially on properties of η and v. We first derive formulas for $\nabla\eta$ and Jv in the next two lemmas. Since η and v are quantities on the neighborhood of Y, the corresponding results are independent of f and are purely geometrical.

Lemma 3. *For $z \in N_0$ lying in a tubular neighborhood of $\partial^* N_0$, we let $z_0 \in \partial^* N_0$ be the unique point such that $|z - z_0| = d(z, \partial^* N_0)$. Then,*

$$\nabla \eta(z) = v(z) = \frac{z - z_0}{|z - z_0|}.$$

Proof. Without loss of generality, we assume that, near z_0, $\partial^* N_0$ is the graph of a function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ so that $N_0 \subset \{\phi \geq 0\}$. Write $z = (x, y)$ with $x \in \mathbb{R}^{n-1}$ and $y \in \mathbb{R}$. Let u be such that $z_0 = (u, \phi(u))$.

Denote $g(w) = (w - x)^2 + (\phi(w) - y)^2$. Then

$$\nabla g(w)|_{w=u} = 2(u - x) + 2(\phi(u) - y)\nabla \phi = 0,$$

and hence,

$$\nabla \phi(u) = -\frac{u - x}{\phi(u) - y}.$$

Since $\eta(z) = \sqrt{(u - x)^2 + (\phi(u) - y)^2}$, it follows that

$$\nabla \eta(z) = \begin{pmatrix} \frac{(u-x)(u_x - I) + (\phi(u) - y)\phi_u u_x}{\sqrt{(u-x)^2 + (\phi(u) - y)^2}} \\ \frac{(u-x)u_y + (\phi(u) - y)(\phi_u u_y - I)}{\sqrt{(u-x)^2 + (\phi(u) - y)^2}} \end{pmatrix} = \begin{pmatrix} \frac{x - u}{\sqrt{(u-x)^2 + (\phi(u) - y)^2}} \\ \frac{y - \phi(u)}{\sqrt{(u-x)^2 + (\phi(u) - y)^2}} \end{pmatrix}.$$

□

Lemma 4. *For any $z \in \partial N_\epsilon$, let $\phi : T_z \partial N_\epsilon \to \mathbb{R}$ be the function whose graph agrees with ∂N_ϵ near z, then $Jv(z)$ is orthogonally similar to*

$$-\begin{pmatrix} Hess(\phi) & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof. From the definition of the neighborhood N_ϵ, we see that $T_z \mathbb{R}^n = V^s(z) \oplus V^{cu}(z)$, where $V^s(z)$ is tangent to the fiber at z and V^{cu} is tangent to $U(z)$. Since $Jv|_{V^s(z)} = 0$, it is sufficient to only consider the center-unstable directions. Therefore, without loss of generality, we assume that $m_0 = n$.

By identifying $T_z \partial N_\epsilon$ with $\mathbb{R}^{n-1}$ and z with the origin, respectively, we have that $\partial^* N_\epsilon$ is locally generated by the graph of a function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ with $\nabla \phi(0) = 0$.

Consider points $(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R}$ lying in a neighborhood of 0. We first compute $v(x, y)$. Let $u : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be a function such that

$$d((x, y), (u(x, y), \phi(u(x, y)))) = d((x, y), \mathrm{graph}(\phi)).$$

Then (x, y) is a normal vector of $\partial^* N_\epsilon$ through the point $(u, \phi(u))$, $u = u(x, y)$. It follows that there exists a λ such that

$$x = u(x, y) - \lambda(\nabla \phi)(u(x, y)), \quad y = \phi(u(x)) + \lambda. \tag{6}$$

Therefore,

$$v(x, y, z) = \frac{1}{\sqrt{|\nabla \phi|^2 + 1}} \begin{pmatrix} -(\nabla \phi)(u(x)) \\ 1 \end{pmatrix},$$

and,

$$x = u(x, y) - (y - \phi(u(x, y)))(\nabla \phi)(u(x, y)).$$

Differentiating the above with respect to (x, y) yields

$$\begin{aligned}Dx &= Du - (\nabla\phi)(u)(\nabla(y-\phi(u)))^\top - (y-\phi(u))D((\nabla\phi)(u))\\ &= Du - (\nabla\phi)(u)(e_n - \nabla(\phi(u))^\top) - (y-\phi(u))\text{Hess}(\phi)Du\\ &= Du - (\nabla\phi)(u)(e_n - (\nabla\phi)^\top(u)Du) - (y-\phi(u))\text{Hess}(\phi)Du,\end{aligned}$$

where $e_n = (0, \cdots, 0, 1)$. Thus,

$$Du = \left(I + (\nabla\phi)(\nabla\phi)^\top - (y-\phi)\text{Hess}(\phi)\right)^{-1}(I, \nabla\phi).$$

Therefore,

$$\begin{aligned}Jv(x,y) &= \begin{pmatrix} -(\nabla\phi)(u(x)) \\ 1 \end{pmatrix}(\nabla(\frac{1}{\sqrt{|\nabla\phi|^2+1}}))^\top\\ &\quad + \frac{1}{\sqrt{|\nabla\phi|^2+1}}J\begin{pmatrix} -(\nabla\phi)(u(x)) \\ 1 \end{pmatrix}\\ &= -\frac{1}{2(|\nabla\phi|^2+1)^{3/2}}\begin{pmatrix} -(\nabla\phi)(u(x)) \\ 1 \end{pmatrix}(\nabla|(\nabla\phi)(u)|^2)^\top\\ &\quad - \frac{1}{\sqrt{|\nabla\phi|^2+1}}\begin{pmatrix} \text{Hess}(\phi)Du \\ 0 \end{pmatrix}\\ &= -\frac{1}{(|\nabla\phi|^2+1)^{3/2}}\begin{pmatrix} -(\nabla\phi)(u(x)) \\ 1 \end{pmatrix}(\nabla\phi)^\top(u)\text{Hess}(\phi)Du\\ &\quad - \frac{1}{\sqrt{|\nabla\phi|^2+1}}\begin{pmatrix} \text{Hess}(\phi)Du \\ 0 \end{pmatrix}\\ &= -\frac{1}{(|\nabla\phi|^2+1)^{3/2}}\begin{pmatrix} -(\nabla\phi)(\nabla\phi)^\top + |\nabla\phi|^2+1 \\ (\nabla\phi)^\top \end{pmatrix}\text{Hess}(\phi)Du.\end{aligned}$$

Using the expression of Du and the facts that $\phi(0) = 0$, $\nabla\phi(0) = 0$, we have

$$Jv(0) = -\begin{pmatrix} I \\ 0 \end{pmatrix}\text{Hess}(\phi)\begin{pmatrix} I & 0 \end{pmatrix} = -\begin{pmatrix} \text{Hess}(\phi) & 0 \\ 0 & 0 \end{pmatrix}.$$

□

The following Gronwall's inequality and projection identities will be frequently used.

