

CENTER MANIFOLDS FOR SMOOTH INVARIANT MANIFOLDS

SHUI-NEE CHOW, WEISHI LIU, AND YINGFEI YI

ABSTRACT. We study dynamics of flows generated from smooth vector fields in \mathbb{R}^n in the vicinity of an invariant and closed smooth manifold Y . By applying the Hadamard graph transform technique, we show that there exists an invariant manifold (called a *center manifold* of Y) based on the information of the linearization along Y , which contains every locally bounded solution and is persistent under small perturbations.

1. INTRODUCTION

This paper is devoted to the study of existence and smoothness of a local center manifold for invariant manifolds of flows. By extending the classical center manifold theory, our primary goal is to build up a geometrical foundation which allows one to study dynamics of a differential equation in the vicinity of an invariant set Y (e.g. a torus or a heteroclinic cycle) in addition to an equilibrium point or a periodic orbit.

The classical center manifolds theory of equilibria, since first introduced by Pliss ([39]) and Kelley ([30]) in the 1960's and later developed by many others (e.g. [7], [26], [48], [51], [53] etc.), has become an important subject and found tremendous applications in the study of flows and diffeomorphisms (see [7], [11], [12], [17], [21], [24] and references therein). Besides generalizations to various cases of infinite dimensional semiflows (e.g. [3], [4], [8], [13], [25], [35], [52]), there have been several important extensions of the classical center manifolds theory in the case of invariant sets. Center manifolds along a trajectory of a diffeomorphism were constructed in [26] and [47]. In [19], as part of the geometric theory of singularly perturbed ordinary differential equations, Fenichel showed existence of center manifolds for invariant manifolds consisting of equilibria (see also [29], [32] for more applications of the theory). Related to perturbation and bifurcation problems, some cases of center manifolds for invariant tori were studied in Chenciner and Iooss ([10]), Chow and Lu ([14]). Center manifolds for skew-product flows were studied in Chow and Yi ([15]). Recently, Homburg ([27]) and Sandstede ([45]) constructed center manifolds for certain homoclinic orbits to study various global bifurcation problems.

We shall show in this paper that, if a smooth flow in \mathbb{R}^n admits a connected, invariant and closed (i.e., compact without boundary) smooth manifold Y , then under certain exponential rate conditions on a splitting of the linearization along Y , the flow has a smooth locally invariant manifold $M_c(Y)$ (a *center manifold* of Y) corresponding to the splitting, which contains Y and all locally bounded solutions, and is persistent under small perturbations (see Section 2 for details).

Key words and phrases. Center manifold, graph transform, overflowing.

The center manifolds theory we shall present is also closely related to the theory of normally hyperbolic invariant manifolds—a subject which has been extensively studied (see [16], [18], [26], [31], [34], [42], [55] and references therein). In the 1970's, Sacker ([42]), and Fenichel ([18]) showed that an r -normally hyperbolic overflowing compact invariant C^r manifold Y of a flow is C^r persistent, and the invariant manifold obtained after perturbation is unique (see [26] for parallel results for diffeomorphisms, [5], [32] for infinite dimensional semiflows, and [23], [25], [44], [56] for non-autonomous systems). On the other hand, it has been shown by Mañé ([34]), and by Bronstein and Kopanskii ([6]) that, if Y is a C^r invariant closed manifold of a C^r flow, then r -normal hyperbolicity is equivalent to C^r persistence and isolation. Recently, Pliss and Sell ([40]) introduced the concept of a weakly, normally hyperbolic invariant set and showed persistence results for such a set.

Our result can be viewed as a generalization to both the theory of normally hyperbolic invariant manifolds and the center manifolds theory of Fenichel ([19]) and Chenciner and Iooss ([10]). Related to the former theory, our work simply provides information when normal hyperbolicity fails. Comparing with [19] in which the invariant manifold consists of equilibria and with [10] in which a subsystem of the linearization along the invariant torus is independent of the points on the torus, our center manifolds theory deals with general flows on an invariant manifold, and the ‘center bundle’ associated to the linearization along the invariant manifold needs not have a constant structure.

Like many studies on invariant manifolds of dynamical systems, our work is based on the standard Hadamard graph transform technique ([22]). However, with a general invariant manifold involved, several technical difficulties need to be resolved when we apply the Hadamard graph transform technique. The first one is to choose a function space to which the graph transform applies. In the equilibrium case, such a function space can be chosen as the space of Lipschitz functions from the center eigenspace to the hyperbolic eigenspaces. For the general case we consider, although an invariant splitting of the linearization along the invariant manifold is assumed, the ‘center subspaces’ associated to the invariant splitting do not form a manifold of desired dimension. Therefore, we need to construct an approximate center manifold which is tangent to the ‘center subspaces’. This is done by introducing a new Riemannian structure on the tangent bundle of the invariant manifold and employing the exponential map. By defining a smooth bundle structure over the approximated center manifold in a neighborhood of the invariant manifold, a function space is then chosen as sections of the bundle. Another key step in applying the graph transform technique is to modify the original vector field near the invariant manifold to satisfy the so called ‘overflowing’ property. Unlike the case of [19], the modified vector field in our case cannot be made to be C^1 close to the original one in general- which results in a *large* perturbation problem. Therefore, a careful choice and estimations on the modified vector field are crucial in our analysis.

Besides the Hadamard graph transform, another fruitful technique which has been frequently used in the classical center manifolds theory as well as other invariant manifolds (such as inertial and integral manifolds) theory is the Lyapunov-Perron method ([33], [38]). This method allows more detailed analysis on and near an invariant manifold especially when a natural coordinate system is available for a particular problem (e.g. [9], [20], [25], [30], [39], [56]). The Lyapunov-Perron

method does not seem to directly apply to our problem. On one hand, to be able to work directly with integral equations as the Lyapunov-Perron method suggested, one more or less needs to construct a center manifold by obtaining its invariant foliations based on the invariant manifold. This is indeed the case of Chow and Lu ([14]) in a nearly integrable system and the case of Chow and Yi ([15]) for flows with skew product structures. However, such foliations of a center manifold cannot be generally expected, for example, in many Hamiltonian systems. On the other hand, to apply the Lyapunov-Perron method in our current situation, a uniform separation rate associated to a linear invariant splitting should be generally assumed. But this would be a major restriction to many interesting applications.

This paper is organized as follows. We state our main results along with several examples in Section 2. In Section 3, we modify the original vector field near an invariant manifold of the flow following the constructions of an approximated center-unstable manifold and a local bundle structure. Section 4 is a rather technical section which is devoted to the estimations of the modified flow. To avoid reading too much technical details, one may skip most materials in this section and only take Proposition 1 for granted. Our main theorem is proved in Section 5.

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2. MAIN RESULTS AND EXAMPLES

2.1. Statement of main theorem. Consider the following ordinary differential equation

$$(1) \quad z' = f(z)$$

where $z \in \mathbb{R}^n$, $f \in C^r$ ($r \geq 3$).

Let Y be a smooth invariant manifold of (1). For simplicity, we denote the induced flow on Y by $y \cdot t$ ($y \in Y$, $t \in \mathbb{R}$). Consider the linearization of (1) along Y :

$$(2) \quad z' = A(y \cdot t)z$$

where $A(y) = Jf(y)$ is the Jacobian of f at $y \in Y$. We let $\Phi(y, t)$ be the principal matrix of (2), i.e., the fundamental matrix solution of (2) with $\Phi(y, 0) = I$ -the identity matrix.

We make the following hypotheses.

(H1.) System (2) admits a continuous, invariant splitting of the tangent spaces $T_y \mathbb{R}^n$ ($y \in Y$), that is,

$$T_y \mathbb{R}^n = V_s(y) \oplus V_c(y) \oplus V_u(y),$$

where $T_y Y \subset V_c(y)$, $V_i(y)$ varies continuously in $y \in Y$ and $\Phi(y, t)V_i(y) = V_i(y \cdot t)$, $i = s, c, u$, for all $t \in \mathbb{R}$ and $y \in Y$.

(H2.) With respect to a fixed Riemannian structure \mathcal{R} on the tangent bundle $T_Y\mathbb{R}^n$, there exist constants α, β and integer d satisfying $0 < d \leq r$, $\alpha < 0$, and $0 \leq \beta < \frac{1}{d}$ such that

$$\begin{aligned} \|\Phi(y, t)|_{V_s(y)}\| &\leq e^{\alpha t}, \quad t \geq 0; \\ \|\Phi(y, t)|_{V_u(y)}\| &\leq e^{-\alpha t}, \quad t \leq 0; \\ m(\Phi(y, t)|_{V_c(y)}) &\geq \|\Phi(y, t)|_{V_s(y)}\|^\beta, \quad t \geq 0; \\ m(\Phi(y, t)|_{V_c(y)}) &\geq \|\Phi(y, t)|_{V_u(y)}\|^\beta, \quad t \leq 0; \end{aligned}$$

for all $y \in Y$, where, for a linear operator L , $\|L\|$ denotes the operator norm of L with respect to \mathcal{R} and $m(L) = \min\{\|Lz\| : \|z\| = 1\}$.

