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Topological Dynamics and Differential Equations

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Abstract.
By reviewing our previous works on lifting dynamics in skew-product semi-flows and also the work of Johnson on almost periodic Floquet theory, we show several significant applications of the abstract theory of topological dynamics to the qualitative study of non-autonomous differential equations. The paper also contains some detailed discussions on a conjecture of Johnson.

1. Introduction

We discuss connections between the abstract theory of topological dynamics, especially the algebraic theory of Ellis, and the qualitative study of non-autonomous differential equations. As remarked by Ellis in his book ([**5**]), the abstract theory of topological dynamics usually plays less of a role in the qualitative study of autonomous differential equations because, not only is the differential structure ignored but the topological properties of the reals are not used in an essential manner. Nevertheless, since the introduction of the notion of a continuous *skew-product flow* by Miller ([**20**]) and Sell ([**28**]) in 1960's, the theory of topological dynamics has found significant applications in many essential ways to the study of non-autonomous ordinary, partial and functional differential equations.

To show how topological dynamics comes into play, let us recall the construction of a skew-product flow from a non-autonomous ordinary differential equation:

$$x' = f(x,t), \quad x \in \mathbb{R}^n, \tag{1.1}$$

where f is C^2 *admissible*, that is, f is C^2 in x and Lipschitz in t, and moreover, for any compact set $K \subset \mathbb{R}^n$, f as well as all its partial derivatives are bounded and uniformly continuous on $K \times \mathbb{R}$. Due to the time dependence, (1.1) does not generate a flow on $\mathbb{R}^n$ itself. One alternative would be to add one dimension and make it autonomous since the system

$$\begin{cases} x' = f(x,t) \\ t' = 1 \end{cases} \tag{1.2}$$

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clearly defines a flow $\tilde{\Pi}$ on $\mathbb{R}^n \times \mathbb{R}$. Unfortunately, in the dynamical system (1.2), all ω, α-limit sets will be empty due to the lack of compactness of solutions.

Therefore, to study the dynamics of (1.1), one has to take into account the effect of its coefficient structure. Such a structure should mainly capture the topological rather than differential natures of the coefficient space because, the differentiability of f with respect to t usually only contributes to the regularity of solutions but not to their dynamical behavior. To be more precise, let the coefficient space be $H(f)$, the *hull* of f, that is, $H(f) = cl\{f_\tau | \tau \in \mathbb{R}\}$ under compact open topology, where $f_\tau(x,t) \equiv f(x, t+\tau)$. The time translate $g \cdot t = g_t$ then defines a natural flow $(H(f), \mathbb{R})$, and moreover, this flow is minimal or almost periodic minimal if f is a minimal or an almost periodic function in t uniformly with respect to other variables. For each $g \in H(f)$, $x_0 \in \mathbb{R}^n$, let $x(x_0, g, t)$ denote the solution of

$$x' = g(x,t) \tag{1.3$_g$}$$

with initial value x_0. By the standard local existence, uniqueness and continuity results for ordinary differential equations, equation (1.1) gives rise to a (local) *skew-product flow* $\Pi : \mathbb{R}^n \times H(f) \times \mathbb{R} \to \mathbb{R}^n \times H(f)$,

$$\Pi(x_0, g, t) = (x(x_0, g, t), g \cdot t), \tag{1.4}$$

where $x(x_0, g, t)$ is C^1 in x_0.

Since the family $(1.3)_g$ $(g \in H(f))$ consists of only translated and limiting equations of (1.1), the (local) skew-product flow (1.4) precisely reflects the 'dynamics' of (1.2) especially when long time behavior of solutions are concerned. Such a topological setting allows one to apply general techniques developed in the abstract theory of topological dynamics since the natural projection $p : \mathbb{R}^n \times H(f) \to H(f)$ induces a flow homomorphism from a compact invariant set of (1.4) to $H(f)$.

Concerning with the qualitative study of a non-autonomous differential equation, there are essentially three types of problems which are closely related to the abstract theory of topological dynamics: a) Global structure of a system (see the almost periodic Floquet theory in Section 3 for example). b) Asymptotic behavior of bounded solutions. In this context, one tends to study the lifting property of an ω-limit set in the associated skew-product flow. c) The existence of certain recurrent or oscillatory solutions. For example, if f in (1.1) is almost periodic in t, then one is often interested in an almost periodic or an almost automorphic minimal lifting of (1.4) from the almost periodic base flow $(H(f), \mathbb{R})$. Clearly, these lifting properties will heavily depend on certain differential structures (e.g. stability, hyperbolicity and monotonicity) in a particular differential equation.

Motivated by applications, there has been a tremendous amount of studies on non-autonomous differential equations and (continuous) skew-product flows or semi-flows in the past twenty years or so. The aim of the current article is however not to give a survey to this broad and active area. Instead, by reviewing several existing works, we are trying to explore the fact that the application of the abstract theory of topological dynamics can be essential in the qualitative study of non-autonomous differential equations.

The paper is organized as follows. In Section 2, we summarize basic concepts and fundamental results from topological dynamics which are used in the current paper. For our particular applications, we shall deal with only real flows, that is, transformation groups with $\mathbb{R}$ as the acting group. In Section 3, we review some early results of Sacker and Sell ([**25**]) and recent results of Shen and Yi ([**34**]) on lifting properties in skew-product semi-flows. Section 4 is a brief review of the almost periodic Floquet theory due to Johnson ([**13**]). We will also give some discussions on a conjecture of Johnson in [**13**] and indicate several fundamental topological dynamics issues related to it. It should be pointed out that the current paper is by no means even a complete review of these works mentioned above. Also, an apology must be made in advance for failure to mention many (in fact most of) important works in the area of non-autonomous differential equations and skew-product (semi)flows.

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2. Algebraic theory of topological dynamics

An algebraic way to study the nature of a compact flow was introduced by R. Ellis ([**5**]). One basic idea of the algebraic theory is to associate a semigroup, the *Ellis semigroup*, or *enveloping semigroup*, to a compact flow.

Let $(X, \mathbb{R})$ be a compact flow. The space X^X of self maps of X, when furnished with the point open topology, is a compact T_2 space, and, composition of maps provides a natural semigroup structure on X^X. For each $t \in \mathbb{R}$, we note that $\Pi_t : X \to X$, $\Pi_t x =: x \cdot t$ defines a homeomorphism, hence an element of X^X.

Definition 2.1. $E(X) = cl\{\Pi_t | t \in \mathbb{R}\} \subset X^X$ is called the *Ellis semigroup* associated to $(X, \mathbb{R})$.

Clearly, $E(X)$ is a sub-semigroup of X^X with identity $e = \Pi_0$, and the composition $\Pi_t \circ \gamma =: \gamma \cdot t$ ($\gamma \in E(X), t \in \mathbb{R}$) defines a compact point flow $(E(X), e, \mathbb{R})$. Throughout the paper, using the identification $\Pi_t x =: x \cdot t$ ($x \in X, t \in \mathbb{R}$), we shall choose the left action of $E(X)$ on a compact flow $(X, \mathbb{R})$ although the action of $\mathbb{R}$ on X has been assumed on the right.

Theorem 2.1. (Ellis [**5**]) *Let $p : (X, \mathbb{R}) \to (Y, \mathbb{R})$ be an epimorphism of compact flows. Then there exists a unique epimorphism $\tilde{p} : (E(X), \mathbb{R}) \to (E(Y), \mathbb{R})$ such that $p(\gamma x) = (\tilde{p}\gamma)(px)$ for all $\gamma \in E(X), x \in X$.*

In what follows, we will write $\tilde{p}$ as p if no confusion occurs.

Definition 2.2. 1) A (left) *ideal* in $E(X)$ is a non-empty subset I in $E(X)$ with $E(X)I \subset I$. A (left) ideal I in $E(X)$ is said to be *minimal* if it contains no non-empty proper (left) sub-ideal in $E(X)$.

2) An *idempotent point* $u \in E(X)$ is such that $u^2 = u$.

It is observed in [**5**] that I is an (left) (minimal) ideal in $E(X)$ if and only if I is an invariant (minimal) subset of the compact flow $(E(X), \mathbb{R})$. It follows that a minimal (left) ideal in $E(X)$ always exists. The structure of a minimal (left) ideal is as follows.

Theorem 2.2. (Ellis [**5**]) *Let I be a minimal (left) ideal in $E(X)$ and $J(I)$ be the set of idempotent points of $E(X)$ in I. Then the following holds:*

1) *$J(I) \neq \emptyset$;*
2) *For each $u \in J(I)$, uI is a group with identity u and the family $\{uI\}_{u \in J(I)}$ forms a partition of I.*

DEFINITION 2.3. 1) Points $x_1, x_2 \in X$ are said to be *distal* (*positively distal, negatively distal*) if there is a pseudo-metric d on X such that

$$\inf_{t\in\mathbb{R}(t\in\mathbb{R}^+,t\in\mathbb{R}^-)} d(x_1\cdot t, x_2\cdot t) > 0.$$

x_1, x_2 are said to be *proximal* (*positively proximal, negatively proximal*) if they are not distal (positively distal, negatively distal).

2) $x \in X$ is a *distal point* if it is only proximal to itself. $(X, \mathbb{R})$ is a *point distal flow* if there is a distal point $x_0 \in X$ with dense orbit. $(X, \mathbb{R})$ is a *distal flow* if every point in X is a distal point.

3) Let $p : (X, \mathbb{R}) \to (Y, \mathbb{R})$ be an epimorphism of compact flows. Then $(X, \mathbb{R})$ is a *distal* (*proximal*) *extension* of $(Y, \mathbb{R})$ if for all $y \in Y$, any two points x_1 and x_2, on a same fiber $p^{-1}(y)$, are distal (proximal).