Lemma 5. *If $\beta(t) \geq 0$, and, $\alpha(t)$ and $\phi(t)$ are continuous functions on $[a, b]$ such that*

$$\phi(t) \leq \alpha(t) + \int_a^t \beta(s)\phi(s)ds, \quad a \leq t \leq b,$$

then

$$\phi(t) \leq \alpha(t) + \int_a^t \beta(s)\alpha(s)e^{\int_s^t \beta(\mu)d\mu}ds, \quad a \leq t \leq b.$$

If, in addition, $\alpha'(t) \geq 0$, then

$$\phi(t) \leq \alpha(t)e^{\int_a^t \beta(s)ds}, \quad a \leq t \leq b.$$

Proof. See [14]. □

Lemma 6. *Let $P_i(y) : T_y\mathbb{R}^n \to V_i(y)$, $i = s, c, u$ be the projections. Then*

$$(P_i(y\cdot t))' = A(y\cdot t)P_i(y\cdot t) - P_i(y\cdot t)A(y\cdot t).$$

Proof. Let $i = s, c, u$. We clearly have

$$(P_i(y\cdot t)\Phi(y,t))' = (P_i(y\cdot t))'\Phi(y,t) + P_i(y\cdot t)A(y\cdot t)\Phi(y,t),$$

$$(\Phi(y,t)P_i(y))' = A(y\cdot t)\Phi(y,t)P_i(y) = A(y\cdot t)P_i(y\cdot t)\Phi(y,t).$$

Using the invariance of $V_i(Y)$, we also have $P_i(y\cdot t)\Phi(y,t) = \Phi(y,t)P_i(y)$, from which the lemma follows. □

The C^0 closeness between the modified and the original flows can be easily obtained as follows.

Lemma 7. *Let $z_1(t)$ and $z_2(t)$ be trajectories of* (1) *and* (5), *with the initial conditions $\bar{z}_1, \bar{z}_2 \in N_0$, respectively. If $z_2(t) \in N_0$ and $\lambda z_2(t) + (1-\lambda)z_1(t) \in N_0$, for $|t| \le T$, $\lambda \in [0,1]$, then*

$$|z_2(t) - z_1(t)| \le \left(|\bar{z}_2 - \bar{z}_1| + C_2\epsilon_0|t| + d_0C_0\epsilon_0^2|t|\right) e^{|Df||t|},$$

where $|Df| = |Df|_{N_0}$.

Proof. We only prove the case $t \ge 0$. Since

$$\begin{aligned}(z_2(t) - z_1(t))' =& f(z_2(t)) - \hat{f}(z_1(t))\\ =& f(z_2(t)) - f(z_1(t)) - C_2\epsilon_0\chi(\eta(z_2(t)))v(z_2(t))\\ &- d_0\chi(\eta(z_2(t)))(z_2(t) - \pi_{cu}z_2(t)),\end{aligned}$$

we have

$$|z_2(t) - z_1(t)| - |\bar{z}_2 - \bar{z}_1| \le |Df|\int_0^t |z_1(s) - z_2(s)|ds + (C_2\epsilon_0 + d_0C_0\epsilon_0^2)t.$$

By Lemma 5, then

$$|z_1(t) - z_2(t)| \le \left(|\bar{z}_2 - \bar{z}_1| + (C_2\epsilon_0 + d_0C_0\epsilon_0^2)t\right) e^{|Df|t}$$

for all $t \le T$. □

The next lemma asserts that any trajectory of the modified flow cannot exit the "top" and the "bottom" parts of N_0 in forward time. This is a necessary condition to define a graph transform in N_0.

Lemma 8. *For given $T > 0$, there exists a $C(T) > 0$ such that if $z(t) = (p(t), q(t))$, $t \le T$, is the solution of* (5) *with $z(0) = (p_0, q_0) \in N_0$, then*

$$|q(t)| \le Ke^{\alpha t}|q_0| + C(T)\epsilon_0^2.$$

In particular, for a fixed $T > -1/\alpha \ln K$, if C_0 is sufficiently large, then $|q_0| \le C_0\epsilon_0^2$ implies that $|q(T)| \le C_0\epsilon_0^2$.

Proof. Let $y = Q(p_0) \in Y$. Lemma 7 implies that $|z(t) - y\cdot t| \le C(T)\epsilon_0$ for some $C(T)$ independent of C_0 since $d_0C_0\epsilon_0$ is bounded. Note that

$$\begin{aligned}(z(t) - y\cdot t)' =& A(y\cdot t)(z(t) - y\cdot t) + F(z(t), y\cdot t)\\ &- C_2\epsilon_0\chi(\eta(z))v(z(t)) - d_0\chi(\eta(z))(z - \pi_{cu}z),\end{aligned} \tag{7}$$

where $F(z,y) = f(z) - f(y) - A(y)(z-y)$. It follows that

$$(P_s(y\cdot t)(z(t) - y\cdot t))' = (A(y\cdot t) - d_0\chi(\eta(z(t))))P_s(y\cdot t)(z(t) - y\cdot t) + C(T)\epsilon_0^2,$$

and therefore,

$$
\begin{aligned}
|P_s(y\cdot t)(z(t)-y\cdot t)| \le & e^{-d_0\int_0^t \chi(\eta(z(s)))\,ds}|\Phi(y,t)P_s(y)(z(0)-y)| \\
& + \int_0^t e^{-d_0\int_s^t \chi(\eta(z(\tau)))\,d\tau}\Phi(y,t)P_s(y)\Phi^{-1}(y,s)C(T)\epsilon_0^2\,ds \\
\le & Ke^{\alpha t}|P_s(y)(z(0)-y)| + C(T)\epsilon_0^2.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
|q(t)| = & |\pi_s z(t)| = |\pi_s z(t) - \pi_s y\cdot t| \\
= & |D\pi_s(y\cdot t)(z(t)-y\cdot t)| + C(T)|z(t)-y\cdot t|^2 \\
= & |P_s(y\cdot t)(z(t)-y\cdot t)| + C(T)|z(t)-y\cdot t|^2 \\
\le & Ke^{\alpha t}|q_0| + C(T)\epsilon_0^2.
\end{aligned}
$$

The lemma clearly follows. □

Next, we consider the C^1 estimates between the two flows.