For $i = s, c, u$, and $y \in Y$, if we denote $P_i(y) : T_y\mathbb{R}^n \rightarrow V_i(y)$ as the projections associated to the invariant splitting, then $\Phi(y, t)|_{V_i(y)} = \Phi(y, t)P_i(y)$. For simplicity, sometimes we shall also denote $\Phi(y, t)|_{V_i(y)}$ as $\Phi_i(y, t)$ for $i = s, c, u$ respectively.

Remark 1. Since $\Phi(y, t)\Phi^{-1}(y, s) = \Phi(y \cdot s, t - s)$, $P_i(y \cdot t)\Phi(y, t) = \Phi(y, t)P_i(y)$, one has $\Phi(y, t)P_i(y)\Phi^{-1}(y, s) = \Phi(y \cdot s, t - s)P_i(y \cdot s)$ ($i = s, c, u$) for all t, s and $y \in Y$. It follows that the first two inequalities in **(H2.)** are equivalent to

$$\begin{aligned} \|\Phi(y, t)P_s(y)\Phi^{-1}(y, s)\| &\leq e^{\alpha(t-s)}, \quad t \geq s, \quad y \in Y; \\ \|\Phi(y, t)P_s(y)\Phi^{-1}(y, s)\| &\leq e^{-\alpha(t-s)}, \quad t \leq s, \quad y \in Y; \end{aligned}$$

respectively.

Definition 1. (a) A submanifold M of \mathbb{R}^n with boundary ∂M is called *locally invariant* under (1), if, for any point $p \in M \setminus \partial M$, there exists an $\epsilon > 0$ such that $z(t, p) \in M$ for $t \in (-\epsilon, \epsilon)$, where $z(t, p)$ is the solution of (1) with $z(0, p) = p$.

(b) A locally invariant C^k ($k \leq r$) manifold M of a C^r vector field f is called *C^k persistent* if there exists a neighborhood $\mathcal{U}(f)$ of f in the space of C^k vector fields with the C^k topology, such that for each $g \in \mathcal{U}(f)$, there exists a locally invariant C^k manifold $M(g)$ (not necessarily unique) of g which is C^k close to M with respect to the Hausdorff metric.

Our main theorem is stated as follows.

Theorem 1. *Suppose Y is a connected, invariant and closed C^r manifold of (1) for which **(H1)** and **(H2)** are satisfied. Then there exists a manifold $M_c(Y)$ with the following properties.*

- (i) $M_c(Y)$ is C^d and locally invariant;
- (ii) $T_y M_c(Y) = V_c(y)$ for all $y \in Y$;
- (iii) $M_c(Y)$ is C^d persistent;
- (iv) $M_c(Y)$ contains all locally bounded solutions, that is, there exists a neighborhood $N(Y)$ of Y such that $M_c(Y)$ contains all solutions lying entirely in $N(Y)$.

The property (iv) above particularly implies that $M_c(Y)$ contains Y .

We refer to a manifold $M_c(Y)$ as a *center manifold* of Y if all properties (i)–(iv) above are satisfied. By (ii) and (iii) above, if the invariant splitting in **(H1)** is ‘optimal’, then $M_c(Y)$ more or less gives a normally hyperbolic invariant manifold containing Y with the least dimension (comparing (iii) with [6], [34]).

We shall leave the proof of this theorem to Section 5.

2.2. Spectral conditions. A natural (but not necessarily optimal) linear invariant splitting satisfying **(H1)** and **(H2)** above can be constructed by the well known Sacker-Sell ([43]) spectral theory (see also [46]).

Consider for each $\lambda \in \mathbb{R}$ the skew-product flow on $T_Y \mathbb{R}^n = \mathbb{R}^n \times Y$:

$$\pi_\lambda(z, y, t) = (\Phi_\lambda(y, t)z, y \cdot t),$$

where $\Phi_\lambda(y, t) = e^{-\lambda t} \Phi(y, t)$ is the principal matrix of

$$z' = (A(y \cdot t) - \lambda)z.$$

The flow π_λ is said to admit an *exponential dichotomy* (ED) over Y , if there exist a continuous family of projections $P(y) : T_y \mathbb{R}^n \rightarrow T_y \mathbb{R}^n$ ($y \in Y$) and constants $K > 0, \alpha < 0$ such that

$$\begin{aligned} |\Phi_\lambda(y, t)P(y)\Phi_\lambda^{-1}(y, s)| &\leq Ke^{\alpha(t-s)}, \quad t \geq s; \\ |\Phi_\lambda(y, t)[I - P(y)]\Phi_\lambda^{-1}(y, s)| &\leq Ke^{-\alpha(t-s)}, \quad t \leq s \end{aligned}$$

where $|\cdot|$ denotes the operator norm with respect to the Euclidean metric on \mathbb{R}^n .

The set

$$\Sigma(Y) = \{\lambda \in \mathbb{R} : \pi_\lambda \text{ admits no ED over } Y\}$$

is called the *S-S spectrum* of (2).

Let

$$\begin{aligned} \mathcal{S}_\lambda(Y) &= \{(z, y) : |\Phi_\lambda(y, t)z| \rightarrow 0 \text{ as } t \rightarrow +\infty\}, \\ \mathcal{U}_\lambda(Y) &= \{(z, y) : |\Phi_\lambda(y, t)z| \rightarrow 0 \text{ as } t \rightarrow -\infty\}. \end{aligned}$$

The following was shown in [43].

1) $\Sigma(Y)$ is a union of k compact intervals, that is,

$$\Sigma(Y) = [a_1, b_1] \cup \dots \cup [a_k, b_k],$$

where $k \leq n$ and $a_1 \leq b_1 < a_2 \leq b_2 < \dots < a_k \leq b_k$.

2) If $\lambda_0, \lambda_1, \dots, \lambda_k$ are chosen so that

$$\lambda_0 < a_1 \leq b_1 < \lambda_1 < \dots \leq b_k < \lambda_k,$$

then for $1 \leq i \leq k$

$$V_i = V_i(Y) = \mathcal{S}_{\lambda_i}(Y) \cap \mathcal{U}_{\lambda_{i-1}}(Y) \neq \{0\} \times Y$$

is an invariant subbundle (called the spectral subbundle associated to the spectral interval $[a_i, b_i]$) of $T_Y \mathbb{R}^n$. Moreover, $V_i(Y) \cap V_j(Y) = \{0\} \times Y$ for $i \neq j$ and

$$T_Y \mathbb{R}^n = V_1(Y) \oplus \dots \oplus V_k(Y).$$

In terms of fibers $V_i(y)$ ($y \in Y$) of V_i , the above is equivalent to

$$T_y \mathbb{R}^n = V_1(y) \oplus \dots \oplus V_k(y)$$

for all $y \in Y$, where $\dim V_i(y) = n_i$ ($y \in Y$, $n_i \geq 1$ and $n_1 + n_2 + \dots + n_k = n$).

Now consider, for $i \leq j$, a union of spectral intervals of form $\Sigma_{i,j} = \bigcup_{p=i}^j [a_p, b_p]$ and denote the corresponding spectral subbundle by $V_{i,j} = V_i \oplus \dots \oplus V_j$. Let

$$i_0 = \max\{i : TY \subset V_{i,k}\}, \quad j_0 = \min\{j : TY \subset V_{1,j}\},$$

where TY is the tangent bundle of Y , that is, V_{i_0, j_0} is the smallest spectral subbundle which contains TY . Clearly, i_0 and j_0 are uniquely defined. We refer to $\Sigma_c = \Sigma_{i_0, j_0}$, $V_c = V_{i_0, j_0}$ as the *generalized center spectrum* and the *generalized center subbundle* of Y , respectively. Note that, since Y is compact and invariant,

$0 \in \Sigma_c$, and Σ_c may contain more than one spectral intervals (see Example 2.2). To unify the notation, we let $a_{j_0+1} = +\infty$ if $i_0 = k$ and let $b_{i_0-1} = -\infty$ if $j_0 = 1$.