Clearly, a distal extension of a distal flow is distal.

DEFINITION 2.4. The set $P(X) = \{(x_1, x_2) \in X \times X | x_1, x_2 \text{ are proximal}\}$ is referred to as the *proximal relation* of $(X, \mathbb{R})$.

Some consequences of distality and proximality are summarized below.

THEOREM 2.3. (Ellis [**5**]) 1) *$(X, \mathbb{R})$ is distal if and only if $E(X)$ is a group.*

2) *If $(X, \mathbb{R})$ is distal, then it laminates into minimal sub-flows, that is, X is a union of minimal sets.*
3) *$(x_1, x_2) \in P(X)$ if and only if there exists a minimal (left) ideal I in $E(X)$ such that $\gamma x_1 = \gamma x_2$ $(\gamma \in I)$.*
4) *$P(X)$ is an equivalence relation if and only if there is only one minimal (left) ideal in $E(X)$.*
5) *If $P(X)$ is closed, then it is an equivalence relation.*

THEOREM 2.4. (Auslander [**1**], Furstenberg [**8**]) *Let $p : (X, \mathbb{R}) \to (Y, \mathbb{R})$ be a homomorphism of distal minimal flows. Then p is an open map.*

DEFINITION 2.5. 1) $(X, \mathbb{R})$ is (uniformly) *almost periodic* if $\{\Pi_t | t \in \mathbb{R}\} \subset X^X$ forms an equicontinuous family.

2) A minimal flow $(X, \mathbb{R})$ is *almost automorphic* if there is $x_0 \in X$ such that whenever t_α is a net with $x_0 \cdot t_\alpha \to x_*$, then also $x_* \cdot (-t_\alpha) \to x_0$.

The notion of almost automorphy, as a generalization to almost periodicity, was first introduced by S. Bochner in 1955 in a work of differential geometry ([**2**]). Fundamental properties of almost automorphic functions on groups and abstract almost automorphic minimal flows were studied by W. A. Veech ([**39**]-[**41**]) and others (see [**7**], [**9**], [**22**], [**37**], [**38**]). Recently, Shen and Yi ([**30**]-[**36**], [**42**]) gave a systematic study on almost automorphic phenomena in almost periodic differential equations (see also Section 3).

THEOREM 2.5. (Ellis [**5**]) *$(X, \mathbb{R})$ is almost periodic if and only if $E(X)$ is a group of continuous maps of X into X.*

If $(X, \mathbb{R})$ is almost periodic minimal, then it is isomorphic to $(E(X), \mathbb{R})$ (see [**5**]). Using Theorem 2.5, it is easy to see that $(X, \mathbb{R})$ is almost periodic minimal if and only if $E(X)$ is a compact abelian topological group ([**42**]).

THEOREM 2.6. (Veech, 3.4 of [**39**]) *A minimal flow* $(X,\mathbb{R})$ *is almost automorphic if and only if it is an* almost automorphic extension *of an almost periodic minimal flow* $(Y,\mathbb{R})$*, that is, there is a* x_0 *with* $p^{-1}p(x_0) = \{x_0\}$*, where* $p : (X,\mathbb{R}) \to (Y,\mathbb{R})$.

We note that if X is a metric space, then the almost automorphic extension in the above theorem becomes an almost 1 to 1 extension ([**40**]).

3. Lifting properties in skew-product semi-flows

In this section, we give a brief review of works by the current authors concerning lifting properties in skew-product semi-flows related to qualitative study of non-autonomous differential equations.

Throughout the section, we assume that $\sigma = (Y,\mathbb{R})$ is a minimal flow with compact metric phase space Y, X is a Banach space, $\Pi = (X \times Y, \mathbb{R}_+)$ is a *skew-product semi-flow*, namely, a semi-flow of the following form:

$$\Pi(x, y, t) = (\phi(x, y, t), y \cdot t), \quad (x, y) \in X \times Y, \quad t \in \mathbb{R}_+. \tag{3.1}$$

We denote $p : X \times Y \to Y$ as the natural projection.

3.1. Skew product semi-flows generated by PDE's and FDE's. In the introduction, we have shown how to construct a skew product flow from a non-autonomous ordinary differential equation. We now give two examples of skew-product semi-flows generated by non-autonomous parabolic equations and delay differential equations.

EXAMPLE 1. Consider a scalar parabolic equation

$$\begin{cases} u_t = \Delta u + f(u, \bigtriangledown u, x, t), & t > 0, \quad x \in \Omega \\ u|_{\partial\Omega} = 0 \quad \text{or} \quad \dfrac{\partial u}{\partial n}|_{\partial\Omega} = 0, & t > 0, \end{cases} \tag{3.2}$$

where Ω is a bounded, connected, smooth domain in $\mathbb{R}^n$, $f : (\mathbb{R}^1 \times \mathbb{R}^n \times \mathbb{R}^n) \times \mathbb{R} \to \mathbb{R}^1$ is a C^2 admissible and (uniform) minimal function. By the standard theory of parabolic equations, for each $U_0 \in C^1(\bar{\Omega})$ which satisfies the boundary condition of (3.2) and for each $g \in H(f)$, the equation

$$\begin{cases} u_t = \Delta u + g(u, \bigtriangledown u, x, t), & t > 0, \quad x \in \Omega \\ u|_{\partial\Omega} = 0 \quad \text{or} \quad \dfrac{\partial u}{\partial n}|_{\partial\Omega} = 0, & t > 0 \end{cases} \tag{3.3$_g$}$$

has (locally) a unique classical solution $u(U_0, g, x, t)$ with initial value U_0.

We now define a (local) skew-product semi-flow over $H(f)$ similarly to the case of (1.1). To do so, the phase space X can be chosen as a suitable fractional power space ([**11**]) which is embedded in $C^1(\bar{\Omega})$. Having chosen such a X, one can show that if $U_0 \in X$, $g \in H(f)$, then $u(U_0, g, \cdot, t) \in X$ is C^2 in U_0 and is continuous in g, t within its (time) interval of existence. In other word, there is a well defined (local) skew-product semi-flow $\Pi : X \times H(f) \times \mathbb{R}^+ \to X \times H(f)$:

$$\Pi(U_0, g, t) = (u(U_0, g, \cdot, t), g \cdot t), \quad t > 0 \tag{3.4}$$

associated to (3.2), where $u(U_0, g, t)$ is C^2 in U_0.

By the standard *a priori* estimates of parabolic equations, if $u(U_0, g, \cdot, t)$ is a bounded solution of $(3.3)_g$ for t in its interval of existence, then $u(U_0, g, \cdot, t)$ exists for all $t > 0$. Furthermore, for $\delta > 0$, $\{u(U_0, g, \cdot, t)|t \geq \delta\}$ is relatively compact, hence its ω-limit set $\omega(U_0, g)$ is well defined and compact. Moreover, the restriction of Π to $\omega(U_0, g)$ is a (global) semi-flow which admits a flow extension $(\omega(U_0, g), \mathbb{R})$. A minimal set E of (3.4) can be defined in the same fashion, that is, $E = \omega(x_0, g)$ for some $(x_0, g) \in E$ and $(E, \mathbb{R})$ is minimal in the usual sense.

We remark that, with the introduction of a skew-product semi-flow (3.4), dynamics of (3.2) is relatively independent of the choice of a phase space X as long as the class of solutions under investigation possess enough regularity. In fact, by the standard *a priori* estimates of parabolic equations, for any $U_0 \in X$, if $u(U_0, g, \cdot, t)$ is X-bounded, then it is $H^{2,p}$ bounded and moreover $\omega(U_0, g)|_{X\times H(f)}$ coincides with $\omega(U_0, g)|_{H^{2,p}(\Omega)\times H(f)}$ $(p > n)$.

Finally, we note that by the comparison principle for scalar parabolic equations, one can easily find a natural condition which guarantees the existence of a C^2-bounded (hence X-bounded) solution for (3.2).

Example 2. Consider a delay differential equation

$$x'(t) = f(x(t), x(t-1), t), \tag{3.5}$$

where $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a C^2 admissible and (uniform) minimal function. Let $X = C([-1, 0], \mathbb{R}^n)$. Then by the standard theory of delay differential equation ([**10**]), for each $\phi \in X$ and each $g \in H(f)$, the equation

$$x'(t) = g(x(t), x(t-1), t) \tag*{$(3.6)_g$}$$

has (locally) a unique solution $x(\phi, g, t)$ with initial value ϕ, that is, $x(\phi, g, t) = \phi(t)$ for $t \in [-1, 0]$. Let $x_t(\phi, g) \in X$ $(t > 0$) be defined as follows: $x_t(\phi, g)(\theta) \equiv x(\phi, g, t+\theta)$ $(\theta \in [-1, 0])$. Then $x_t(\phi, g)$ is C^2 in $\phi \in X$ and Lipschitz in $g \in H(f)$ (see [**10**]). Therefore, there is a well defined (local) skew-product semi-flow Π : $X \times H(f) \times \mathbb{R}^+ \to X \times H(f)$,

$$\Pi(\phi, g, t) = (x_t(\phi, g), g \cdot t) \tag{3.7}$$

associated to (3.5).

It can be proved that if $x(\phi, g, t) \in \mathbb{R}$ is a bounded solution of $(3.6)_g$ for t in the existence interval, then $x_t(\phi, g)$ is defined for all $t > 0$ and $\{x_t(\phi, g)|t \geq 1 + \delta\}$ is relatively compact in X for any $\delta > 0$, hence $\omega(\phi, g)$ is well defined (see [**35**]). Moreover, under certain conditions (see [**35**]), Π restricted to $\omega(\phi, g)$ extends to a usual flow.