Let $(p_i,q_i)\in N_0$ and $z_i(t)$ be solutions of the modified flow (5) with $z_i(0)=(p_i,q_i)$, for $i=1,2$, and $|p_2-p_1|\le C\epsilon_0$. Let $y=Q(p_1)\in Y$. By Lemma 7, $|z_i(t)-y\cdot t|\le C(T)\epsilon_0$ for $t\in[0,T]$. Also, as in (7),

$$
\begin{aligned}
(z_i(t)-y\cdot t)' = & \hat f(z_i(t)) - f(y\cdot t) = A(y\cdot t)(z_i(t)-y\cdot t) + F(z_i(t), y\cdot t) \\
& - C_1\epsilon_0\chi(\eta(z_i))v(z_i(t)) - d_0\chi(\eta(z_i))(z_i(t)-\pi_{cu}z_i(t)).
\end{aligned}
$$

Set $w(t)=z_2(t)-z_1(t)$. Then

$$
w' = A(y\cdot t)w + B(y\cdot t)w + O(\epsilon_0 w), \tag{8}
$$

where

$$
\begin{aligned}
B(y\cdot t)w = & - C_2\epsilon_0\chi'(\eta(\bar z))\langle\nabla\eta(\bar z), w\rangle v(\bar z) - C_2\epsilon_0\chi(\eta(\bar z)))Jv(\bar z)w \\
& - d_0\chi'(\eta(\bar z))\langle\nabla\eta(\bar z), w\rangle(\bar z - \pi_{cu}(\bar z)) - d_0\chi(\eta(\bar z)))(I-P_{cu}(\bar z))w
\end{aligned}
\tag{9}
$$

with $\bar z(t) = \lambda(t)z_2(t) + (1-\lambda(t))z_1(t)$ for some $\lambda(t)\in[0,1]$.

The following result gives an estimate on the decay rate of $|q_2(t)-q_1(t)|$.

Lemma 9. *Given $T>0$, $C>0$ and ϵ_0 small. For $i=1,2$, let $z_i(t)$ be solutions of* (5) *with $z_i(0)=(p_i,q_i)$, where $(p_i,q_i)\in N_0$ and $|p_2-p_1|<C\epsilon_0$, then there exist $C(T)>0$, $K>0$ such that, as long as $z_i(t)\in N_0$ for $t\le T$,*

$$
\begin{aligned}
|q_2(T)-q_1(T)| \le & \left(e^{-d_0\int_0^T\chi\,dt + C(T)\epsilon_0}||\Phi_s(y,T)|| + C(T)\epsilon_0\right)|q_2-q_1| \\
& + C(T)\epsilon_0|p_2-p_1|,
\end{aligned}
$$

where $y=Q(p_1)$. In particular, if $y=Q\pi_{cu}z$, then

$$
||D_z(\pi_s\phi^T)|_{TS_{\epsilon_0}(p)}|| \le Ke^{-d_0\int_0^T\chi\,dt + C(T)\epsilon_0}||\Phi_s(y,T)|| + C(T)\epsilon_0
$$

and

$$
||D_z(\pi_s\phi^T)|_{TU_0}|| \le C(T)\epsilon_0.
$$

Proof. For simplicity, all constants below depending only on T will be denoted by $C(T)$.

Let y, $w(t)$ and $B(y\cdot t)w(t)$ be as above. By Lemma 7,

$$
|q_2(t)-q_1(t)| = |\pi_s z_2(t) - \pi_s z_1(t)| = |D\pi_s(\tilde z(t))(z_2(t)-z_1(t))|,
$$

where $|\tilde{z}(t) - y \cdot t| \le C(T)\epsilon_0$. It follows that

$$|q_2(t) - q_1(t)| \le |P_s(y \cdot t)w(t)| + C(T)\epsilon_0 |w(t)|.$$

Using the identity in Lemma 6 and the equation (8), we have

$$\begin{aligned}(P_s(y\cdot t)w)' =& A(y\cdot t)P_s(y\cdot t)w + P_s(y\cdot t)B(y\cdot t)w + O(\epsilon_0 P_s(y\cdot t)w)\\ =& \left(A(y\cdot t) - d_0\chi(\eta)\right)P_s(y\cdot t)w\\ & - C_2\epsilon_0\chi'(\eta)\langle\nabla\eta, w\rangle P_s(y\cdot t)v(\tilde{z}(t)) - C_2\epsilon_0\chi(\eta)P_s(y\cdot t)Jvw\\ & - d_0\chi'(\eta)\langle\nabla\eta, w\rangle P_s(y\cdot t)\tilde{z}(t) + O(\epsilon_0 w).\end{aligned} \tag{10}$$