Corollary 1. *Let Y be as in Theorem 1 and let $\Sigma_c = \Sigma_{i_0, j_0}$ be the generalized center spectrum of Y for which the following spectrum gap conditions are satisfied: there is a positive integer $d \leq r$ such that*

$$-a_{i_0}d < -b_{i_0-1}, \quad b_{j_0}d < a_{j_0+1}.$$

Then there exists a manifold $M_c(Y)$ which satisfies all properties (i)–(iv) stated in Theorem 1.

Proof. Without loss of generality, we assume that a_{j_0+1} and b_{i_0-1} are finite.

Consider (2) and denote $V_s(y) = V_{1, i_0-1}(y)$, $V_u(y) = V_{j_0+1, k}(y)$ ($y \in Y$). Let $P_i(y) : T_y\mathbb{R}^n \rightarrow V_i(y)$, $y \in Y$, $i = s, c, u$, be the associated projections. Then

$$T_y\mathbb{R}^n = V_s(y) \oplus V_c(y) \oplus V_u(y), \quad y \in Y$$

is a continuous invariant splitting of (2).

We now define an equivalent metric on $T_Y\mathbb{R}^n$ following the ideas of [26], [28], [42]. Fix a $0 < \lambda < \min\{(a_{i_0} - b_{i_0-1})/2, (a_{j_0+1} - b_{j_0})/2\}$. For $y \in Y$ and $z_1, z_2 \in T_y\mathbb{R}^n$, we let

$$\begin{aligned} \langle z_1, z_2 \rangle_y &= \int_{-\infty}^0 e^{-2(a_{j_0+1}-\lambda)s} \langle \Phi(y, s)P_u(y)z_1, \Phi(y, s)P_u(y)z_2 \rangle ds \\ &\quad + \int_{-\infty}^0 e^{-2(a_{i_0}-\lambda)s} \langle \Phi(y, s)P_c(y)z_1, \Phi(y, s)P_c(y)z_2 \rangle ds \\ &\quad + \int_0^{+\infty} e^{-2(b_{j_0}+\lambda)s} \langle \Phi(y, s)P_c(y)z_1, \Phi(y, s)P_c(y)z_2 \rangle ds \\ &\quad + \int_0^{+\infty} e^{-2(b_{i_0-1}+\lambda)s} \langle \Phi(y, s)P_s(y)z_1, \Phi(y, s)P_s(y)z_2 \rangle ds. \end{aligned}$$

Since

$$\begin{aligned} |\Phi(y, t)P_s(y)\Phi^{-1}(y, s)| &\leq Ke^{(b_{i_0-1}+\lambda/2)(t-s)}, \quad t \geq s; \\ |\Phi(y, t)P_c(y)\Phi^{-1}(y, s)| &\leq Ke^{(b_{j_0}+\lambda/2)(t-s)}, \quad t \geq s; \\ |\Phi(y, t)P_c(y)\Phi^{-1}(y, s)| &\leq Ke^{(a_{i_0}-\lambda/2)(t-s)}, \quad t \leq s; \\ |\Phi(y, t)P_u(y)\Phi^{-1}(y, s)| &\leq Ke^{(a_{j_0+1}-\lambda/2)(t-s)}, \quad t \leq s, \end{aligned}$$

for some $K > 0$, the above integral converges and clearly defines an inner product $\langle \cdot, \cdot \rangle_y$ on $T_y\mathbb{R}^n$ ($y \in Y$). We denote $\|\cdot\|_y$ and $\|\cdot\|$, respectively, as the induced norm on $T_y\mathbb{R}^n$ ($y \in Y$) and its associated operator norm. To show the uniform equivalence between $\|\cdot\|_y$ and the Euclidean norm $|\cdot|$ on $T_y\mathbb{R}^n$ ($y \in Y$), on one

hand, we have

$$\begin{aligned}
 \|z_0\|_y^2 &= \int_{-\infty}^0 e^{-2(a_{j_0+1}-\lambda)s} |\Phi(y, s)P_u(y)z_0|^2 ds \\
 &\quad + \int_{-\infty}^0 e^{-2(a_{i_0}-\lambda)s} |\Phi(y, s)P_c(y)z_0|^2 ds \\
 &\quad + \int_0^{+\infty} e^{-2(b_{j_0}+\lambda)s} |\Phi(y, s)P_c(y)z_0|^2 ds \\
 &\quad + \int_0^{+\infty} e^{-2(b_{i_0-1}+\lambda)s} |\Phi(y, s)P_s(y)z_0|^2 ds \\
 &\leq \int_{-\infty}^0 e^{-2(a_{j_0+1}-\lambda)s} K^2 e^{2(a_{j_0+1}-\lambda/2)s} |z_0|^2 ds \\
 &\quad + \int_{-\infty}^0 e^{-2(a_{i_0}-\lambda)s} K^2 e^{2(a_{i_0}-\lambda/2)s} |z_0|^2 ds \\
 &\quad + \int_0^{+\infty} e^{-2(b_{j_0}+\lambda)s} K^2 e^{2(b_{j_0}+\lambda/2)s} |z_0|^2 ds \\
 &\quad + \int_0^{+\infty} e^{-2(b_{i_0-1}+\lambda)s} K^2 e^{2(b_{i_0-1}+\lambda/2)s} |z_0|^2 ds \\
 &= \left(2K^2 \int_{-\infty}^0 e^{\lambda s} ds + 2K^2 \int_0^{+\infty} e^{-\lambda s} ds \right) |z_0|^2 \\
 &= \frac{4K^2}{\lambda} |z_0|^2.
 \end{aligned}$$

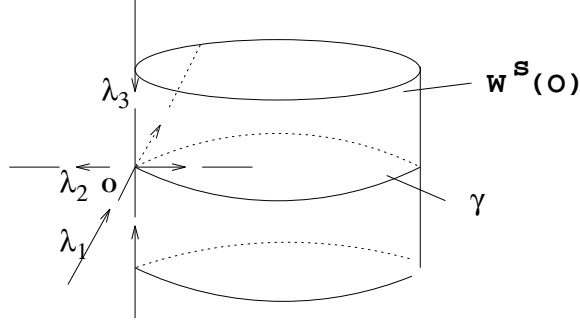
On the other hand, since Y is compact and $\Phi(y, 0) = I$, there exists a $\delta > 0$ such that $|\Phi(y, s)| \geq 1/2$ as $|s| \leq \delta$. It follows that

$$\begin{aligned}
 \|z_0\|_y^2 &\geq \int_{-\delta}^0 e^{-2(a_{j_0+1}-\lambda)s} 1/4 |P_u(y)z_0|^2 ds + \int_{-\delta}^0 e^{-2(a_{i_0}-\lambda)s} 1/4 |P_c(y)z_0|^2 ds \\
 &\quad + \int_0^{\delta} e^{-2(b_{j_0}+\lambda)s} 1/4 |P_c(y)z_0|^2 ds + \int_0^{\delta} e^{-2(b_{i_0-1}+\lambda)s} 1/4 |P_s(y)z_0|^2 ds \\
 &\geq c_2^{-1} (|P_u(y)z_0|^2 + |P_c(y)z_0|^2 + |P_s(y)z_0|^2) \geq c_2^{-1} |z_0|^2,
 \end{aligned}$$

for some $c_2 > 0$ independent of $y \in Y$. This verifies the equivalence of the two metrics.

Since

$$\begin{aligned}
 \|\Phi(y, t)P_s(y)z_0\|_{y,t}^2 &= \int_0^{+\infty} e^{-2(b_{i_0-1}+\lambda)s} |\Phi(y \cdot t, s)\Phi(y, t)P_s(y)z_0|^2 ds \\
 &= \int_0^{+\infty} e^{-2(b_{i_0-1}+\lambda)s} |\Phi(y, s+t)P_s(y)z_0|^2 ds \\
 &= \int_t^{+\infty} e^{-2(b_{i_0-1}+\lambda)(s-t)} |\Phi(y, s)P_s(y)z_0|^2 ds \\
 &\leq e^{2(b_{i_0-1}+\lambda)t} \int_0^{\infty} e^{-2(b_{i_0-1}+\lambda)s} |\Phi(y, s)P_s(y)z_0|^2 ds \\
 &= e^{2(b_{i_0-1}+\lambda)t} |z_0|_y^2,
 \end{aligned}$$

FIGURE 1. Stable set $W^s(0)$ of 0 along γ

for any $t \geq 0$, we have

$$\|\Phi(y, t)P_s(y)\Phi^{-1}(y, s)\| \leq e^{(b_{i_0-1}+\lambda)(t-s)}$$

for $t \geq s$. Similarly,

$$\begin{aligned} \|\Phi(y, t)P_c(y)\Phi^{-1}(y, s)\| &\leq e^{(b_{j_0}+\lambda)(t-s)}, \quad t \geq s; \\ \|\Phi(y, t)P_c(y)\Phi^{-1}(y, s)\| &\leq e^{(a_{i_0}-\lambda)(t-s)}, \quad t \leq s; \\ \|\Phi(y, t)P_u(y)\Phi^{-1}(y, s)\| &\leq e^{(a_{j_0+1}-\lambda)(t-s)}, \quad t \leq s. \end{aligned}$$

Now let $\alpha = \max\{b_{i_0-1} + \lambda, \lambda - a_{j_0+1}\}$ and choose β such that $\frac{a_{i_0}-\lambda}{b_{i_0-1}+\lambda} \leq \beta < \min\left\{\frac{b_{j_0}}{a_{j_0+1}}, \frac{a_{i_0}}{b_{i_0-1}}\right\}$. It is easy to see from the above inequalities that **(H2)** is satisfied with such α and β . \square

2.3. Examples. We now give some examples to illustrate certain fundamental aspects of our results. The first is an example to which Corollary 1 is applicable.