3.2. Lifting dynamics. We now consider the skew-product semi-flow (3.1) and assume that $\sigma = (Y, \mathbb{R})$ is minimal and distal. Throughout rest of the section, we let $M \subset X \times Y$ be a compact invariant set of Π (that is, the semi-flow on M admits a flow extension).

Motivated by the qualitative study of differential equations, one is often interested in the lifting properties of $p : (M, \mathbb{R}) \to (Y, \mathbb{R})$. In case that $(Y, \mathbb{R})$ is almost periodic, one would like to ask when the lifted flow $(M, \mathbb{R})$ is also almost periodic or almost automorphic of finite type (this is closely related to the classical study of the existence of a harmonic or sub-harmonic almost periodic solution in differential equations with almost periodic time dependence). It turns out that such a lifting is not generally possible without a stability condition.

DEFINITION 3.1. 1) $(M,\mathbb{R})$ is *uniformly stable* if for any $\epsilon > 0$, there is a $\delta = \delta(\epsilon) > 0$ such that whenever $(\hat{x}, y) \in M$, $(x,y) \in X \times Y$ satisfy $\|\hat{x} - x\| < \delta(\epsilon)$, then

$$\|\phi(\hat{x}, y, t) - \phi(x, y, t)\| < \epsilon \quad \text{for all } t \geq 0.$$

2) $(M,\mathbb{R})$ is *uniformly asymptotically stable* if it is uniformly stable and there is a $\delta_0 > 0$ such that if $(\hat{x}, y) \in M$, $(x,y) \in X \times Y$ and $\|\hat{x} - x\| \leq \delta_0$, then

$$\lim_{t\to\infty} \|\phi(\hat{x}, y, t) - \phi(x, y, t)\| = 0.$$

THEOREM 3.1. (Sacker-Sell [**24**][**25**]) *If $(M,\mathbb{R})$ is uniformly stable, then $p : (M,\mathbb{R}) \to (Y,\mathbb{R})$ is a distal extension.*

PROOF. First, we claim that $(M,\mathbb{R})$ is negatively distal, that is, for any (x_1, y), $(x_2, y) \in M$, $x_1 \neq x_2$,

$$\inf_{t\leq 0} \|\phi(x_1, y, t) - \phi(x_2, y, t)\| > 0. \tag{3.8}$$

Suppose (x_1, y), $(x_2, y) \in M$ are two points for which (3.8) fails. Let $\epsilon_0 = \frac{1}{2}\|x_1 - x_2\|$ and let $\delta(\epsilon_0)$ be as in Definition 3.1. Then there is a $\tau < 0$ such that $\|\phi(x_1, y, \tau) - \phi(x_2, y, \tau)\| < \delta(\epsilon_0)$. Thus, $\|\phi(x_1, y, t+\tau) - \phi(x_2, y, t+\tau)\| < \epsilon_0$ $(t \geq 0)$. In particular, for $t = -\tau$, one has $\|x_1 - x_2\| = 2\epsilon_0 < \epsilon_0$, a contradiction.

Let $E(M)$ denote the Ellis semigroup of $(M,\mathbb{R})$ and e be the identity of $E(M)$. Since the α-limit set $E_-(M) \equiv \alpha(e)$ of e is compact invariant, it contains a minimal set I, which is a minimal (left) ideal of $E(M)$. Let $u \in J(I)$ be an idempotent point of I. For any $(x,y) \in M$, since $u(u(x,y)) = u(x,y)$, (x,y) and $u(x,y)$ are negatively proximal. Hence $u(x,y) = (x,y)$. Since (x,y) is arbitrary, $u = e$, that is, $I = eI$ is a group (Theorem 2.2). Now $E(M) = E(M)e \subset E(M)I \subset I$. One has that $E(M) = I$ is a group, that is, $(M,\mathbb{R})$ is distal (Theorem 2.3 1)). □

THEOREM 3.2. (Sacker-Sell [**25**]) *If $(M,\mathbb{R})$ is uniformly asymptotically stable, then there is a positive integer N such that $p : (M,\mathbb{R}) \to (Y,\mathbb{R})$ is an N to 1 extension, that is, $\operatorname{card} M \cap p^{-1}(y) = N$ $(y \in Y)$. Moreover, M is an N-fold covering of Y if $(Y,\mathbb{R})$ is distal.*

PROOF. Suppose that $\operatorname{card} M \cap p^{-1}(y_0) = \infty$ for some $y_0 \in Y$. Since $M \cap p^{-1}(y_0)$ is compact, it must contain an accumulation point, say (x_0, y_0). Let δ_0 be as in Definition 2.1 2). Then there exists a $(x_*, y_0) \in M \cap p^{-1}(y_0) \setminus \{(x_0, y_0)\}$ such that $\|x_* - x_0\| \leq \delta_0$. It follows that $\|\phi(x_*, y_0, t) - \phi(x_0, y_0, t)\| \to 0$ as $t \to \infty$, that is, (x_*, y_0), (x_0, y_0) are proximal, a contradiction to Theorem 3.1. Thus $\operatorname{card} M \cap p^{-1}(y_0) < \infty$ for all $y \in Y$.

Let $y_1, y_2 \in Y$ be arbitrary and let $t_n \to \infty$ be a sequence such that $y_1 \cdot t_n \to y_2$. Since $p : (M,\mathbb{R}) \to (Y,\mathbb{R})$ is a distal extension according to Theorem 3.1, points on $M \cap p^{-1}(y_1)$ converge to distinct points on $M \cap p^{-1}(y_2)$ following a subsequence of $\{t_n\}$. This implies that $\operatorname{card} M \cap p^{-1}(y_2) \geq \operatorname{card} M \cap p^{-1}(y_1)$. Similarly, $\operatorname{card} M \cap p^{-1}(y_1) \geq \operatorname{card} M \cap p^{-1}(y_2)$. Therefore, there exists a positive integer N for which $\operatorname{card} M \cap p^{-1}(y) = N$ $(y \in Y)$.

In the case that $(Y,\mathbb{R})$ is distal, one can apply Theorem 2.4 to conclude that M is an N-fold covering of Y. □

REMARK 3.1. It is shown in [**25**] that if $p : (M, \mathbb{R}) \to (Y, \mathbb{R})$ is an N to 1 extension (e.g., $(M, \mathbb{R})$ satisfies the condition of Theorem 3.2) and $(Y, \mathbb{R})$ is almost periodic, then $(M, \mathbb{R})$ is almost periodic. It is also shown in [**25**] that if M is a uniformly stable ω-limit set, then $p : (M, \mathbb{R}) \to (Y, \mathbb{R})$ is in fact a minimal extension.

We now assume that ϕ is C^2 in x in (3.1) and denote $\Phi(x, y, t) \equiv \phi_x(x, y, t)$. Then Φ defines a *semicocycle* on $X \times Y$. Due to the fact that dynamics of a conservative system (e.g., a Hamiltonian) is usually complicated, we pay our particular attentions to lifting properties in skew-product semi-flows of strongly monotone natures.

DEFINITION 3.2. Π is *strongly monotone* if the following conditions hold:

1) The Banach space X is strongly ordered, that is, there is a closed convex cone $X_+ \subset X$ with $IntX_+ \neq \emptyset$ and $X_+ \cap (-X_+) = \{0\}$;
2) If $v \in X_+$, then for any $(x, y) \in X \times Y$, $\Phi(x, y, t)v \in IntX_+$ $(t > 0)$.

DEFINITION 3.3. $(M, \mathbb{R})$ is *linearly stable* if its upper Lyapunov exponent λ_M is non-positive, that is,

$$\lambda_M = \sup_{(x,y)\in M} \limsup_{t\to\infty} \frac{\ln \|\Phi(x, y, t)\|}{t} \leq 0.$$

Typical strongly monotone skew-product semi-flows are those generated from almost periodic cooperative systems of ordinary and delay differential equations (see [**35**]). Also, a skew-product semi-flow of type (3.4) generated by a parabolic equation is always strongly monotone (see [**35**]).

By [**34**], a strongly monotone skew-product flow Π is *strongly order preserving* in the following sense:

Define a strong partial ordering $\geq$ on $X \times Y$ as follows:

$$\begin{aligned}
(x_1, y_1) \geq (x_2, y_2) \quad &\iff \quad y_1 = y_2 \text{ and } x_1 - x_2 \in X_+;\\
(x_1, y_1) > (x_2, y_2) \quad &\iff \quad (x_1, y_1) \geq (x_2, y_2) \text{ and } x_1 \neq x_2;\\
(x_1, y_1) \gg (x_2, y_2) \quad &\iff \quad y_1 = y_2 \text{ and } x_1 - x_2 \in \operatorname{Int} X_+.
\end{aligned}$$

Then $u(x_1, y, t) \gg u(x_2, y, t)$ $(t > 0)$ if $(x_1, y) > (x_2, y)$.

THEOREM 3.3. (Shen-Yi [**34**]) 1) *If $(M, \mathbb{R})$ is linearly stable and minimal, then there exists a distal and N to 1 extension $p_* : (Y_*, \mathbb{R}) \to (Y, \mathbb{R})$ for some positive integer N such that $\tilde{p} : (M, \mathbb{R}) \to (Y_*, \mathbb{R})$ is almost 1 to 1, where $p = p_* \circ \tilde{p}$.*

2) *If $(M, \mathbb{R})$ is also uniformly stable, then $p : (M, \mathbb{R}) \to (Y, \mathbb{R})$ is an N to 1 extension for some positive integer N.*

PROOF. 2) is an easy consequence of 1) and Theorem 3.1.

To prove 1), we denote $O(M) \subset M \times M$ as the order relation on M, that is,

$$O(M) = \{(x_1, y_1), (x_2, y_2)) \in M \times M | (x_1, y_1) \geq (x_2, y_2) \text{ or } (x_2, y_2) \geq (x_1, y_1)\}.$$

$O(M)$ is clearly a closed and positively invariant relation.