Applying the variation of constant formula to (10) yields

$$\begin{aligned}P_s(y\cdot t)w(t) =& e^{-d_0\int_0^t \chi(\eta)\,ds}\Phi(y,t)P_s(y)w(0)\\ & -C_2\epsilon_0\int_0^t e^{-d_0\int_s^t \chi(\eta)\,d\tau}\Phi(y,t)\Phi^{-1}(y,s)\chi'(\eta)\langle\nabla\eta, w\rangle P_s(y\cdot s)v(\tilde{z}(s))\,ds\\ & -C_2\epsilon_0\int_0^t e^{-d_0\int_s^t \chi(\eta)\,d\tau}\Phi(y,t)\Phi^{-1}(y,s)\chi(\eta)P_s(y\cdot s)Jv(\tilde{z}(s))w(s)\,ds\\ & -d_0\int_0^t e^{-d_0\int_s^t \chi(\eta)\,d\tau}\Phi(y,t)\Phi^{-1}(y,s)\chi'(\eta)\langle\nabla\eta, w\rangle P_s(y\cdot s)\tilde{z}(s)\,ds\\ & +\int_0^t e^{-d_0\int_s^t \chi(\eta)\,d\tau}\Phi(y,t)\Phi^{-1}(y,s)O(\epsilon_0 w(s))\,ds.\end{aligned}$$

Thus,

$$\begin{aligned}|e^{d_0\int_0^t \chi(\eta)\,ds}\Phi^{-1}(y,t)P_s(y\cdot t)w(t)| \le& |P_s(y)w(0)|\\ & +C_2\epsilon_0^2\int_0^t e^{d_0\int_0^s \chi(\eta),d\tau}|\Phi^{-1}(y,s)||\chi'(\eta)||\langle\nabla\eta, w\rangle|\,ds\\ & +C_2\epsilon_0^2\int_0^t e^{d_0\int_0^s \chi(\eta),d\tau}|\Phi^{-1}(y,s)|\chi(\eta)|P_s(y\cdot s)w(s)|\,ds\\ & +d_0C_0\epsilon_0^2\int_0^t e^{d_0\int_0^s \chi(\eta),d\tau}|\Phi^{-1}(y,s)||\chi'(\eta)||\langle\nabla\eta, w\rangle|\,ds\\ & +C(T)\epsilon_0\int_0^t |P_{cu}(y\cdot s)w(s)|\,ds.\end{aligned}$$

It follows from Lemma 5 that

$$\begin{aligned}&|e^{d_0\int_0^t \chi(\eta)\,ds}\Phi^{-1}(y,t)P_s(y\cdot t)w(t)|\\ &\le \left(|P_s(y)w(0)| + C(T)\epsilon_0\int_0^t |P_{cu}(y\cdot s)w(s)|\,ds\right)e^{C(T)\epsilon_0}.\end{aligned} \tag{11}$$

Using

$$(P_{cu}(y\cdot t)w)' = A(y\cdot t)P_{cu}(y\cdot t)w + P_{cu}(y\cdot t)B(y\cdot t)w + O(\epsilon_0 P_{cu}(y\cdot t)w) \tag{12}$$

and a similar argument as above, we see that

$$|P_{cu}(y\cdot t)w(t)| \le C(T)|w(0)| \tag{13}$$

for all $t \in [0,T]$.

Combining the estimates (11) and (13), and applying Lemma 5, we have

$$\begin{aligned}|q_2(T)-q_1(T)| \le &\left(e^{-d_0\int_0^T \chi(\eta(\tilde{z}(t)))\,dt + C(T)\epsilon_0}||\Phi_s(y,T)|| + C(T)\epsilon_0\right)|q_2-q_1| \\ &+ C(T)\epsilon_0|p_2-p_1|.\end{aligned}$$

This completes the lemma. ☐

Finally, we give an estimate on the decay rate of $|p_2(t)-p_1(t)|$.

Lemma 10. *Given $T>0$, $C>0$ and ϵ_0 small. For $i=1,2$, let $z_i(t)$ be solutions of* (5) *with $z_i(0)=(p_i,q_i)$, where $(p_i,q_i)\in N_0$ and $|p_2-p_1|$ is small, then there exist $C(T)>0$, $K>0$ such that, as long as $z_i(t)\in N_{\epsilon_0}$ for $t\le T$,*

$$\begin{aligned}|p_2(T)-p_1(T)| \ge &K^{-1}\left(e^{-C(T)\int_0^T \chi(\eta)\,ds - C(T)\epsilon_0}||\Phi_s(y,t)||^\beta - C(T)\epsilon_0\right)|p_2-p_1| \\ &- C(T)\epsilon_0|q_2-q_1|,\end{aligned}$$

where $y=Q(p_1)$. In particular, if $y=Q(p)$, then

$$|D_p(\pi_{cu}\phi^T)| \ge K^{-1}e^{-C(T)\int_0^T \chi(\eta)\,ds - C(T)\epsilon_0}||\Phi_s(y,t)||^\beta - C(T)\epsilon_0.$$

Proof. Let y, $w(t)$ and $B(y\cdot t)$ be as in Lemma 9. Similar to the proof of Lemma 9, we have

$$\begin{aligned}|p_2(t)-p_1(t)| =&|\pi_{cu}z_2(t)-\pi_{cu}z_1(t)| = |D\pi_{cu}(\tilde{z}(t))(z_2(t)-z_1(t))| \\ \ge&|P_{cu}(y\cdot t)w(t)| - C(T)\epsilon_0|w(t)|,\end{aligned} \tag{14}$$

where $\tilde{z}(t)$ is such that $|\tilde{z}(t)-y\cdot t|\le C(T)\epsilon_0$.

By Lemma 6 and (8), we have

$$\begin{aligned}(P_{cu}(y\cdot(T-t))w(T-t))' =& -A(y\cdot(T-t))P_{cu}(y\cdot(T-t))w(T-t) \\ &+ P_{cu}(y\cdot(T-t))B(y\cdot(T-t))w(T-t) \\ &+ O(\epsilon_0 P_{cu}(y\cdot(T-t))w(T-t)).\end{aligned}$$