Example 2.1. Consider a flow

$$z' = f(z), \quad z \in \mathbb{R}^3, \quad f \in C^r,$$

where $z = 0$ is a saddle-node equilibrium with eigenvalues λ_i , $i = 1, 2, 3$, satisfying $\lambda_3 < \lambda_1 = 0 < \lambda_2$. Assume that $\gamma(t)$ is a homoclinic orbit to $z = 0$ which approaches $z = 0$ along the eigendirection of λ_1 as $t \rightarrow \pm\infty$ (see Fig. 1).

Let $Y = \{\gamma(t); t \in \mathbb{R}\} \cup \{0\}$. We further assume that, in a neighborhood of $\gamma(t)$, the closure of the stable set $W^s(0)$ of $z = 0$ is a manifold (i.e. a cylinder or a Möbius band). By the Lambda Lemma ([1], [49]), $W^u(0)$ is also a manifold of the same topology type as $W^s(0)$, and, on the stable set of 0, there exists a unique locally invariant stable foliation of $W^s(0)$ with fibers transversal to Y (see [1], [2]). By the Lambda Lemma again, the foliation can be extended to the entire $W^s(0)$ by using the backward flow. Similarly, there exists an invariant unstable foliation on $W^u(0)$. Therefore, for $y \in Y$, the tangent lines to the fibers at y of the stable and unstable foliations together with $T_y Y$ form an invariant splitting of $T_y \mathbb{R}^3$ of the linearization along Y . It can be easily verified that the S-S spectrum of the linearization along Y is $\Sigma = \{\lambda_1, \lambda_2, \lambda_3\}$ and the spectral subbundle corresponding to $\{\lambda_1 = 0\}$ is TY . Thus, conditions in Corollary 1 are satisfied with $V_c(y) = T_y Y$ ($y \in Y$). We then conclude that the flow admits a one-dimensional C^r center manifold of Y , that is, Y itself.

Next, we adopt an example from [43] to indicate the necessity of choosing the generalized center spectrum when applying Corollary 1.

Example 2.2. Let $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the two-dimensional torus. As usual, we coordinate $x \in T^2$ by $x = (x_1, x_2) \in \mathbb{R}^2 \pmod{1}$. Let $\bar{x} = Ax$ be the Anosov diffeomorphism on T^2 , where

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

The eigenvalues of A are $\sigma_{\pm} = (3 \pm \sqrt{5})/2$.

Let $z' = F(z)$ denote the Anosov flow on a 3-manifold Y generated by the standard suspension of the Anosov diffeomorphism. Recall that Y can be identified as the collection of all $z = (x, s)$ where $x = (x_1, x_2) \in \mathbb{R}^2$, $s \in \mathbb{R}$, and $(x, s) = (\hat{x}, \hat{s})$ if and only if $\hat{x} = A^n x \pmod{1}$ and $\hat{s} = s + n$ for some integer n . Since the Anosov diffeomorphism on T^2 is a Poincaré map of the Anosov flow, the S-S spectrum of the linearization of the Anosov flow along Y is simply $\Sigma(Y) = \{\lambda, -\lambda, 0\}$ where $\lambda = \ln \sigma_+$.

We now consider the following flow on the four-dimensional manifold $W = Y \times S^1$:

$$\begin{aligned} z' &= F(z) + h(z, \theta) \\ \theta' &= \alpha \sin \theta + g(z, \theta), \end{aligned}$$

where $(z, \theta) \in W$, α is a parameter, h and g are smooth functions satisfying $h(z, 0) = 0$ and $|g(z, \theta)| = O(|\theta|^2)$. Clearly, Y is an invariant manifold of W given by $\theta = 0$ and the S-S spectrum of the linearization of this flow along Y is $\{\alpha, \lambda, -\lambda, 0\}$. Moreover, it follows from the suspension procedure that the spectral subbundle corresponding to 0 is of dimension one. But Y is of dimension three, which implies that its center manifold is at least three dimensional. Therefore, considering the spectral interval $\{0\}$ alone is not sufficient to generate a desired center manifold of Y . In fact, if $|\alpha| > \lambda$, then Y is normally hyperbolic and the center manifold of Y is just Y itself. If $|\alpha| < \lambda$, then Y is not normally hyperbolic and a center manifold of Y is of dimension four, that is, a neighborhood of Y in W .

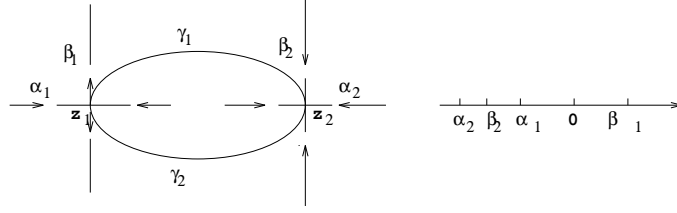
We now give an example in which Corollary 1 fails but Theorem 1 can be still applied to obtain an optimal center manifold for an invariant manifold.

Example 2.3. Consider a flow

$$z' = f(z), \quad z \in \mathbb{R}^2, \quad f \in C^r,$$

which has two heteroclinic orbits $\gamma_1(t)$ and $\gamma_2(t)$ connecting a saddle point $z_1 = (-1, 0)$ to a stable point $z_2 = (1, 0)$ (see Fig. 2). Denote by α_i, β_i as the eigenvalues of z_i ($i = 1, 2$) respectively, where $\alpha_1 < 0 < \beta_1$, $\alpha_2 < \beta_2 < 0$. For simplicity, we assume that, for $i = 1, 2$, the eigendirections corresponding to α_i and β_i are $e_1 = (1, 0)$ and $e_2 = (0, 1)$ respectively.

With the above assumptions, the closure Y of $\gamma_1 \cup \gamma_2$ is clearly a one dimensional invariant manifold of the flow. We denote by Σ_c, Σ respectively as the generalized center and S-S spectrum of the linearization of the flow along Y . Since $\gamma_1'(t)$ is a solution of the linearization along γ_1 with β_1 and β_2 as the asymptotic rates, one has $[\beta_2, \beta_1] \subset \Sigma_c$. Hence, if $\alpha_1 \geq \beta_2$, then $\Sigma_c = \Sigma = [\alpha_2, \beta_1]$ and the dimension of the generalized center spectral subbundle will be two. In this case, a center manifold of Y obtained from Corollary 1 will be a neighborhood of Y in \mathbb{R}^2 .

FIGURE 2. The invariant manifold Y and the eigenvalues of the equilibria

However, we can apply Theorem 1 to show that, in any case, Y is a center manifold of itself. Similar to the argument in Example 2.1, there exists an invariant splitting of the linearization along Y , that is,

$$T_y \mathbb{R}^2 = V_s(y) \oplus V_c(y),$$

where $V_c(y) = T_y Y$ and $V_s(y)$ is transversal to $T_y Y$ ($y \in Y$). Moreover, it is easy to see that there are positive constants $K(y)$ and $K(z_1)$ such that, for $y \in Y \setminus \{z_1\}$,

$$\begin{aligned} |\Phi_s(y, t)| &\leq K(y)e^{\alpha_2 t}, \quad t \geq 0; \\ |\Phi_c(y, t)| &\geq K(y)e^{\beta_2 t}, \quad t \geq 0; \end{aligned}$$

and

$$\begin{aligned} |\Phi_s(z_1, t)| &\leq K(z_1)e^{\alpha_1 t}, \quad t \geq 0; \\ |\Phi_c(z_1, t)| &\geq K(z_1)e^{\beta_1 t}, \quad t \geq 0; \end{aligned}$$

where $\Phi_i(y, t) = \Phi(y, t)|_{V_i(y)}$ ($i = s, c$) and $\Phi(y, t)$ denotes the principal matrix of the linearization along Y .