Let 2^M endow with the Hausdorff metric and consider map $q : Y \to 2^M$, $y \mapsto M \cap p^{-1}(y)$. Since q is upper semi-continuous, the set Y_0 of continuous points of q is a residual subset of Y.

We now claim that the following *non-ordering principle* holds: for any $y \in Y_0$, there is no ordered pair on $M \cap p^{-1}(y)$. Suppose for contradiction that there is an ordered pair (x_0^1, y_0), $(x_0^2, y_0) \in M \cap p^{-1}(y_0)$ for some $y_0 \in Y_0$. Without loss of generality, we may assume that $(x_0^1, y_0) \gg (x_0^2, y_0)$. By the Zorn's lemma, one can find a maximal element (x_M, y_0) on $M \cap p^{-1}(y_0)$ with respect to '$\geq$'. Now let $t_n \to -\infty$ be a sequence such that $(x_M, y_0) \cdot t_n \to (x_0^2, y_0)$. By the lower continuity of q at y_0, there is a sequence $\{(x_n, y_0)\} \subset M \cap p^{-1}(y_0)$ such that $(x_n, y_0) \cdot t_n \to (x_0^1, y_0)$. Since $(x_0^1, y_0) \gg (x_0^2, y_0)$, there is an n_0 sufficiently large such that $(x_{n_0}, y_0) \cdot t_{n_0} > (x_M, y_0) \cdot t_{n_0}$. This implies that $(x_{n_0}, y_0) > (x_M, y_0)$, a contradiction with the maximality of (x_M, y_0).

Let $P(M)$ be the proximal relation on $(M, \mathbb{R})$. The above discussion implies that $O(M) \subset P(M)$. To see this, we let $(x_1, y), (x_2, y) \in O(M)$ with $(x_1, y) > (x_2, y)$. For a fixed $y_0 \in Y_0$, by the minimality of $(Y, \mathbb{R})$, there is a sequence $t_n \to +\infty$ such that $y \cdot t_n \to y_0$. Without loss of generality, we assume that $(x_i, y) \cdot t_n \to (x_i^0, y_0)$ $(i = 1, 2)$ as $n \to \infty$. Now, if $(x_1, y), (x_2, y)$ are distal, then $(x_1^0, y_0) \neq (x_2^0, y_0)$. Since Π is strongly order preserving and $O(M)$ is a closed relation, $(x_1^0, y_0) > (x_2^0, y_0)$, a contradiction to the non-ordering principle.

Next, using linear stability and the strong monotonicity of Π, one can show the following: There are ϵ_0, δ_0, $K > 0$ such that if (x_1, y), $(x_2, y) \in M$, $\|x_1 - x_2\| < \epsilon_0$, and $(x_1, y) \cdot t$, $(x_2, y) \cdot t$ are not ordered for t in a finite interval $[0, t_0]$, then

$$\|\phi(x_1, y, t) - \phi(x_2, y, t)\| \leq K e^{-\delta_0 t} \|x_1 - x_2\| \tag{3.9}$$

for all $t \in [0, t_0]$ (see [**34**]).

By (3.9), the non-ordering principle and the fact $O(M) \subset P(M)$, one can further show that $P(M)$ is an equivalence relation. Therefore the Ellis semigroup $E(M)$ contains a unique minimal (left) ideal I. Thus, $I \subset \alpha(e) \cap \omega(e)$, where e is the identity of $E(M)$. It follows that if $((x_1, y), (x_2, y)) \in P(M) \setminus O(M)$, then they are both positively and negatively proximal. Now let $t_n \to -\infty$ be such that $\|\phi(x_1, y, t_n) - \phi(x_2, y, t_n)\| \to 0$ as $n \to \infty$. Then as n sufficiently large, $\|\phi(x_1, y, t_n) - \phi(x_2, y, t_n)\| < \epsilon_0$ and $(x_1, y) \cdot (t + t_n)$, $(x_2, y) \cdot (t + t_n)$ are not ordered for $t \in [0, -t_n]$. By (3.9),

$$\|x_1 - x_2\| = \|\phi((x_1, y) \cdot t_n, -t_n) - \phi((x_2, y) \cdot t_n, -t_n)\| \leq \epsilon_0 K e^{\delta_0 t_n}.$$

We let $n \to \infty$ to conclude that $x_1 = x_2$. This shows that $O(M) = P(M)$. Therefore, both $O(M)$ and $P(M)$ are closed and invariant relations.

Consider $Y_* = M/P(M) = M/O(M)$ and denote $(Y_*, \mathbb{R})$ as the induced flow. A similar argument using (3.9) and the proof of Theorem 3.2 shows that $p_* : (Y_*, \mathbb{R}) \to (Y, \mathbb{R})$ is a distal and N to 1 extension for some positive integer N. Let $\tilde{p} : (M, \mathbb{R}) \to (Y_*, \mathbb{R})$ be the projection and let $Y_*^0 = \tilde{p}^{-1}(Y_0)$. Clearly, $\operatorname{card} M \cap \tilde{p}^{-1}(y) = 1$ $(y \in Y_*^0)$. □

By Remark 3.1 and Theorem 2.6, if $(Y, \mathbb{R})$ is almost periodic, then $(M, \mathbb{R})$ in the above theorem is almost automorphic (almost periodic) in the case of linear stability (uniform stability). We note that a linearly stable ω-limit set of Π need not be minimal.

All results above are sharp. In the case of strong monotonicity, a linearly stable minimal set need not be almost periodic and N can be bigger than 1 (see [**35**]).

We refer the readers to [**23**]-[**25**] for more discussions on almost periodic lifting dynamics and to [**30**]-[**36**] for almost automorphic lifting dynamics in skew-product flows and semi-flows.

4. Almost periodic Floquet theory

In [**13**], using topological dynamics techniques, Johnson derived an analogue of the classical Floquet theory for two dimensional, almost periodic linear system of ordinary differential equations which provided a clear qualitative picture of such a system. We now give a brief review on the Johnson's theory along with some discussions on a case which was left open in Johnson's original work [**13**].

Consider

$$x' = a(y\cdot t)x, \quad x\in\mathbb{R}^2, \quad y\in Y \tag{$(4.1)_y$}$$

where $tr(a)\equiv 0$, $(Y,\mathbb{R})$ is an almost periodic minimal flow with compact metric phase space Y.

By using

$$a(y) = \begin{pmatrix} 0 & -\sigma(y) \\ \sigma(y) & 0 \end{pmatrix} + \begin{pmatrix} \delta(y) & \epsilon(y) \\ \epsilon(y) & -\delta(y) \end{pmatrix},$$

one can rewrite $(4.1)_y$ into the following polar coordinate form

$$r' = f(\theta, y\cdot t)r \tag{$(4.2)_{\theta,y}$}$$

$$\theta' = L(\theta, y\cdot t), \tag{$(4.3)_y$}$$

where $r=|x|\in\mathbb{R}^+$, $\theta = arg(x)\in S^1$ and

$$f(\theta,y) = \delta(y)\cos 2\theta + \epsilon(y)\sin 2\theta$$

$$L(\theta,y) = \sigma(y) + \epsilon(y)\cos 2\theta - \delta(y)\sin 2\theta.$$

Let $p_0 : S^1\times Y \to P^1\times Y$ be the natural projection, where P^1 denotes the projective 1-space (= the set of lines through the origin in $\mathbb{R}^2$). Then the relation $f_0(p_0(\theta,y))\equiv f(\theta,y)$, $L_0(p_0(\theta,y))\equiv L(\theta,y)$ induces functions f_0, L_0 on $P^1\times Y$. Therefore, $(4.3)_y$ generates skew-product flows $\tilde{\Pi}$ and Π on both $S^1\times Y$ and $P^1\times Y$ respectively. We also denote $p : P^1\times Y\to Y$, $\tilde{p} = p\circ p_0 : S^1\times Y\to Y$ as natural projections.

Definition 4.1. 1) The system

$$x' = (a(y\cdot t) - \lambda I)x \tag{$(4.1)_{y,\lambda}$}$$

is said to admits an *exponential dichotomy* (ED) over Y if there are constants $\alpha, K>0$ and a continuous family of projections $P_\lambda(y):\mathbb{R}^2\to\mathbb{R}^2$ such that for all $y\in Y$ one has

$$\|\Phi_\lambda(y,t)P_\lambda(y)\Phi_\lambda^{-1}(y,s)\| \le Ke^{-\alpha(t-s)}, \quad s\le t,$$

$$\|\Phi_\lambda(y,t)(I-P_\lambda(y))\Phi_\lambda^{-1}(y,s)\| \le Ke^{-\alpha(t-s)}, \quad t\le s,$$

where $\Phi_\lambda(y,t)$ denotes the principle matrix of $(4.1)_{y,\lambda}$.

2) The *dynamical spectrum* Σ of the linear skew-product flow generated by $(4.1)_y$ is $\Sigma = \{\lambda\in\mathbb{R} | (4.1)_{y,\lambda}$ does not admit an exponential dichotomy (ED) over $Y\} = \{\lambda\in\mathbb{R}|(4.1)_{y,\lambda}$ has a nontrivial bounded solution for some $y\in Y\}$.

According to a general spectral theory due to Sacker-Sell ([**26**]) and Selgrade ([**25**]), either $\Sigma = \{-\beta,\beta\}$ for some $\beta>0$ or $[-\beta,\beta]$ for some $\beta\ge 0$.

DEFINITION 4.2. Let $(Y,\mathbb{R})$ be a minimal flow. A continuous map $U: Y \to GL(2,\mathbb{R})$ defines a *strong Perron transformation* if, for each $y \in Y$, the map $\mathbb{R} \to GL(2,\mathbb{R})$, $t \mapsto U(y\cdot t)$ is continuously differentiable, and the map $Y \to L(\mathbb{R}^2)$, $y \mapsto \frac{d}{dt}U(y\cdot t)|_{t=0}$ is continuous.