Since $\Phi(y,T-t)\Phi^{-1}(y,T)$ is the principal matrix solution of

$$x' = -A(y\cdot(T-t))x,$$

the variation of constant formula yields

$$\begin{aligned}P_{cu}(y\cdot(T-t))w(T-t) =& \Phi(y,T-t)\Phi^{-1}(y,T)P_{cu}(y\cdot T)w(T) \\ &+ \int_0^t \Phi(y,T-t)\Phi^{-1}(y,T-s)P_{cu}(y\cdot(T-s))B(y\cdot(T-s))w(T-s)\,ds \\ &+ \int_0^t \Phi(y,T-t)\Phi^{-1}(y,T-s)O(\epsilon_0 w(T-s))\,ds,\end{aligned}$$

or equivalently,

$$\begin{aligned}\Phi(y,T)\Phi^{-1}(y,T-t)P_{cu}(y\cdot(T-t))w(T-t) =& P_{cu}(y\cdot T)w(T) \\ &+ \int_0^t \Phi(y,T)\Phi^{-1}(y,T-s)P_{cu}(y\cdot(T-s))B(y\cdot(T-s))w(T-s)\,ds \\ &+ \int_0^t \Phi(y,T)\Phi^{-1}(y,T-s)O(\epsilon_0 w(T-s))\,ds.\end{aligned}$$

It then follows from (9) that

$$\begin{aligned}|\Phi(y,T)\Phi^{-1}(y,T-t)P_{cu}(y\cdot(T-t))w(T-t)| &\le |P_{cu}(y\cdot T)w(T)|\\ &+C(T)\epsilon_0\int_0^t |\chi'(\eta)||\Phi(y,T)\Phi^{-1}(y,T-s)P_{cu}(y\cdot(T-s))w(T-s)|\,ds\\ &+C(T)\int_0^t \chi(\eta)|\Phi(y,T)\Phi^{-1}(y,T-s)P_{cu}(y\cdot(T-s))w(T-s)|\,ds\\ &+C(T)\epsilon_0\int_0^t |\chi'(\eta)||\Phi(y,T)\Phi^{-1}(y,T-s)P_{cu}(y\cdot(T-s))w(T-s)|\,ds\\ &+C(T)\epsilon_0\int_0^t |P_s(y\cdot(T-s))w(T-s)|\,ds.\end{aligned}$$

By Lemma 5, Lemma 9 and the estimate (4), we have

$$|\Phi(y,T)P_{cu}(y)w(0)| \le e^{C(T)\int_0^T \chi(\eta)\,ds + C(T)\epsilon_0}(|P_{cu}(y\cdot T)w(T)| + C(T)\epsilon_0|w(0)|),$$

or equivalently,

$$\begin{aligned}|p_2(T)-p_1(T)| \ge & e^{-C(T)\int_0^T \chi(\eta)\,ds - C(T)\epsilon_0}|\Phi(y,T)P_{cu}(y)w(0)|\\ & - C(T)\epsilon_0|P_{cu}(y)w(0)| - C(T)\epsilon_0|P_s(y)w(0)|\\ \ge & \left(e^{-C(T)\int_0^T \chi(\eta)\,ds - C(T)\epsilon_0} m(\Phi_{cu}(y,T)) - C(T)\epsilon_0\right)|p_2-p_1|\\ & - C(T)\epsilon_0|q_2-q_1|.\end{aligned}$$

By (H3), then

$$\begin{aligned}|p_2(T)-p_1(T)| \ge & K^{-1}\left(e^{-C(T)\int_0^T \chi(\eta)\,ds - C(T)\epsilon_0}||\Phi_s(y,t)||^\beta - C(T)\epsilon_0\right)|p_2-p_1|\\ & - C(T)\epsilon_0|q_2-q_1|.\end{aligned}$$

This completes the lemma. □

5. Proof of main results

With the estimates in Lemma 9 and Lemma 10, the proof of Theorem 1 is exactly the same as that for the smooth invariant manifolds in [6]. For the reader's convenience, we outline the main steps below.

5.1. **Center-unstable manifold.** We first choose a function space on which the graph transform will be performed. Define

$$\Gamma := \{h : U_0 \to N_0;\ h(p) \in S_{\epsilon_0}(p), \forall p \in U_0,\ |h|_{C^0} < \infty\},$$

that is, Γ is the set of sections of the fiber bundle N_0 with base space U_0. For $0 < \rho \le 1$, let

$$\Gamma_\rho := \{h : U_0 \to N_0;\ h(p) \in S_{\epsilon_0}(p), \forall p \in U_0,\ \mathrm{Lip}(h) \le \rho\}, \tag{15}$$

where

$$\mathrm{Lip}(h) = \sup_{p\in U_0} \mathrm{Lip}_p(h),\ \ \mathrm{Lip}_p(h) = \limsup_{p'\to p} \frac{|h(p')-h(p)|}{|p'-p|},\ \ p' \in U_0.$$

It follows from [6] that Γ_ρ is closed. Thereafter, the graph $\{(p,h(p)) :\ p \in U_0\}$ of $h \in \Gamma_\rho$ will be denoted as graph(h).

For fixed $T > 0$, let φ^T be the time T map of the modified flow (5). We define the *graph transform* as follows:

$$\varphi^* : \Gamma_\rho \to \Gamma_\rho : \varphi^* h = H,$$

where $\text{graph}(H) = \varphi^T(\text{graph}(h))$, provided that $\varphi^T(\text{graph}(h))$ is a graph.

Note that $H = \pi_s \cdot \varphi^T \cdot (id, h) \cdot \sigma_h^{-1}$ where $\sigma_h(p) = \pi_{cu} \cdot \varphi^T \cdot (id, h)(p)$, for $p \in U_0$.

The existence of a center-unstable manifold of (5) is due to the following result.

Proposition 2. *There exist $T > 0$, $\epsilon_0(T) > 0$ and $0 < \rho \le 1$, such that the following holds.*

(i) *$\varphi^* : \Gamma_\rho \to \Gamma_\rho$ is well-defined;*
(ii) *φ^* is a contraction mapping with respect to the C^0-norm;*
(iii) *Let $h \in \Gamma_\rho$ be the fixed point of φ^*. Then $M_{cu}(Y) := graph(h)$ is an invariant Lipschitz manifold of* (5).

Proof. Let $h \in \Gamma_\rho$. Fix a $(p_0, q_0) \in \phi^T(\text{graph}(h)) \cap N_{\epsilon_0}$ and define $(p^*, q^*) = \varphi^{-T}(p, q)$ for $(p, q) \in \varphi^T(\text{graph}(h)) \cap N_{\epsilon_0}$. Then there exists a $\delta > 0$ such that $|(p, q) - (p_0, q_0)| \le \delta$ implies that $|(p^*, q^*) - (p_0^*, q_0^*)| \le \epsilon_0$.