By the **Uniformity Lemma** in [18], there are constants α, β, K with $\alpha < 0, 0 < \frac{1}{\beta} < \frac{\alpha_2}{\beta_2}, K > 0$ such that $|\Phi_s(y, t)| \leq Ke^{\alpha t}$, $|\Phi_c(y, t)| \geq K|\Phi_s(y, t)|^\beta$ for all $y \in Y$ and $t \geq 0$. Similar to the proof of Corollary 1, one can show that, for $y \in Y$ and $z_1, z_2 \in T_y \mathbb{R}^2$,

$$\begin{aligned} \langle z_1, z_2 \rangle_y &= \int_0^{+\infty} e^{2(\alpha+\lambda)s} \langle \Phi_s(y, s)z_1, \Phi_s(y, s)z_2 \rangle ds \\ &\quad + \int_0^{+\infty} \|\Phi_s(y, s)\|^{-2\beta} \langle \Phi_c(y, s)z_1, \Phi_c(y, s)z_2 \rangle ds, \end{aligned}$$

defines an equivalent metric on $T_Y \mathbb{R}^2$, where $\lambda > 0$ is a small constant. Moreover, if $\|\cdot\|$ denotes the operator norm with respect to the new metric, then

$$\begin{aligned} \|\Phi_s(y, t)\| &\leq e^{(\alpha+\lambda)t}; \\ m(\Phi_c(y, t)) &\geq \|\Phi_s(y, t)\|^\beta \end{aligned}$$

for $t \geq 0$ (we note that, since $\dim V_c(y) = 1$, $m(\Phi_c(y, t)) = \|\Phi_c(y, t)\|$). This verifies the condition **(H2)**.

3. MODIFICATION OF THE VECTOR FIELD

We shall apply the graph transform induced by a time T -map ϕ^T of the flow to show the existence of a local center-unstable manifold. The existence of a local

center-stable manifold can be obtained similarly after reversing time, and the intersection of the center-stable and the center-unstable manifolds in a neighborhood of Y gives a desired local center manifold.

Our construction of a local center-unstable manifold will be based on the following crucial steps: 1) To find an approximate center-unstable manifold which will serve as a base space of graphs; 2) To define a bundle structure in a neighborhood of Y in which graphs can be defined as sections; 3) To modify the original vector field locally so that the overflowing property is satisfied. These constructions are closely tied up together in a way that each later step depends heavily on the former ones.

3.1. An approximate center-unstable manifold. By an *approximate center-unstable manifold* of Y , we mean a smooth manifold which is tangent to $V_{cu}(y) = V_c(y) \oplus V_u(y)$ at all points $y \in Y$.

Since Y is smooth, such an approximate manifold can be obtained easily by using the exponential map (see [26] for example). Let \mathcal{R}_0 be a new Riemannian metric on $T_Y \mathbb{R}^n$ under which $V_s(y)$ is orthogonal to $V_{cu}(y)$ at each $y \in Y$. For a fixed neighborhood $N(Y)$ of Y , this metric can be extended smoothly to $T_{N(Y)} \mathbb{R}^n$ (we denote the extended metric again by \mathcal{R}_0). Let $V'(Y)$ be the orthogonal complement of $T_Y Y$ in $V_{cu}(Y)$. For $\epsilon > 0$ sufficiently small, we let $B(\epsilon)$ be the ϵ -neighborhood of Y in $T_Y \mathbb{R}^n$. Then $\hat{M}_{cu}(Y) = \exp(V'(Y) \cap B(\epsilon))$ defines a desired approximate center-unstable manifold since $T_y \hat{M}_{cu}(Y) = V_{cu}(y)$ for all $y \in Y$, where $\exp : T_{N(Y)} \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the exponential map with respect to \mathcal{R}_0 .

3.2. A local bundle structure. We now construct a fiber bundle structure over $\hat{M}_{cu}(Y)$ in a neighborhood of Y .

By the tubular neighborhood theorem, there exists a neighborhood $N_0(Y)$ of $\hat{M}_{cu}(Y)$ in $N(Y)$ with the property that for any $z \in N_0(Y)$, there exists a unique $p_z \in \hat{M}_{cu}(Y)$ with $dist(z, \hat{M}_{cu}(Y)) = dist(z, p_z)$. Define $\pi_{cu} : N_0(Y) \rightarrow \hat{M}_{cu}(Y)$ by $\pi_{cu}(z) = p_z$, $\pi_s : N_0(Y) \rightarrow N_0(Y)$ by $\pi_s(z) = z - \pi_{cu}(z) := q$, and $Q : \hat{M}_{cu}(Y) \rightarrow Y$ by $Q(p) = y_p$, where $y_p \in Y$ is the unique point satisfying $dist(p, Y) = dist(p, y_p)$. By jiggling the bundle slightly if necessary, we can assume by **Whitney's Embedding Theorem** ([54]) that the bundle and the maps defined above are C^r (see also [18]). Denote the differential of π_{cu} , π_s at z by $D\pi_{cu}(z)$ and $D\pi_s(z)$ respectively. We then have $D\pi_{cu}(y) = P_{cu}(y)$, $D\pi_s(y) = P_s(y)$, $y \in Y$.

The new Riemannian metric \mathcal{R}_0 under which $V_{cu}(y)$, $V_s(y)$ are orthogonal will play an important role later in simplifying our analysis. Without loss of generality, we may assume that \mathcal{R}_0 agrees with the Euclidean metric in \mathbb{R}^n . This is because of the **Nash's Embedding Theorem** ([36]), which says that *any Riemannian n -manifold with C^r metric, $r \geq 3$, has a C^r isometric embedding in $n_0 = 3/2n^3 + 7n^2 + 11/2n$ dimensional Euclidean space*. To be more precise, let $(N_0(Y), \mathcal{R}_0)$ be as above. By Nash's Embedding Theorem, it can be C^r isometrically embedded in \mathbb{R}^{n_0} . By considering a tubular neighborhood \mathcal{N} of $N_0(Y)$ in \mathbb{R}^{n_0} , we can extend our vector field to \mathcal{N} in such a way that the extended flow on the fibers of \mathcal{N} is unstable with sufficiently large rate. That is, the new flow in \mathcal{N} is essentially the same as in $N_0(Y)$ with some extra unstable directions, which, of course does not affect the existence of a center manifold.

In the sequel, we identify \mathcal{R}_0 with the Euclidean metric in \mathbb{R}^n and denote $\langle \cdot, \cdot \rangle$, d and $|\cdot|$ as the Euclidean metric, distance and norm, respectively.

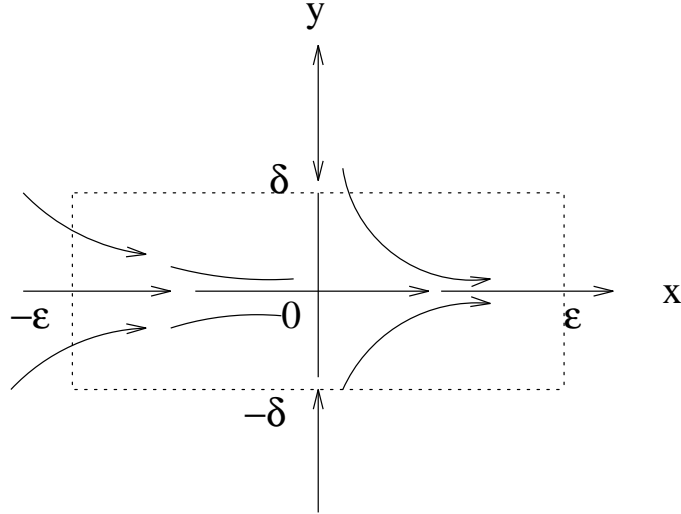


FIGURE 3. Lack of overflowing property

3.3. A modified vector field.

Definition 2. Let $N \subset \mathbb{R}^n$ be a submanifold of dimension n with smooth boundary ∂N , and $A \subset \partial N$. We say that a vector field f satisfies *the overflowing property with respect to (N, A)* if at each point $z \in A$ the vector field f is tangent to or points outward to ∂N .

The overflowing property is essential to perform a graph transform induced by a flow in a neighborhood of an invariant manifold (see [18], [19], [47] etc.). This is simply because the image of a graph under a graph transform should lie entirely in the neighborhood. For center manifolds problems, due to the existence of center directions in a vector field, overflowing property cannot be generally expected. For example, even a simple vector field like

$$\begin{cases} x' = x^2 \\ y' = -y \end{cases}$$

does not satisfy the overflowing property near $O = (0, 0)$ for any choices of ϵ and δ (see Fig. 3). Specifically, on $x = -\epsilon$, the vector field points inward to the interior of the box $[-\epsilon, \epsilon] \times [-\delta, \delta]$.