Similar to the classical Floquet theory, the almost periodic Floquet theory of Johnson concerns with finding a suitable strong Perron transformation which transforms $(4.1)_y$ into a canonical form (e.g., an upper triangular or a diagonal system). Such a theory depends on a detailed classification of minimal sets of $\Pi = (P^1 \times Y, \mathbb{R})$ in all situations of the dynamical spectrum Σ (see also [**21**] for a measure theoretical classification of Π). To give an idea, let $M \subset P^1 \times Y$ be a minimal set of Π and $\tilde{M}$ be a minimal lift of M in $\tilde{\Pi} = (S^1 \times Y, \mathbb{R})$. Note that $\tilde{M}$ is either a 1-cover or a 2-cover of M. Define $\tilde{a}: \tilde{M} \to L(\mathbb{R}^2)$, $\tilde{a}(m) = a(\tilde{p}(m))$ $(m \in \tilde{M})$. Then

$$x' = \tilde{a}(m\cdot t)x \qquad (4.4)_m$$

coincides with $(4.1)_y$ if $y = \tilde{p}(m)$. Let $U: \tilde{M} \to SL(\mathbb{R}^2)$,

$$m = (\theta, y) \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Then U is a strong Perron transformation and $x = U(m\cdot t)z$ transforms $(4.4)_m$ into the form

$$z' = \begin{pmatrix} u(m\cdot t) & v(m\cdot t) \\ 0 & -u(m\cdot t) \end{pmatrix} z. \qquad (4.5)_m$$

If either for some $y \in Y$, $M \cap p^{-1}(y)$ admits a distal pair or Π admits precisely two minimal sets, then another strong Perron transformation would further transform $(4.5)_m$ into a diagonal system (see [**13**]).

Differing from the classical Floquet theory of periodic systems, a fundamental fact of Johnson's theory is that a minimal set of Π is often almost automorphic (see [**13**]-[**14**]). Thus, an almost automorphic (not necessary almost periodic) strong Perron transformation has to be introduced to transform $(4.1)_y$ into an almost automorphic canonical form (which need not be a constant coefficient system either).

4.1. $\Sigma = \{-\beta\} \cup \{\beta\}$ **for some** $\beta > 0$**.** In this case, $(4.1)_y$ $(y \in Y)$ admits an ED with precisely two invariant line bundles $\mathcal{B}_1$, $\mathcal{B}_2$ of $P^1 \times Y$ associated to the ED (or to the Lyapunov exponents $-\beta$, β). That is, Π admits precisely two minimal sets $M_i = \{(\theta, y) \in P^1 \times Y|$ the line with direction θ is in $\mathcal{B}_i\}$ $(i = 1, 2)$ which are 1-covers of Y.

THEOREM 4.1. (Johnson, 6.7 of [**13**]) *Suppose that* $\Sigma = \{-\beta, \beta\}$. *There is an almost periodic minimal set* $\tilde{M}$ *of* $\tilde{\Pi}$ *which is either a* 1*-cover or a* 2*-cover of* Y *and a strong Perron transformation* $x = U(m\cdot t)z$ *(*$m \in \tilde{M}$*) which transforms* $(4.4)_m$ *into*

$$z' = \begin{pmatrix} b(m\cdot t) & 0 \\ 0 & -b(m\cdot t) \end{pmatrix} z.$$

4.2. $\Sigma = \{0\}$**.** By the definition of Σ, there is $y \in Y$ such that $(4.1)_y$ admits a nontrivial bounded solution. We therefore have the following two cases.

CASE 1. All solutions of $(4.1)_y$ for all y are bounded.

This case was originally studied by Cameron ([**3**]) for almost periodic systems and extensions were made in Johnson [**13**] and Ellis and Johnson [**6**] for distal and recurrent systems respectively. We note that these works all deal with linear systems with arbitrary dimensions.

Restricting to two dimensions, then either $P^1 \times Y$ itself is minimal or it contains at least three minimal sets (therefore infinitely many minimal sets by [**13**], [**25**]). Below we only summarize the result and outline a proof from [**13**].

THEOREM 4.2. (Cameron [**3**], Johnson [**13**], Ellis-Johnson [**6**]) *Suppose Case* 1. *Then there is a strong Perron transformation* $U : Y \to GL(2, \mathbb{R})$ *such that the change of variable* $x = U(y \cdot t)z$ *takes* $(4.1)_y$ *into*

$$(4.6)_y \qquad z' = \begin{pmatrix} 0 & -b(y \cdot t) \\ b(y \cdot t) & 0 \end{pmatrix} z.$$

PROOF. Define a flow $\hat{\Pi}_t$ on $SL_1(2, \mathbb{R}) \times Y = \{A \in GL(2, \mathbb{R}) | detA = 1\} \times Y$ as follows:

$$\hat{\Pi}_t(A, y) = (\Phi(y, t)A, y \cdot t),$$

where $\Phi(y, t)$ denotes the principle matrix of $(4.1)_y$. It can be shown that $\hat{\Pi}_t$ is distal. Therefore, for fixed $y_0 \in Y$, $X = cl\{\hat{\Pi}_t(I, y_0)\} \subset SL_1(2, \mathbb{R}) \times Y$ is minimal and distal. Let $\eta : X \to Y$, $(A, y) \mapsto y$ be the natural projection and $\eta_* : E(X) \to E(Y)$ be the projection induced by η according to Theorem 2.1, where $E(X)$ and $E(Y)$ are the Ellis semi-groups of X and Y respectively. Since both $E(X)$ and $E(Y)$ are groups (Theorem 2.3 1)), $G = \{\gamma \in E(X) | \eta_* \gamma(y_0) = y_0\}$ is a subgroup, and therefore $G_0 = G|_{\eta^{-1}(y_0)}$ is a group of self maps of $X_0 = \eta^{-1}(y_0)$. Let $(A, y_0) \in X_0$ and let $\hat{\Pi}_{t_\alpha} \to \beta_0 \in G_0$ in $E(X)$. Assume that $\lim_\alpha \Phi(y_0, t_\alpha)A = A_{\beta_0}$. It can be shown that the map $G_0 \to \eta^{-1}(y_0) \subset SL_1(2, \mathbb{R})$, $\beta_0 \mapsto A_{\beta_0}$ is a group isomorphism. Thus $\eta^{-1}(y_0)$ is a compact subgroup of $SL_1(2, \mathbb{R})$. We identify G_0 with $\eta^{-1}(y_0)$.

Since X, Y are distal, the map $y \mapsto \eta^{-1}(y) : Y \to 2^X$ is continuous with respect to the Hausdorff metric (Theorem 2.4). Therefore, for $(A, y) \in \eta^{-1}(y)$, if $\{t_\alpha\}$ is such that $\hat{\Pi}_{t_\alpha}(I, y_0) \to (A, y)$, then $\eta^{-1}(y) = A \cdot G_0$.

Define inner product $\langle \cdot, \cdot \rangle_*$ on $\mathbb{R}^2$:

$$\langle x_1, x_2 \rangle_* = \int_{\eta^{-1}(y_0)} \langle Ax_1, Ax_2 \rangle dA,$$

where dA is the normalized Haar measure on $\eta^{-1}(y_0)$. Then $\eta^{-1}(y_0)$ is a subgroup of the unitary group with respect to $\langle \cdot, \cdot \rangle_*$ and there is a positive definite operator Q_0 on $\mathbb{R}^2$ such that $\langle Q_0 x_1, Q_0 x_2 \rangle_* = \langle x_1, x_2 \rangle$ $(x_1, x_2 \in \mathbb{R}^2)$.

Define $R : Y \to GL(2, \mathbb{R})$, $R(y) = AA_*$, where (A, y) is some element of $\eta^{-1}(y)$ and A_* is the adjoint of A with respect to $\langle \cdot, \cdot \rangle_*$. One can check that R is well defined, continuous, positive definite and self-adjoint with respect to $\langle \cdot, \cdot \rangle_*$. Let $U_1(y)$ be the unique positive definite, self-adjoint square root of $R(y)$ and let $U(y) = U_1(y)Q_0$ $(y \in Y)$. Then $U : Y \to GL(2, \mathbb{R})$ defines a strong Perron transformation. Consider the transformed equation

$$(4.7)_y \qquad z' = \hat{a}(y \cdot t)z$$

under $x = U(y \cdot t)z$. If $z_1(t)$, $z_2(t)$ are two solutions of $(4.7)_y$, then it follows from the identity $R(y \cdot t) = \Phi(y,t)AA_*\Phi(y,t)_*$ (for some $A \in \eta^{-1}(y)$) that

$$\begin{aligned}\langle z_1(t), z_2(t)\rangle &= \langle Q_0^{-1}U_1^{-1}(y \cdot t)\Phi(y,t)x_1, Q_0^{-1}U_1^{-1}(y \cdot t)\Phi(y,t)x_2\rangle \\ &= \langle U_1^{-1}(y \cdot t)\Phi(y,t)x_1, U_1^{-1}(y \cdot t)\Phi(y,t)x_2\rangle_* \\ &= \langle R^{-1}(y \cdot t)\Phi(y,t)x_1, \Phi(y,t)x_2\rangle_* \\ &= \langle R^{-1}(y)x_1, x_2\rangle_* = \text{constant}.\end{aligned}$$

Thus, $\hat{a}(y)$ must be skew symmetric. □

In fact, if some equation $(4.1)_y$ admits an almost periodic solution, then all solutions to $(4.1)_y$ $(y \in Y)$ are almost periodic and $b(y)$ in $(4.6)_y$ may be assumed to be a constant b_0 ([**13**]).