Let $(p, q) \in \phi^T(graph(h)) \cap N_{\epsilon_0}$ be such that $|(p, q) - (p_0, q_0)| \le \delta$ and denote $(p_1, q_1) = (p_0^*, q_0^*), (p_2, q_2) = (p^*, q^*)$. By the overflowing property of the modified flow, we have $\varphi^t(p_i, q_i) \in N_{\epsilon_0}$, $i = 1, 2$, for all $0 \le t \le T$. By Lemmas 9, 10,

$$\begin{aligned}
\frac{|q - q_0|}{|p - p_0|} =& \frac{|q_2(T) - q_1(T)|}{|p_2(T) - p_1(T)|} \\
\le& \frac{\left(K e^{-d_0 \int_0^T \chi} ||\Phi_s(y, T)|| + C(T)\epsilon_0\right) |q_2 - q_1| + C(T)\epsilon_0 |p_2 - p_1|}{\left(e^{-C(T)\int_0^T \chi} ||\Phi_s(y, T)||^\beta - C(T)\epsilon_0\right) |p_2 - p_1| - C(T)\epsilon_0 |q_2 - q_1|} \\
\le& \left(K e^{(-d_0 + C(T))\int_0^T \chi} ||\Phi_s(y, T)||^{1-\beta} + C(T)\epsilon_0\right) \rho.
\end{aligned}$$

Using (H3), we can choose T sufficiently large then ϵ_0 sufficiently small so that the right hand side of the above is less or equal to ρ. This proves (i).

To prove (ii), we take $h_1, h_2 \in \Gamma_\rho$ and denote $H_i = \phi^*(h_i)$, for $i = 1, 2$. For any $p \in U_{\epsilon_0}$, let $p_1, p_2 \in U_{\epsilon_0}$ be such that $\pi_{cu} \cdot \varphi^T(p_i, h_i(p_i)) = p$, $i = 1, 2$. It follows from Lemma 9 and Lemma 10 that

$$\begin{aligned}
|H_2(p) - H_1(p)| =& |\pi_s \varphi^T(p_2, h_2(p_2)) - \pi_s \varphi^T(p_1, h_1(p_1))| \\
\le& |\pi_s \varphi^T(p_1, h_1(p_1)) - \pi_s \varphi^T(p_1, h_2(p_1))| \\
&+ |\pi_s \varphi^T(p_1, h_2(p_1)) - \pi_s \varphi^T(p_2, h_2(p_2))| \\
\le& \left(e^{-d_0 \int_0^T \chi} ||\Phi_s(y, T)|| + C(T)\epsilon_0\right) |h_2(p_1) - h_1(p_1)| + C(T)\epsilon_0 |p_2 - p_1| \\
\le& \left(e^{-d_0 \int_0^T \chi} ||\Phi_s(y, T)|| + C(T)\epsilon_0\right) \left(1 + \frac{C(T)\epsilon_0}{e^{C(T)\int_0^T \chi} ||\Phi_s(y, T)||^\beta - C(T)\epsilon_0}\right) \\
&\cdot |h_2 - h_1|.
\end{aligned}$$

Again, by choosing T sufficiently large then ϵ_0 sufficiently small in the above, we can make the above coefficient of $|h_2 - h_1|$ smaller than 1. This proves (ii).

(iii) follows from the fact that φ^T is the time-T map of (5). □

The C^d smoothness of $M_{cu}(Y)$ follows from Lemmas 9, 10 and the C^r section theorem in [16], [30]. See [6] for details.

5.2. **Proof of Theorem 1.** Let $M_{cu}(Y)$ be the center-unstable manifold obtained in Proposition 2. Since $\hat{f}$ agrees with f in a small neighborhood of Y, $M_{cu}(Y)$ is also a locally invariant C^d manifold of (1) and $T_y M_{cu}(Y) = V_{cu}(y)$ for all $y \in Y$. By reversing time, we similarly obtain a locally invariant C^d center-stable manifold $M_{cs}(Y)$ of (1) in a neighborhood of Y with $T_y M_{cs}(Y) = V_{cs}(y)$ for all $y \in Y$. The intersection $M_c(Y) = M_{cu}(Y) \cap M_{cs}(Y)$ then gives a desired locally invariant C^d center manifold of (1) with $T_y M_c(Y) = V_c(y)$ for all $y \in Y$. Since $M_{cu}(Y)$ carries all solutions of (1) locally bounded in backward time and $M_{cs}(Y)$ carries all solutions of (1) locally bounded in forward time, $M_c(Y)$ carries all solutions of (1) which are locally bounded in $\mathbb{R}$.

The C^d persistent of $M_c(Y)$ follows from arguments of [10] and [16].

The proof of Theorem 1 is now complete.

5.3. **Proof of Theorem 2.** Without loss of generality, we assume that a_{j_0+1} and b_{i_0-1} are finite.

Consider (3) and denote $V_s(y) = V_{1,i_0-1}(y)$, $V_u(y) = V_{j_0+1,k}(y)$ $(y \in Y)$. Let $P_i(y) : T_y\mathbb{R}^n \to V_i(y)$, $y \in Y$, $i = s, c, u$, be the associated projections. Then

$$T_y\mathbb{R}^n = V_s(y) \oplus V_c(y) \oplus V_u(y), \quad y \in Y,$$

is a continuous invariant splitting of (3). It also follows from the Sacker-Sell spectral theory ([27]) that there is a $K > 0$ such that

$$\begin{aligned}
|\Phi(y,t)P_s(y)\Phi^{-1}(y,s)| &\leq Ke^{(b_{i_0-1}+\lambda/2)(t-s)}, \quad t \geq s;\\
|\Phi(y,t)P_c(y)\Phi^{-1}(y,s)| &\leq Ke^{(b_{j_0}+\lambda/2)(t-s)}, \quad t \geq s;\\
|\Phi(y,t)P_c(y)\Phi^{-1}(y,s)| &\leq Ke^{(a_{i_0}-\lambda/2)(t-s)}, \quad t \leq s;\\
|\Phi(y,t)P_u(y)\Phi^{-1}(y,s)| &\leq Ke^{(a_{j_0+1}-\lambda/2)(t-s)}, \quad t \leq s.
\end{aligned}$$

Let $\alpha = \max\{b_{i_0-1} + \lambda, \lambda - a_{j_0+1}\}$ and choose β to be such that

$$\frac{a_{i_0} - \lambda}{b_{i_0-1} + \lambda} \leq \beta < \min\left\{\frac{b_{j_0}}{a_{j_0+1}}, \frac{a_{i_0}}{b_{i_0-1}}\right\}.$$

Then (H3) is satisfied with the constants α and β.