Therefore, for the sake of performing a graph transform, we need to modify the vector field (1) so that the overflowing property will be satisfied in an appropriate neighborhood of Y .

For $\epsilon_0 > 0$ small and $0 < \epsilon \leq \epsilon_0$, we consider the following sets

$$\begin{aligned} U_\epsilon &= \{z \in \hat{M}_{cu}(Y) : d(z, Y) \leq \epsilon\}, \\ U(z_0) &= \{z \in N_0(Y) : d(z, \hat{M}_{cu}(Y)) = d(z_0, \hat{M}_{cu}(Y))\}, \quad z_0 \in N_0(Y), \\ S(z_0) &= \{z \in N_0(Y) : d(z, \hat{M}_{cu}(Y)) = d(z, z_0)\}, \quad z_0 \in \hat{M}_{cu}(Y), \\ S_\epsilon(z_0) &= \{z \in S(z_0) : d(z, z_0) < \epsilon\}, \quad z_0 \in \hat{M}_{cu}(Y), \\ N_\epsilon &= \bigcup_{z_0 \in U_\epsilon} S_{\epsilon_0}(z_0), \\ \partial^* N_\epsilon &= \bigcup_{z_0 \in \partial U_\epsilon} S_{\epsilon_0}(z_0). \end{aligned}$$

Note that $\partial^* N_\epsilon$ is the portion of ∂N_ϵ over ∂U_ϵ , and $\dim \partial^* N_\epsilon = n - 1$.

Our goal is to construct a modified vector field of (1) which satisfies the overflowing property with respect to $(N_{\epsilon_0}, \partial^* N_{\epsilon_0})$. To do so, we let $\chi_i : \mathbb{R} \rightarrow \mathbb{R}$, $i = 1, 2$, be cut-off functions satisfying the following properties:

$$\chi_1(x) = \begin{cases} 1, & x \geq \epsilon_0/2 \\ 0, & x \leq 0 \end{cases}$$

with $\chi'_1(x) \geq 0$, $|\chi'_1(x)| \leq 4\epsilon_0^{-1}$, and

$$\chi_2(x) = \begin{cases} 0, & x \leq -\epsilon_0^2 \\ 1, & x \geq 0 \end{cases}$$

with $\chi'_2(x) \geq 0$, $|\chi'_2(x)| \leq 2\epsilon_0^{-2}$.

For $z \in \partial^* N_\epsilon$, let $\{v_j(z)\}_{j=1}^n$ be an orthonormal basis of $T_z N_\epsilon$ such that $v(z) = v_1(z), v_2(z), \dots, v_{m_0}(z) \in T_z U(z)$, where $v(z)$ is the inward normal vector to $T_z \partial^* N_\epsilon$ and $v_{m_0+1}, \dots, v_n(z) \in T_z S_{\epsilon_0}(\pi_{cu} z)$. Using the orthonormal basis, we can decompose $f(z) = D\pi_s(z)f(z) + D\pi_{cu}(z)f(z)$ as

$$\begin{aligned} D\pi_{cu}(z)f(z) &= \sum_{j=1}^{m_0} \xi_j(z)v_j(z), \\ D\pi_s(z)f(z) &= \sum_{j=m_0+1}^n \xi_j(z)v_j(z), \end{aligned}$$

where $\xi_j(z)$'s are coordinates of $f(z)$ with respect to the basis. Note that $\xi(z) = \xi_1(z) = \langle f(z), v(z) \rangle$ is independent of the choices of $v_j(z)$ for $j = 2, \dots, n$.

We now modify the vector field (1) to the following:

$$(3) \quad z' = \hat{f}(z) = f(z) - (1 - \chi_1(\eta(z)))\chi_2(\xi(z))\xi(z)v(z),$$

where $\eta(z) = d(\pi_{cu} z, \partial U_{\epsilon_0})$. This new vector field clearly coincides with (1) in $N_{\epsilon_0/2}$. Moreover, this modified vector field satisfies the overflowing property with respect to $(N_{\epsilon_0}, \partial^* N_{\epsilon_0})$. Indeed, if $z \in \partial^* N_{\epsilon_0}$, then $\chi_1(\eta(z)) = 0$, and therefore

$$\begin{aligned} \langle \hat{f}(z), v(z) \rangle &= \langle f(z), v(z) \rangle - \chi_2(\xi(z))\xi(z)\langle v(z), v(z) \rangle \\ &= (1 - \chi_2(\xi(z)))\xi(z). \end{aligned}$$

Since $\chi_2(\xi(z)) = 1$ if $\xi(z) \geq 0$, and $(1 - \chi_2(\xi(z))) \geq 0$ if $\xi(z) < 0$, we see that $\langle \hat{f}(z), v(z) \rangle \leq 0$ for $z \in \partial^* N_{\epsilon_0}$, that is, the vector field (3) is tangent to or points outward to $\partial^* N_{\epsilon_0}$.

Remark 2. Unlike the equilibrium case, the modified vector field (3) is in general not a regular perturbation of the original one in the sense that, within N_{ϵ_0} , it differs from the original vector field by an order of ϵ_0 with respect to the C^0 but not C^1 norm. To give an example, let us consider

$$\begin{cases} x' &= \lambda_1 x + g_1(x, y) \\ y' &= \lambda_2 y + g_2(x, y), \end{cases}$$

where $\lambda_2 < \lambda_1 < 0$ and g_1, g_2 are higher order terms of x and y . It is known that, near $O = (0, 0)$, there exists a locally invariant manifold Y which is tangent to the x -axis at $O = (0, 0)$. If the above vector field is to be modified similarly to (3), say in a small box $[-\epsilon, \epsilon] \times [-\delta, \delta]$ near Y , then the modified vector field would be

$$\begin{cases} x' &= \chi(x)(\lambda_1 x + g_1(x, y)) \\ y' &= \lambda_2 y + g_2(x, y) \end{cases}$$

where $\chi(x) = 0$ for $|x| \geq \epsilon$. Now, the Jacobian of the new vector field is

$$\begin{pmatrix} \chi'(x)(\lambda_1 x + g_1) + (\lambda_1 + \partial_x g_1)\chi(x) & \chi(x)\partial_y g_1 \\ \partial_x g_2 & \lambda_2 + \partial_y g_2 \end{pmatrix},$$

which, at $(\pm\epsilon, 0)$, reduces to

$$\begin{pmatrix} 0 + h.o.t & 0 + h.o.t \\ \partial_x g_2 & \lambda_2 + \partial_y g_2 \end{pmatrix}.$$

But the Jacobian of the original vector field at $(\pm\epsilon, 0)$ is simply

$$\begin{pmatrix} \lambda_1 + h.o.t & h.o.t \\ \partial_x g_2 & \lambda_2 + \partial_y g_2 \end{pmatrix},$$

where *h.o.t* stands for higher order terms of x and y . Therefore, Jacobians for the two vector fields are not close to each other, that is, the new vector field is not C^1 close to the original one.

Although non-regular perturbations seem to be an unavoidable problem in the modification of vector fields of this type, our modified vector field (3) has been chosen to single out the non-regular factors. Roughly speaking, near $\partial^* N_{\epsilon_0}$, we have projected the vector field $f(z)$ to $T_z \partial^* N_{\epsilon_0}$ for $z \in N_{\epsilon_0}$ and leaved the component along $V_s(Y)$ and Y unchanged. Therefore, the components of the new vector field along the stable directions still remain as a regular perturbation to the original ones, that is, these components and their associated derivatives differ from the original ones by an order of ϵ_0^2 , and order of ϵ_0 , respectively (see Proposition 1 a) and Lemma 4). Consequently, the lower bound of the decay rate along these non-regular directions is close to the one associated to the original vector field (see Proposition 1 b)). The non-regular perturbation will only arise in the modification of the center-unstable directions, which however results in a slowing down of the modified flow when entering U_{ϵ_0} . As what we shall see in Section 4, such a ‘slowing down’ will actually provide a positive effect to our analysis.

4. ANALYSIS ON THE MODIFIED VECTOR FIELD

Recall that for $0 < \epsilon \leq \epsilon_0$, N_ϵ admits a bundle structure $N_\epsilon = \cup_{z_0 \in U_\epsilon} S_{\epsilon_0}(z_0)$ over the approximated center-unstable manifold $\hat{M}_{cu}(Y)$. Throughout rest of the paper, for $z = p + q \in N_\epsilon$ where $p \in \hat{M}_{cu}(Y)$ and $q \in S_{\epsilon_0}(p)$, we simply use the notation $z = (p, q)$.