CASE 2. There is $y \in Y$ such that $(4.1)_y$ admits an unbounded solution.

In this case, Π admits no more than two minimal sets. Johnson ([**13**]) had shown the following: If Π admits precisely two minimal sets $M_i (i = 1, 2)$, then $(M_i, \mathbb{R}) \to (Y, \mathbb{R})$ $(i = 1, 2)$ are almost automorphic extensions. If Π admits only one minimal set M which is not proximal extension of Y, then there is an almost periodic minimal flow $(Y_*, \mathbb{R})$, a 2-cover of $(Y, \mathbb{R})$, such that $(M, \mathbb{R}) \to (Y_*, \mathbb{R})$ is an almost automorphic extension.

THEOREM 4.3. (Johnson, 6.12 of [**13**]) *Suppose Case* 2 *and let some fiber* $p^{-1}(y)$ *contain a distal pair. Then there is* (i) *an almost automorphic minimal set* $\tilde{M}$ *of* $\tilde{\Pi}$ *which is either an almost* 1*-cover or an almost* 2*-cover of* Y*;* (ii) *a strong Perron transformation* $x = U(m \cdot t)z$ *(*$m \in \tilde{M}$*) which transforms* $(4.4)_m$ *into*

$$z' = \begin{pmatrix} b(m \cdot t) & 0 \\ 0 & -b(m \cdot t) \end{pmatrix} z.$$

4.3. Discussion. In the case that $\Sigma = [-\beta, \beta]$, $\beta > 0$, Johnson had shown the following.

THEOREM 4.4. (Johnson [**13**], [**14**]) *If* $\Sigma = [-\beta, \beta]$, $\beta > 0$*, then*
1) $\Pi = (P^1 \times Y, \mathbb{R})$ *admits a unique minimal set* M*;*
2) M *supports exactly two ergodic measures;*
3) $p : (M, \mathbb{R}) \to (Y, \mathbb{R})$ *is a proximal extension.*

By citing examples ([**14**]) originally due to Millionšćikov and Vinograd, Johnson ([**13**]) conjectured the following: If $\Sigma = [-\beta, \beta]$, with $\beta \geq 0$, and if Π admits a unique minimal set M with $p : (M, \mathbb{R}) \to (Y, \mathbb{R})$ being a proximal extension, then p is an almost automorphic extension. Later, Johnson ([**15**]) proved that there is a large class of equations of form $(4.1)_y$ which admit an unbounded solution for all $y \in Y$, the corresponding flow $\Pi = (P^1 \times Y, \mathbb{R})$ is minimal and $p : (P^1 \times Y, \mathbb{R}) \to (Y, \mathbb{R})$ is a proximal extension. Therefore, the original conjecture in [**13**] fails in general. Below, we will discuss some cases in which the conjecture of Johnson is still valid.

Let $(\tilde{M}, \mathbb{R})$ be a minimal lift of $(M, \mathbb{R})$ in $\tilde{\Pi} = (S^1 \times Y, \mathbb{R})$. If the conjecture were true, then $\tilde{p} : (\tilde{M}, \mathbb{R}) \to (Y, \mathbb{R})$ would be either an almost 1 to 1 or an almost 2 to 1 extension. Suppose that $(\tilde{M}, \mathbb{R})$ were also almost automorphic (thus $(\tilde{M}, \mathbb{R})$ would be an almost automorphic extension of either a 1-cover or a 2-cover of $(Y,\mathbb{R})$). Then there would exist an almost automorphic strong Perron transformation $x = U(m{\cdot}t)z$ $(m \in \tilde{M})$ which transforms

$$x' = a(\tilde{p}(m \cdot t))x$$

into an upper triangular form:

$$z' = \begin{pmatrix} b(m \cdot t) & c(m \cdot t) \\ 0 & -b(m \cdot t) \end{pmatrix} z. \qquad (4.8)_m$$

Even so, $(4.8)_m$ can not be of diagonal form for otherwise $p^{-1}(y)$ would admit a distal pair for some $y \in Y$.

As far as completion of the almost periodic Floquet theory is concerned, there are two fundamental questions which are related to the Johnson's conjecture: 1) *When is* $(M,\mathbb{R})$ *almost automorphic?* 2) *If* $(M,\mathbb{R})$ *were almost automorphic, would* $(\tilde{M},\mathbb{R})$ *also be almost automorphic?*

The study of these issues is also interesting from other aspects (e.g., linear almost periodic oscillation problems). We note that since $\dfrac{\partial L(\theta,y)}{\partial\theta} \equiv -2f(\theta,y)$ in $(4.2)_{\theta,y}$ and $(4.3)_y$, both $(M,\mathbb{R})$ and $(\tilde{M},\mathbb{R})$ admit a positive Lyapunov exponent if $\Sigma = [-\beta,\beta]$, with $\beta > 0$. Therefore, in the case of interval spectrum, even though both $(M,\mathbb{R})$ and $(\tilde{M},\mathbb{R})$ are almost automorphic, they are not almost periodic by the non-unique ergodicity, and the flows on them can be complicated or even chaotic (see [**14**] for an example of topological complication of such a minimal flow).

Below, we present some discussions related to the Johnson's conjecture. For this purpose, we parameterize both S^1 and P^1 by the angular variable θ. For simplicity, $\theta(y,t)$ shall be denoted as the solution of $(4.3)_y$ with $\theta(y,0) = \theta$.

Theorem 4.5. $(\tilde{M},\mathbb{R})$ *is almost automorphic if and only if* $P(\tilde{M})$ *is a closed relation.*

Proof. We first note that since $tra(y) \equiv 0$, the following Liouville's formula holds

$$e^{\int_0^t f(\theta_1(y,s),y\cdot s)ds} e^{\int_0^t f(\theta_2(y,s),y\cdot s)ds} \sin(\theta_1(y,t) - \theta_2(y,t)) \equiv \sin(\theta_1 - \theta_2) \qquad (4.9)$$

$(t \in \mathbb{R}$, $(\theta_i, y) \in S^1 \times Y$, $i = 1,2)$. Therefore, if $\theta_1, \theta_2 \in S^1$ is such that $0 < \theta_1 - \theta_2 < \pi$, then $0 < \theta_1(y,t) - \theta_2(y,t) < \pi$ for any $y \in Y$ and $t \in \mathbb{R}$.

By Theorem 2.6, if $(\tilde{M},\mathbb{R})$ is almost automorphic, then $P(\tilde{M})$ is a closed relation. Conversely, suppose that $P(\tilde{M})$ is a closed relation. Let $Y_0 \subset Y$ be the residual set for which $h : Y \to 2^{\tilde{M}} : y \mapsto \tilde{M} \cap \tilde{p}^{-1}(y)$ is continuous in Hausdorff metric (see the proof of Theorem 3.3). Fix a $y_0 \in Y_0$. We first show that $\tilde{M} \cap \tilde{p}^{-1}(y_0)$ contains no more than two points. Suppose not. Then one can find two points $(\theta_1^0, y_0), (\theta_2^0, y_0)$ on $\tilde{M} \cap \tilde{p}^{-1}(y_0)$ with $0 < \theta_1^0 - \theta_2^0 < \pi$. Let $\tilde{I}$ be the unique minimal (left) ideal in $E(\tilde{M})$ and $u \in J(\tilde{I})$ be an idempotent point such that $u(\theta_1^0, y_0) = (\theta_1^0, y_0)$. Denote σ as the reflection: $S^1 \times Y \to S^1 \times Y$, $(\theta,y) \mapsto (\theta+\pi, y)$. Since $p : (M,\mathbb{R}) \to (Y,\mathbb{R})$ is a proximal extension, for any $(\theta, y_0) \in \tilde{M} \cap \tilde{p}^{-1}(y_0)$, either $u(\theta,y_0) = (\theta_1^0, y_0)$ or $u(\theta, y_0) = \sigma(\theta_1^0, y_0)$.

Let $\{t_\alpha | \alpha \in \Lambda\}$ be a net in $\mathbb{R}$ such that $\tilde{\Pi}_{t_\alpha} \to u$ in $E(\tilde{M})$. By the lower continuity of h at y_0, there is a net $\{(\theta_\alpha, y_0) | \alpha \in \Lambda\} \subset \tilde{M} \cap \tilde{p}^{-1}(y_0)$ with $(\theta_\alpha, y_0) \cdot t_\alpha \to (\theta_2^0, y_0)$. It follows that there is an α_0 such that $0 < \theta_1^0(y_0, t_\alpha) - \theta_\alpha(y_0, t_\alpha) < \pi$ $(\alpha > \alpha_0)$, that is, $0 < \theta_1^0 - \theta_\alpha < \pi$ $(\alpha > \alpha_0)$ by (4.9). By taking a subnet, we assume without loss of generality that $\{\theta_\alpha\}$ is (strictly) monotonically increasing with respect to the positive (counterclockwise) orientation of S^1, $0 < \theta_1^0 - \theta_\alpha < \pi$ for all α and $\theta_0 = \lim_\alpha \theta_\alpha$.

Fix an α. Since $0 < \theta_{\alpha'} - \theta_\alpha < \pi$, by (4.9), $0 < \theta_{\alpha'}(y_0, t_{\alpha'}) - \theta_\alpha(y_0, t_{\alpha'}) < \pi$ ($\alpha' > \alpha$). It follows that $0 \le \theta_2^0 - \lim_{\alpha'} \theta_\alpha(y_0, t_{\alpha'}) \le \pi$. Therefore, $u(\theta_\alpha, y_0) \ne (\theta_1^0, y_0)$ for any α. This is already a contradiction if $\tilde{M}$ is a 1-cover of M.