6. Applications

Our result is not just an abstract extension to the classical center manifolds theorem and other existing results in this area. It is strongly motivated by problems of higher dimensional, global bifurcations, and perturbations (especially singular perturbations). Below, we give three examples arising in problems of homoclinic bifurcations, singular perturbations, and bifurcations from tori.

Example 6.1. Consider a sufficiently smooth vector field

$$x' = f(x), \; x \in R^n, \tag{16}$$

which admits a homoclinic orbit Γ to a hyperbolic equilibrium P.

We group the eigenvalues of P into three disjoint sets E_s, E_c, E_u, where E_c consists of real parts of the eigenvalues associated to the homoclinic orbit Γ, i.e., Γ approaches P along the eigendirections of eigenvalues with real parts contained in E_c, and, E_s (E_u resp.) integrates positive (negative resp.) real parts of the rest of eigenvalues. Let V_c be the linear subspace spanned by eigenvectors associated to

E_c and denote $m = \dim V_c$, $\alpha = \max E_s$, $\beta_- = \min E_c$, $\beta_+ = \max E_c$, $\gamma = \min E_u$, $d = \max\{[\frac{\beta_-}{\alpha}], [\frac{\gamma}{\beta_+}]\}$, here $[\cdot]$ means integral parts.

Consider the linearization of (16) along $Y = \Gamma \cup \{P\}$:

$$x' = Jf(y \cdot t)x, \ y \in Y. \tag{17}$$

Since the end points of Sacker-Sell spectral intervals are Lyapunov exponents ([18]), these end points must be contained in $E_s \cup E_c \cup E_u$, and moreover, $[\beta_-, \beta_+]$ belongs to the spectral interval containing 0.

Corollary 1. *Assume that the spectral interval containing* 0 *is precisely* $[\beta_-, \beta_+]$. *Then* Y *admits a* m*-dimensional* C^d *center manifold* M *with* $T_P M = V_c$.

Proof. We first note that, as a homoclinic loop, Y is admissible. Let $V_s(Y)$, $V_c(Y)$, $V_u(Y)$ be the stable, center, unstable spectral subbundles associated to (17). Then $V_c(Y)$ is associated to the spectral interval $[\beta_-, \beta_+]$ and $V_c(P) = V_c$. Using the definition of GTB and V_c, it is easy to see that $V_c(Y)$ is also a GTB of Y. The lemma follows immediately from Theorem 2. □

Remark 4. (a) Using the above corollary, one can certainly have very low dimensional center manifolds for homoclinic loops. For two dimensional center manifolds, let us consider a vector field in R^3. Then the dynamical condition assumed in the corollary can be replaced by the following weaker condition (see [6]): The closure of the stable manifold $W_s(P)$ of P in a neighborhood of Y is a manifold, that is, a topological cylinder or Mobius band (no matter how many twists occur when evolving along Γ). Typical models for three dimensional center manifolds would be those containing a Shilnikov orbit.

(b) For heteroclinic cycles, a similar dynamical condition as in the corollary is however not sufficient to guarantees the existence of a center manifold with the same dimension as the asymptotic subspaces (similarly defined as V_c above) associated to the hyperbolic equilibria (see [6]).

Example 6.2. In singular perturbation problems, turning points play the dominate role in understanding the dynamics in the vicinity of slow manifolds. Presence of turning points destroys the normal hyperbolicity of the slow manifold at the relevant place. However, due to the special structure of singular perturbation problems, the center manifolds theorem remedies the geometric difficulty, and, as an advantage, all hypotheses (H1)–(H3) can be trivially verified.

Consider a singularly perturbed system in the slow time scale τ

$$\begin{aligned} \epsilon\dot{x}(\tau) &= f(x, y; \epsilon), \\ \dot{y}(\tau) &= g(x, y; \epsilon), \end{aligned} \tag{18}$$

where $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$, f and g are C^r in their arguments. Let $\mathcal{M}_0 = \{(x, y) : x = H(y), \ y \in D\}$ be a smooth, compact portion of the slow manifold $\{(x, y) : f(x, y; 0) = 0\}$, which we assume to be the graph of a smooth function $H : D \subset \mathbb{R}^n \to \mathbb{R}^m$. In the fast time scale $t = \tau/\epsilon$, the system (18) becomes

$$\begin{aligned} x'(t) &= f(x, y; \epsilon), \\ y'(t) &= \epsilon g(x, y; \epsilon). \end{aligned} \tag{19}$$

Then the manifold $\mathcal{M}_0$ consists of equilibria of (19) at $\epsilon = 0$, and, the linearization of (19) along $\mathcal{M}_0$ is

$$\begin{pmatrix} f_x(H(y), y; 0) & f_y(H(y), y; 0) \\ 0 & 0 \end{pmatrix}, \ y \in D.$$

If $\mathrm{Re}\lambda(y) \neq 0$ for all eigenvalues $\lambda(y)$ of $f_x(H(y), y; 0)$, then $\mathcal{M}_0$ is normally hyperbolic and hence is persistent for $\epsilon \neq 0$. If $\mathrm{Re}\lambda(y) = 0$ for a $y \in D$, then $\mathcal{M}_0$ is no longer normally hyperbolic and y becomes a *turning point* of $\mathcal{M}_0$.