In this section, we shall prove the following proposition which gives the key estimates to the modified flow.

Proposition 1. *Given $T > 0$, $C > 0$ and ϵ_0 small. If $(p_i, q_i) \in N_{\epsilon_0}$ and $z_i(t) = (p_i(t), q_i(t))$ are solutions of (3) with $z_i(0) = (p_i, q_i)$, ($i = 1, 2$) and $|p_2 - p_1| < C\epsilon_0$, then there exists a $C(T) > 0$ such that, as long as $z_i(t) \in N_{\epsilon_0}$ for $t \in [0, T]$, the following holds.*

(a) (stable contracting rate)

$$(4) \quad \begin{aligned} |q_2(T) - q_1(T)| &\leq \left(e^{C(T)\epsilon_0} \|\Phi_s(y, T)\| + C(T)\epsilon_0 \right) |q_2 - q_1| \\ &\quad + C(T)\epsilon_0 |p_2 - p_1|, \end{aligned}$$

where $y = Q(p_1)$.

In particular,

$$\|D_z(\pi_s \phi^T)|_{TS_{\epsilon_0}(z)}\| \leq e^{C(T)\epsilon_0} \|\Phi_s(y, T)\| + C(T)\epsilon_0,$$

and

$$\|D_z(\pi_s \phi^T)|_{TU(z)}\| \leq C(T)\epsilon_0,$$

where $y = Q\pi_{cu}z$.

(b) (center-unstable expanding rate)

$$(5) \quad \begin{aligned} |p_2(T) - p_1(T)| &\geq \left(e^{C(T)\epsilon_0} \|\Phi_s(y, T)\|^\beta - C(T)\epsilon_0 \right) |p_2 - p_1| \\ &\quad - C(T)\epsilon_0 |q_2 - q_1|, \end{aligned}$$

where $y = Q(p_1)$.

In particular,

$$\|D_z(\pi_{cu} \phi^T)|_{TU(z)}\| \geq e^{C(T)\epsilon_0} \|\Phi_s(y, T)\|^\beta - C(T)\epsilon_0,$$

where $y = Q\pi_{cu}z$.

We first study some properties of $v(z)$, $\eta(z)$ and $\xi(z)$.

Lemma 1. *There is a $C > 0$ depending only on Y such that for any $z \in \partial^* N_{\epsilon_0}$ the following holds.*

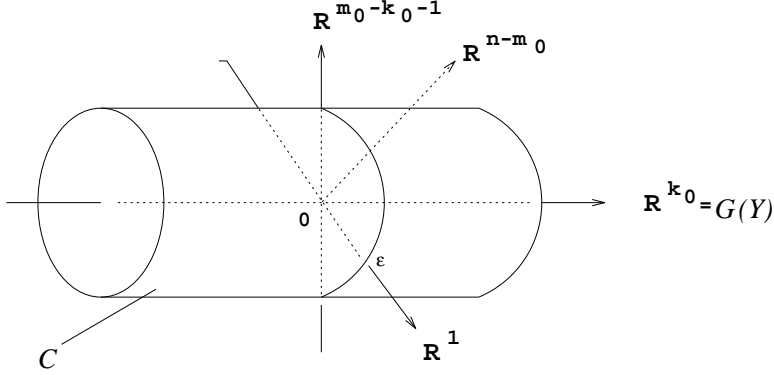
- (i) $v^\top(z)Jv(z) = 0$, $|Jv(z)v_j(z)| \leq C\epsilon_0$ for $j \leq k_0$ or $j > m_0$,
 $|v_j^\top(z)Jv(z)v_j(z) - \epsilon_0^{-1}| \leq C$ for $k_0 < j \leq m_0$;
- (ii) $|\nabla\eta(z) - v(z)| \leq C\epsilon_0$;
- (iii) $|\xi_j(z)| \leq C\epsilon_0$ for $j = 1$ and $j > k_0$;
- (iv) $|\nabla\xi(z)| \leq C$.

Proof. We identify $T_{y_0}Y$, $V'(y_0)$ and $V_s(y_0)$ with \mathbb{R}^{k_0} , $\mathbb{R}^{m_0-k_0}$ and \mathbb{R}^{n-m_0} , respectively. Let

$$\mathcal{G} : N(y_0) \subset \mathbb{R}^n \rightarrow \mathbb{R}^{k_0} \times \mathbb{R}^{m_0-k_0} \times \mathbb{R}^{n-m_0}$$

be the inverse of $\exp_{y_0} : T_{y_0}\mathbb{R}^n \rightarrow \mathbb{R}^n$. Then $D\mathcal{G}$ is ϵ_0 close to the identity and $D^2\mathcal{G}$ is bounded. Moreover,

$$\begin{aligned} \mathcal{G}(y_0) &= 0, \mathcal{G}(z) = (0, \dots, 0, \epsilon_0, 0, \dots, 0), \\ \mathcal{G} : N(y_0) \cap Y &\rightarrow \mathbb{R}^{k_0} \times \{0\} \times \{0\}, \text{ and} \\ \mathcal{G} : N(y_0) \cap U_{\epsilon_0} &\rightarrow \mathbb{R}^{k_0} \times \mathbb{R}^{m_0-k_0} \times \{0\}, \\ \mathcal{G}(\partial U_{\epsilon_0}) &= \mathcal{C}_{\epsilon_0}, \text{ where} \end{aligned}$$

FIGURE 4. The image of \mathcal{G}

$$\mathcal{C}_\epsilon = \left\{ x = (x_1, x_2, \dots, x_{m_0}, 0, \dots, 0) : \sum_{j=k_0+1}^{m_0} x_j^2 = \epsilon^2 \right\}.$$

To prove (i), we note that the inward normal vector at $x \in \mathcal{C}_\epsilon$ is

$$n(x) = \frac{1}{|x|} (0, \dots, 0, -x_{k_0+1}, \dots, -x_{m_0}, 0, \dots, 0).$$

Therefore,

$$Jn(x) = \frac{1}{|x|^3} \begin{pmatrix} O_{k_0 \times k_0} & O & O \\ O & D_{(m_0 - k_0) \times (m_0 - k_0)} & O \\ O & O & O_{(n - m_0) \times (n - m_0)} \end{pmatrix},$$

where $D = (d_{ij})$ with $d_{ii} = x_i^2 - |x|^2$ and $d_{ij} = x_i x_j$ for $i \neq j$, $k_0 + 1 \leq i, j \leq m_0$. The matrix D has zero as a simple eigenvalue with $n(x)$ as the corresponding eigenvector and has $-\frac{1}{|x|}$ as the other eigenvalues. If we assume, without loss of generality, that $x_j = 0$ for $j \neq k_0 + 1$, then $d_{ij} = 0$ for $i \neq j$, $d_{k_0+1, k_0+1} = 0$ and $d_{ii} = -|x|^{-1}$ for $i \neq k_0 + 1$, that is, D has a Jordan form with the desired eigenvalues. The property (i) easily follows.

Next, let $\bar{\eta}(x) = \epsilon_0 - |x|$, for $x \in \mathcal{C}_\epsilon$, which is related to η by \mathcal{G} . Since

$$\nabla \bar{\eta}(x) = -\frac{1}{|x|} (0, \dots, 0, x_{k_0+1}, \dots, x_{m_0}, 0, \dots, 0)^\top = n^\top(x),$$

and $D\mathcal{G}$ is ϵ_0 close to the identity, we have $|\nabla(\eta(z)) - v(z)| \leq C\epsilon_0$. The property (ii) is proved.

The property (iii) clearly holds by the choice of v'_j 's.

It remains to prove (iv). Note that

$$f(z) = \xi(z)v(z) + \sum_{j=2}^n \xi_j(z)v_j(z).$$

Then

$$\begin{aligned} Jf(z) &= v(z)(\nabla \xi(z))^\top + \xi(z)Jv(z) \\ &\quad + \sum_{j=2}^n (v_j(z)(\nabla \xi_j(z))^\top + \xi_j(z)Jv_j(z)). \end{aligned}$$

Multiplying $v^\top(z)$ to the left of above, we have

$$v^\top(z)A(z) = (\nabla\xi(z))^\top + \xi(z)v^\top(z)Jv(z) + \sum_{j=2}^n \xi_j(z)v^\top(z)Jv_j(z).$$

Since $\langle v_i(z), v_j(z) \rangle = \delta_{ij}$, we have

$$v^\top(z)Jv(z) = 0 \text{ and } (Jv_j(z))^\top v(z) = -(Jv(z))^\top v_j(z).$$

Therefore,

$$\begin{aligned} \nabla\xi(z) &= A^\top(z)v(z) - \sum_{j=2}^n \xi_j(z)(Jv_j(z))^\top v(z) \\ (6) \qquad &= A^\top(z)v(z) + \sum_{j=2}^n \xi_j(z)(Jv(z))^\top v_j(z). \end{aligned}$$

By (i) and (iii), $|\nabla\xi(z)| \leq C$. \square

The following generalized Gronwall's inequality will be frequently used later.