In the case that $\tilde{M}$ is a 2-cover of M, we must have $u(\theta_\alpha, y_0) = \sigma(\theta_1^0, y_0)$ for all α. Without loss of generality, we assume that $0 < \theta_0 - \theta_\alpha < \pi$ for all α. By (4.9) again, $0 < \theta_0(y_0, t_\alpha) - \theta_\alpha(y_0, t_\alpha) < \pi$ for all α, that is, $0 \le \lim_\alpha \theta_0(y_0, t_\alpha) - \theta_2^0 \le \pi$. It follows that $u(\theta_0, y_0) = (\theta_1^0, y_0)$. Therefore, $((\theta_1^0, y_0), (\theta_0, y_0)) \in P(\tilde{M})$ and $(\sigma(\theta_1^0, y_0), (\theta_\alpha, y_0)) \in P(\tilde{M})$ for all α. By the closeness of $P(\tilde{M})$, $(\sigma(\theta_1^0, y_0), (\theta_0, y_0)) \in P(\tilde{M})$. On the other hand, since $P(\tilde{M})$ is also an equivalence relation (Theorem 2.3, 5)), $((\theta_1^0, y_0), (\sigma(\theta_1^0, y_0)) \in P(\tilde{M})$, a contradiction.

Above all, for any $y_0 \in Y_0$, since $p : (M, \mathbb{R}) \to (Y, \mathbb{R})$ is a proximal extension, $\tilde{M} \cap \tilde{p}^{-1}(y_0)$ either consists of one point $\{(\theta_0, y_0)\}$ or two points $\{(\theta_0, y_0), \sigma(\theta_0, y_0)\}$. By the definition of Y_0, it is easy to see that either $\operatorname{card} \tilde{M} \cap \tilde{p}^{-1}(y_0) = 1$ for all $y_0 \in Y_0$ or $\operatorname{card} \tilde{M} \cap \tilde{p}^{-1}(y_0) = 2$ for all $y_0 \in Y_0$.

Now, let $Y_* = \tilde{M}/P(\tilde{M})$. Then $(Y_*, \mathbb{R})$ is either a 1 to 1 or a 2 to 1 extension of $(Y, \mathbb{R})$ and $(\tilde{M}, \mathbb{R}) \to (Y^*, \mathbb{R})$ is an almost automorphic extension. □

By the above theorem, if $\tilde{M}$ is a 1-cover of M, then without any condition, $(\tilde{M}, \mathbb{R})$, $(M, \mathbb{R})$ are both almost automorphic extensions of $(Y, \mathbb{R})$. Also, the proof of the theorem implies that in the current case an almost automorphic minimal set $\tilde{M}$ of $\tilde{\Pi}$ is at most an almost 2-cover of Y. Thus, if the entire space $P^1 \times Y$ or $S^1 \times Y$ is minimal (see [15]), neither $\tilde{\Pi}_t$ nor Π_t can be almost automorphic. We now restrict ourselves to the case that $\tilde{M}$ is a 2-cover of M.

THEOREM 4.6. *If $\tilde{M}$ is a 2-cover of M, then the following are equivalent:*
1) *$P(\tilde{M})$ is an equivalence relation;*
2) *For any $(\theta_1, y), (\theta_2, y) \in \tilde{M}$, $|\theta_1(y,t) - \theta_2(y,t)|$ either stay away from 0 for all t or from π for all t;*
3) *For any idempotent point $u \in E(M)$, $p_0^{-1}(u)$ contains a unique idempotent point in $E(\tilde{M})$.*

If, in addition, $\Sigma = [-\beta, \beta]$, $\beta > 0$, then any of 1)-3) above implies that
4) *$(\tilde{M}, \mathbb{R}) \to (Y, \mathbb{R})$ is an almost 2 to 1 extension and therefore $(M, \mathbb{R}) \to (Y, \mathbb{R})$ is an almost automorphic extension.*

PROOF. Since $(M, \mathbb{R}) \to (Y, \mathbb{R})$ is a proximal extension, $E(M)$ contains a unique minimal (left) ideal I (Theorem 2.3 4)). It follows that if $\tilde{I} \subset E(\tilde{M})$ is a minimal (left) ideal, then for any $\gamma \in \tilde{I}$ and any (θ_1, y), $(\theta_2, y) \in \tilde{M}$, either $\gamma(\theta_1, y) = \gamma(\theta_2, y)$ or $\gamma(\theta_1, y) = \sigma\gamma(\theta_2, y)$, where σ is the reflection defined in the proof of Theorem 4.5. We note that an alternative of the condition in 2) is that for any (θ_1, y), $(\theta_2, y) \in \tilde{M}$, either (θ_1, y), (θ_2, y) are distal or (θ_1, y), $\sigma(\theta_2, y)$ are distal.

1) $\Longrightarrow$ 2): If there are (θ_1, y), $(\theta_2, y) \in \tilde{M}$ such that $(\theta_1, y), (\theta_2, y) \in P(\tilde{M})$ and $(\theta_1, y), \sigma(\theta_2, y) \in P(\tilde{M})$, then $(\theta_2, y), \sigma(\theta_2, y) \in P(\tilde{M})$, a contradiction.

2) $\Longrightarrow$ 1): Let u be any idempotent point in $E(\tilde{M})$. If $\{(\theta_1, y), (\theta_2, y)\} \in P(\tilde{M})$, then $(\theta_1, y), \sigma(\theta_2, y)$ are distal, that is, $u(\theta_1, y) \ne u\sigma(\theta_2, y)$. It follows that

$\sigma u(\theta_1, y) = u\sigma(\theta_2, y) = \sigma u(\theta_2, y)$, that is, $u(\theta_1, y) = u(\theta_2, y)$. Clearly, $P(\tilde{M})$ is an equivalence relation.

1) $\Longleftrightarrow$ 3): Define a relation $\sim$ on $E(\tilde{M})$ as follows: $\gamma_1 \sim \gamma_2$ if and only if for any $(\theta, y) \in \tilde{M}$ either $\gamma_1(\theta, y) = \gamma_2(\theta, y)$ or $\gamma_1(\theta, y) = \sigma\gamma_2(\theta, y)$. It turns out that $\sim$ is a closed, invariant (with respect to both $(E(\tilde{M}), \mathbb{R})$ and the semigroup structure on $E(\tilde{M})$) relation, and moreover $\gamma_1 \sim \gamma_2$ if and only if $p_0(\gamma_1) = p_0(\gamma_2)$. Now fix a $u \in J(I)$ and take a $\gamma \in p_0^{-1}(u)$ and an idempotent $\tilde{u} \in E(\tilde{M})$ which covers u. Then $\tilde{u} \sim \gamma \sim \gamma^2$. Note for any $(\theta, y) \in \tilde{M}$, if $\tilde{u}(\theta, y) = \gamma(\theta, y)$, then $\tilde{u}\gamma(\theta, y) = \tilde{u}^2(\theta, y) = \tilde{u}(\theta, y) = \gamma(\theta, y)$, and if $\tilde{u}(\theta, y) = \sigma\gamma(\theta, y)$, then $\tilde{u}\gamma(\theta, y) = \sigma\tilde{u}^2(\theta, y) = \sigma\tilde{u}(\theta, y) = \gamma(\theta, y)$. That is, $\tilde{u}\gamma = \gamma$. A similar argument shows that $\gamma^3 = \gamma$. It follows that $p_0^{-1}(I)$ is fiber-wise distal and laminates into minimal (left) ideals. Now let $\tilde{I}$ be a minimal (left) ideal of $E(\tilde{M})$. Then $\tilde{I} \subset p_0^{-1}(I)$ and $G \equiv \tilde{I} \cap p_0^{-1}(u) = \{\gamma \in \tilde{u}\tilde{I} | \gamma^2 = \tilde{u}\}$, where $\tilde{u}$ is an idempotent point in G. Thus G is an abelian, (normal) subgroup of $\tilde{u}\tilde{I}$, and therefore G contains exactly one idempotent point $\tilde{u} \in J(\tilde{I})$ (by [**4**], G is in fact a topological group). We omit the rest of the proof by noting that $P(\tilde{M})$ is an equivalence relation if and only if $p_0^{-1}(I)$ is a minimal (left) ideal.

We now assume that $\Sigma = [-\beta, \beta]$, with $\beta > 0$, and show that if $P(\tilde{M})$ is an equivalence relation, then $(\tilde{M}, \mathbb{R}) \to (Y, \mathbb{R})$ is an almost 2 to 1 extension.

By Theorem 4.4 and [**21**], $(M, \mathbb{R})$ admits two ergodic measures μ_+, μ_- which concentrate on two separate ergodic sheets. More precisely, there is a full measure set $Y_* \subset Y$ (with respect to the Haar measure μ_0) and measurable functions $\theta_{+,-} : Y \to M$ such that $\int_Y f(\theta_\pm(y), y) d\mu_0 = \pm\beta$ $(y \in Y_*)$. Note that $0 < |\theta_+(y) - \theta_-(y)| < \pi$ for all $y \in Y_*$.

We now fix a $y_* \in Y_*$ and without loss of generality assume that $0 < \theta_-(y_*) - \theta_+(y_*) < \pi$. Let I_i $(i = 1, 2, 3, 4)$ denote positively oriented open intervals $((\theta_+(y_*), y_*), (\theta_-(y_*), y_*))$, $((\theta_-(y_*), y_*), \sigma(\theta_+(y_*), y_*))$, $(\sigma(\theta_+(y_*), y_*), \sigma(\theta_-(y_*), y_*))$, $(\sigma(\theta_-(y_*), y_*), \theta_+(y_*), y_*))$ respectively.