Assume that there exist $\alpha_1 < 0 < \alpha_2$, and integers p, q, r with $p + q + r = m$ such that the eigenvalues $\lambda_i(y)$, $i = 1, 2, \cdots, m$, of $f_x(H(y), y; 0)$ satisfy

$$\mathrm{Re}\lambda_i(y) \leq \alpha_1 < \mathrm{Re}\lambda_j(y) < \alpha_2 \leq \mathrm{Re}\lambda_k(y)$$

for all $y \in D$, $i = 1, \cdots, p$, $j = p + 1, \cdots, p + q$ and $k = p + q + 1, \cdots, m$. In particular, we have allowed the real parts of q eigenvalues above to undergo sign changes.

Corollary 2. *Under the above assumption, if $\epsilon \neq 0$ is sufficiently small, then* (18) *or* (19) *admits a q dimensional, C^r normally hyperbolic invariant manifold $\mathcal{M}_\epsilon$, varying C^r in ϵ. Moreover,* $\lim_{\epsilon \to 0} \mathcal{M}_\epsilon = \mathcal{M}_0$ *exists in C^r sense, and $T_{(H(y),y)} M_0$ coincides with the generalized eigenspace associated to the eigenvalues $\lambda_j(y)$ for all $y \in D$ and $j = p + 1, \cdots, p + q$.*

Proof. Consider the extended system

$$\begin{aligned} x'(t) =& f(x, y; \epsilon), \\ y'(t) =& \epsilon g(x, y; \epsilon), \\ \epsilon' =& 0. \end{aligned} \tag{20}$$

Then $Y = \mathcal{M}_0 \times \{0\}$ is a compact invariant set of (20) consisting of equilibria. Moreover, the linearization of (20) along Y has the coefficient matrix

$$\begin{pmatrix} f_x(H(y), y; 0) & f_y(H(y), y; 0) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ y \in D.$$

It is then easy to see that the hypotheses (H1)–(H3) hold with $V_s(y) \times \{0\}$, $V_c(y) \times \{0\}$, $V_u(y) \times \{0\}$, where $V_s(y)$, $V_c(y)$, $V_u(y)$ are generalized eigenspaces associated to the eigenvalues $\lambda_i(y)$, $i = 1, 2, \cdots, m$, $\lambda_j(y)$, $j = p + 1, \cdots, p + q$, and $\lambda_k(y)$, $k = p + q + 1, \cdots, m$ respectively. ◻

Using Corollary 2, one can reduce (18) near the manifolds M_ϵ as follows, via a special coordinates change mimicking the one in [11].

Corollary 3. *There exists a local coordinates system $(u, v, w, y) \in \mathbb{R}^p \times \mathbb{R}^r \times \mathbb{R}^q \times \mathbb{R}^n$ in a neighborhood of $\mathcal{M}_0$ such that the system* (18) *is reduced to*

$$\begin{aligned} \epsilon\dot{u} &= U(u, v, w, y; \epsilon)u, \\ \epsilon\dot{v} &= V(u, v, w, y; \epsilon)v, \\ \epsilon\dot{w} &= W(u, v, w, y; \epsilon), \\ \dot{y} &= g(u, v, w, y; \epsilon), \end{aligned}$$

where $U(0, 0, 0, y; 0)$, $V(0, 0, 0, y; 0)$ and $W_w(0, 0, 0, y; 0)$ admit eigenvalues $\lambda_i(y)$, $\lambda_k(y)$ and $\lambda_j(k)$ for $i = 1, \cdots, p$, $j = p+1, \cdots, p+q$ and $k = p+q+1, \cdots, p+q+r$, respectively. ◻

We remark that both corollaries can be modified slightly to fit in the so-called singular singularly perturbed systems of the form:

$$\epsilon x' = f(x).$$

Example 6.3. The last example concerns the center manifolds of invariant tori which arises naturally in quasi-periodic bifurcations.

Consider a smooth vector field

$$\begin{aligned} x' =& A(\theta,\lambda)x + F(x,\theta,\lambda), \\ \theta' =& \omega + G(x,\theta,\lambda), \end{aligned} \tag{21}$$

where $(x,\theta) \in \mathbb{R}^n \times T^k$, ω is a constant vector, λ is a parameter. We assume that at a particular parameter value λ_0, $A(\theta,\lambda_0) \equiv A_0$ – a constant matrix, and $F(x,\theta,\lambda_0) = O(|x|^2)$, $G(x,\theta,\lambda_0) = O(|x|^2)$. It is clear that $\{0\} \times T^k$ is an invariant torus of (21) for $\lambda = \lambda_0$.

Let V_s, V_c, V_u be the stable, center and unstable subspaces associated to eigenvalues of A_0 having negative, zero, and positive real parts, respectively. We denote the dimensions of V_s, V_c, V_u by p, q, r $(p+q+r=n)$ respectively.

Corollary 4. *For λ sufficiently close to λ_0, the system* (21) *admits a smooth varying family of $q+k$ dimensional smooth invariant manifolds M_λ in the vicinity of $\{0\} \times T^k$ with $TM_{\lambda_0} = V_c \times T^k$.*

Proof. Consider the extended vector field

$$\begin{aligned} x' =& A(\theta,\lambda)x + F(x,\theta,\lambda), \\ \theta' =& \omega + G(x,\theta,\lambda), \\ \lambda' =& 0. \end{aligned} \tag{22}$$

Then $Y = \{0\} \times T^k \times \{\lambda_0\}$ is an invariant torus of the extended vector field with linearized invariant subbundles $V_i(Y) = V_i \times T^k \times \{\lambda_0\}$, $i = s, c, u$. By the constancy of the linearization along Y, the hypotheses (H1)–(H3) are trivially satisfied. □

If

$$A(\theta, 0) = \begin{pmatrix} A_u & 0 & 0 \\ 0 & A_s & 0 \\ 0 & 0 & A_c \end{pmatrix},$$

where A_i is associated to V_i for $i = u, s, c$, respectively. Then, by the triviality of the linearized subbundles at λ_0, in a vicinity of $\{0\} \times T^k$, one can use the standard coordinate $w \in R^r$ to rewrite the flow on M_λ as

$$\begin{aligned} w' =& A_c(\theta,\lambda)w + F_c(w,\theta,\lambda), \\ \theta' =& \omega + G_c(w,\theta,\lambda), \end{aligned}$$

where $|A_c(\theta,\lambda) - A_c| = O(|\lambda - \lambda_0|)$.

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