Let $(\theta, y_*) \in I_1 \cup I_4$ and denote $x(t)$, $x_\pm(t)$ as solutions of $(4.1)_{y_*}$ with $\arg x(0) = \theta$, $\arg x_\pm(0) = \theta_\pm(y_*)$. Expressing $x_-(t)$ as a linear combination of $x_+(t)$ and $x(t)$, say, $x_-(t) = c_+ x_+(t) + c x(t)$, then $c_+ c < 0$. Since $\lim_{t\to\infty} \dfrac{\ln \|x_\pm(t)\|}{t} = \pm\beta$ (Birkhoff ergodic theorem), one has that $\lim_{t\to\pm\infty} \|x_\pm(t)\| = \infty$, $\lim_{t\to\mp\infty} \|x_\pm(t)\| = 0$. Applying (4.9) with $\theta_1 := \theta$, $\theta_2 := \theta_-(y_*)$ or $\theta_1 := \theta$, $\theta_2 := \theta_+(y_*)$, one also has $\lim_{t\to\pm\infty} \|x(t)\| = \infty$. Now

$$\begin{aligned}\|x_-(t)\|^2 =& c_+^2 \|x_+(t)\|^2 + c^2 \|x(t)\|^2 \\ & - 2|c| \cdot |c_+| \cdot \|x_+(t)\| \cdot \|x(t)\| \cos(\theta(y_*, t) - \theta_+(y_*, t)).\end{aligned}$$

One must have $\lim_{t\to+\infty} |\theta(y_*, t) - \theta_+(y_*, t)| = 0$. Similarly, $\lim_{t\to+\infty} |\theta(y_*, t) - (\theta_+ + \pi)(y_*, t)| = 0$ $((\theta, y_*) \in I_2 \cup I_3)$, $\lim_{t\to-\infty} |\theta(y_*, t) - \theta_-(y_*, t)| = 0$ $((\theta, y_*) \in I_1 \cup I_2)$, and $\lim_{t\to-\infty} |\theta(y_*, t) - (\theta_- + \pi)(y_*, t)| = 0$ $((\theta, y_*) \in I_3 \cup I_4)$. It follows that if $(\theta_1, y_*) \in I_1$, $(\theta_4, y_*) \in I_4$, then $\lim_{t\to+\infty} |\theta_1(y_*, t) - \theta_4(y_*, t)| = 0$ and $\lim_{t\to-\infty} |\theta_1(y_*, t) - \theta_4(y_*, t)| = \pi$. By 2), either $I_1 \cap \tilde{M} \cap \tilde{p}^{-1}(y_*) = \emptyset$ or $I_4 \cap \tilde{M} \cap \tilde{p}^{-1}(y_*) = \emptyset$. Without loss of generality, assume that $I_4 \cap \tilde{M} \cap \tilde{p}^{-1}(y_*) = \emptyset$. Let Y_0 be as in Theorem 4.5 and let $y_0 \in Y_0$. If $\operatorname{card} \tilde{M} \cap \tilde{p}^{-1}(y_0) > 2$, then a similar argument to the proof of Theorem 4.5 shows that $\operatorname{card} \tilde{M} \cap \tilde{p}^{-1}(y_*) = \infty$, in particular, $I_1^* \equiv I_1 \cap \tilde{M} \cap \tilde{p}^{-1}(y_*) \neq \emptyset$ and $I_3^* = I_3 \cap \tilde{M} \cap \tilde{p}^{-1}(y_*) \neq \emptyset$. Now, a point in I_1^* is positively proximal to $(\theta_+(y_*), y_*)$ and negatively proximal to

$(\theta_-(y_*), y_*)$. It follows that $(\theta_-(y_*), y_*)$, $(\theta_+(y_*), y_*)$ are proximal. Above all, two points (θ_i, y_*) $(i = 1, 2)$, are proximal if and only if they lie in the same compact set $I_1^* \cup \{\theta_+(y_*), y_*)\} \cup \{(\theta_-(y_*), y_*)\}$ or $I_3^* \cup \{\sigma(\theta_+(y_*), y_*)\} \cup \{\sigma(\theta_-(y_*), y_*)\}$. Thus, $P(\tilde{M})$ when restricted to $\tilde{M} \cap \tilde{p}^{-1}(y_*)$ is closed. But a similar argument to Theorem 4.5 would give a contradiction. Therefore, card $\tilde{M} \cap \tilde{p}^{-1}(y_0) = 2$ $(y_0 \in Y_0)$, that is, $(\tilde{M}, \mathbb{R}) \to (Y, \mathbb{R})$ is almost 2 to 1. □

We remark that, by the above proof, if one of the conditions 1)-3) holds, then the two ergodic sheets $\{(\theta_+(y), y) | y \in Y_*\}$, $\{(\theta_-(y), y) | y \in Y_*\}$ give an 'upper' and 'lower' bounds of $M \cap p^{-1}(y)$ $(y \in Y_*)$. This is a typical scenario of flow $\pi = (P^1 \times Y, \mathbb{R})$ in the interval spectrum case (see the example in [**14**]). In general, by a perturbation theorem of dynamical spectrum ([**26**]), a perturbation of system $(4.1)_y$ often gives rise to a linear system with point spectrum $\Sigma_\epsilon = \{\beta + \epsilon, -(\beta + \epsilon)\}$ $(0 < |\epsilon| \ll 1)$ (see [**17**], [**18**]), in which the induced flow $\Pi_\epsilon = (P^1 \times Y, \mathbb{R})$ admits exactly two minimal sets M_i^ϵ $(i = 1, 2)$ which are all 1-covers of Y. When passing a limit, M_i^ϵ clips together on Y_0 $(\mu_0(Y_0) = 0)$ and converge to the two ergodic sheets of M on Y_* $(\mu_0(Y_*) = 1)$.

Reviewing conditions in 2) of the above theorem, it seems that for $P(\tilde{M})$ being an equivalence relation, solutions of $(4.1)_y$ should have certain 'regular' oscillatory behavior. Nevertheless, it is possible to show that $(M, \mathbb{R})$ is almost automorphic without knowing that $P(\tilde{M})$ is an equivalence relation.

Since Σ contains 0, there is a $y \in Y$ such that $(4.1)_y$ admits a non-trivial bounded solution. This bounded solution must be unique up to linear dependence of solutions, for otherwise, all solutions of $(4.1)_y$ for all $y \in Y$ would be bounded.

Let $\hat{Y} = \{y \in Y | (4.1)_y$ admits a unique non-trivial bounded solution$\}$. Then $\hat{Y}$ is an invariant set of Y. In the case that $\Sigma = [-\beta, \beta]$ for $\beta > 0$, $\hat{Y}$ is both measure theoretically and topologically small (according to [**12**], there is an invariant residual subset $Y^* \subset Y$ such that for any $y \in Y^*$, solutions of $(4.1)_y$ all oscillate in magnitude between 0 and ∞. Let Y_* be the full measure set defined in the proof of Theorem 4.6. It is easy to see that Y_*, Y^* and $\hat{Y}$ are disjoint).

Now for $y \in \hat{Y}$, let $\hat{\theta}(y) \in P^1$ be such that solution $x(y, t)$ with $arg x(y, 0) = \hat{\theta}(y)$ is bounded. Using (4.9) and arguments of Theorem 4.6 4), one can show that if θ_1, θ_2 lie in the same positively oriented interval $(\hat{\theta}(y), \hat{\theta}(y) + \pi)$ or $(\hat{\theta}(y) + \pi, \hat{\theta}(y))$, then $|\theta_1(y, t) - \theta_2(y, t)|$ stays away from π for all $t \in \mathbb{R}$, and if $\theta_1 \in (\hat{\theta}(y), \hat{\theta}(y) + \pi)$ and $\theta_2 \in (\hat{\theta}(y) + \pi, \hat{\theta}(y))$, then $|\theta_1(y, t) - \theta_2(y, t)|$ stays away from 0 for all $t \in \mathbb{R}$.

Proposition 4.7. *If for some $\hat{y} \in \hat{Y}$, $(\hat{\theta}(\hat{y}), \hat{y}) \notin M$, then card $\tilde{M} \cap \tilde{p}^{-1}(y) = 2$ for all $y \in Y_0$, that is, $p : (M, \mathbb{R}) \to (Y, \mathbb{R})$ is an almost automorphic extension.*

Proof. By the above discussion, it is easy to see that if $\hat{y} \in \hat{Y}$ is such that $(\hat{\theta}(\hat{y}), \hat{y}) \notin M$, then the restriction of $P(\tilde{M})$ on $\tilde{M} \cap \tilde{p}^{-1}(\hat{y})$ is closed. It follows from arguments in the proof of Theorem 4.5 that card $\tilde{M} \cap \tilde{p}^{-1}(y_0) = 2$ $(y_0 \in Y_0)$, that is, card $M \cap p^{-1}(\hat{y}) = 1$ $(y_0 \in Y_0)$. □

Of course, we do not know whether the condition in the above proposition is valid in general. Suppose it does for some systems, then it is still a question whether $P(\tilde{M})$ is an equivalence relation, or whether $(\tilde{M}, \mathbb{R})$ is almost automorphic (note this is needed as far as an almost automorphic strong Perron transformation is considered).

In summary, the study of the conjecture of Johnson leads to three dynamical questions which may also be of general interests.

Let $p : (X,T) \to (Y,T)$ be a homomorphism of minimal transformation groups.

QUESTION 1. If (Y,T) is almost periodic and p is an almost N to 1 ($N \geq 2$) extension, when is (X,T) almost automorphic?

QUESTION 2. If (Y,T) is almost automorphic and p is an N to 1 ($N \geq 2$) extension, when is (X,T) almost automorphic?

QUESTION 3. If $P(Y)$ is an equivalence (closed) relation and p is an N to 1 ($N \geq 2$) extension, when is $P(X)$ an equivalence (closed) relation?

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