Lemma 2. *If $\beta(t) \geq 0$, $\alpha(t)$ and $\phi(t)$ are continuous real functions on $[a, b]$ and*

$$\phi(t) \leq \alpha(t) + \int_a^t \beta(s)\phi(s)ds, \quad a \leq t \leq b$$

holds, then

$$\phi(t) \leq \alpha(t) + \int_a^t \beta(s)\alpha(s)e^{\int_s^t \beta(\mu)d\mu}ds, \quad a \leq t \leq b.$$

If, in addition, $\alpha'(t) \geq 0$, then $\phi(t) \leq \alpha(t)e^{\int_a^t \beta(s)ds}$, $a \leq t \leq b$.

Proof. See [24]. \square

Lemma 3. *Let $z_1(t)$ and $z_2(t)$ be trajectories of (1) and (3) respectively. For any T , if $z_2(t) \in N_{\epsilon_0}$ and $\lambda z_2(t) + (1-\lambda)z_1(t) \in N_0(Y)$, for all $|t| \leq T$ and $\lambda \in [0, 1]$, then*

$$|z_1(t) - z_2(t)| \leq (|z_2(0) - z_1(0)| + C\epsilon_0|t|)e^{|Df||t|}$$

for all $|t| \leq T$, where $|Df| = |Df|_{N_0(Y)}$ and C is the constant defined in Lemma 1.

Proof. We only prove the case when $t \geq 0$. By (1) and (3),

$$\begin{aligned} (z_1(t) - z_2(t))' &= f(z_1(t)) - \hat{f}(z_2(t)) \\ &= f(z_1(t)) - f(z_2(t)) \\ &\quad + (1 - \chi_1(\eta(z_2(t))))\chi_2(\xi(z_2(t)))\xi(z_2(t))v(z_2(t)). \end{aligned}$$

It follows that

$$\begin{aligned} &|z_1(t) - z_2(t)| - |z_2(0) - z_1(0)| \\ &\leq \int_0^t |f(z_1(s)) - f(z_2(s))|ds + \int_0^t |\xi(z_2(s))v(z_2(s))|ds \\ &\leq |Df| \int_0^t |z_1(s) - z_2(s)|ds + \int_0^t |\xi(z_2(s))v(z_2(s))|ds. \end{aligned}$$

Since $|\xi(z_2(s))| \leq C\epsilon_0$, Lemma 2 implies that

$$|z_1(t) - z_2(t)| \leq (|z_2(0) - z_1(0)| + C\epsilon_0 t)e^{|Df|t}, \quad 0 \leq t \leq T.$$

This completes the proof. \square

Lemma 4. *For a given $T > 0$, there exists $C(T) > 0$ such that if*

$$z(t) = (p(t), q(t)) \equiv (\pi_{cu}z(t), \pi_s z(t))$$

is the solution of (3) with $z(0) = (p_0, q_0) \in N_{\epsilon_0}$, then

$$|q(t)| \leq e^{\alpha t} |z(0) - y| + C(T)\epsilon_0^2$$

for $0 \leq t \leq T$, where $y = Q(p_0) \in Y$.

Proof. Below, we fix a $T > 0$ and let $0 \leq t \leq T$.

Denote $F(z, y) = f(z) - f(y) - A(y)(z - y)$. Then

$$\begin{aligned} z'(t) &= f(y \cdot t) + A(y \cdot t)(z(t) - y \cdot t) + F(z(t), y \cdot t) \\ &\quad - (1 - \chi_1(\eta(z)))\chi_2(\xi(z))\xi(z(t))v(z(t)). \end{aligned}$$

The variation of constant formula yields that

$$\begin{aligned} z(t) - y \cdot t &= \Phi(y, t)(z(0) - y) + \int_0^t \Phi(y, t)\Phi^{-1}(y, s)F(z(s), y \cdot s)ds \\ &\quad - \int_0^t \Phi(y, t)\Phi^{-1}(y, s)(1 - \chi_1(\eta(z)))\chi_2(\xi(z))\xi(z(s))v(z(s))ds. \end{aligned}$$

Therefore, by noting that $y \cdot t \in Y \subset \hat{M}_{cu}(Y)$, there exists a $C_0(T) > 0$ such that

$$\begin{aligned} |q(t)| &= |\pi_s z(t)| = |\pi_s z(t) - \pi_s(y \cdot t)| \\ &\leq |D\pi_s(y \cdot t)(z(t) - y \cdot t)| + C_0(T)|z(t) - y \cdot t|^2 \\ &= |P_s(y \cdot t)(z(t) - y \cdot t)| + C_0(T)|z(t) - y \cdot t|^2 \\ &\leq |P_s(y \cdot t)\Phi(y, t)(z(0) - y)| \\ &\quad + \left| \int_0^t \Phi(y, t)P_s(y)\Phi^{-1}(y, s)F(z(s), y \cdot s)ds \right| \\ &\quad + \left| \int_0^t \Phi(y, t)P_s(y)\Phi^{-1}(y, s)(1 - \chi_1(\eta(z(s))))\chi_2(\xi(z(s)))\xi(z(s)) \right. \\ &\quad \left. \cdot P_s(y \cdot s)v(z(s))ds \right| + C_0(T)|z(t) - y \cdot t|^2. \end{aligned}$$

By Lemma 3, $|z(s) - y \cdot s| = O(\epsilon_0)$ and hence $|F(z(s), y \cdot s)| = O(\epsilon_0^2)$. Since v is tangent to $\hat{M}_{cu}(Y)$, we also have $|P_s(y \cdot s)v(z(s))| = O(\epsilon_0)$. The lemma then follows easily from the above inequality, **(H2)** and Remark 1, and Lemma 1. \square

Let $T > 0$ and $z_i(t) = (p_i(t), q_i(t))$ be solutions of (3) with $z_i(0) = (p_i, q_i) \in N_{\epsilon_0}$ ($i = 1, 2$) such that $|p_2 - p_1| = O(\epsilon_0^2)$. Denote $y = Q(p_1) \in Y$. By Lemma 3, there is a $C(T)$ such that $|y \cdot t - z_i(t)| \leq C(T)\epsilon_0$ for $t \in [0, T]$.

Set $w(t) = z_2(t) - z_1(t)$. Since by (3),

$$\begin{aligned} (z_i(t) - y \cdot t)' &= \hat{f}(z_i(t)) - f(y \cdot t) \\ &= A(y \cdot t)(z_i(t) - y \cdot t) + F(z_i(t), y \cdot t) \\ &\quad - (1 - \chi_1(\eta(z_i)))\chi_2(\xi(z_i))\xi(z_i(t))v(z_i(t)), \end{aligned}$$

where $F(z, y) = f(z) - f(y) - A(y)(z - y)$, we have

$$(7) \quad w' = A(y \cdot t)w + B(y \cdot t)w + O(\epsilon_0 w),$$

where

$$\begin{aligned}
B(y \cdot t)w &= \langle \nabla(\chi_1(\eta(\bar{z}))), w \rangle \chi_2(\xi(\bar{z})) \xi(\bar{z}) v(\bar{z}) \\
&\quad - (1 - \chi_1(\eta(\bar{z}))) \langle \nabla(\chi_2(\xi(\bar{z}))), w \rangle \xi(\bar{z}) v(\bar{z}) \\
&\quad - (1 - \chi_1(\eta(\bar{z}))) \chi_2(\xi(\bar{z})) \langle \nabla \xi(\bar{z}), w \rangle v(\bar{z}) \\
&\quad - (1 - \chi_1(\eta(\bar{z}))) \chi_2(\xi(\bar{z})) \xi(\bar{z}) Jv(\bar{z}) w,
\end{aligned}$$

with $\bar{z}(t) = \lambda(t)z_2(t) + (1 - \lambda(t))z_1(t)$ for some $\lambda(t) \in [0, 1]$.

Since, for $i = s, c, u$,

$$\begin{aligned}
(P_i(y \cdot t)\Phi(y, t))' &= (P_i(y \cdot t))' \Phi(y, t) + P_i(y \cdot t)A(y \cdot t)\Phi(y, t), \\
(\Phi(y, t)P_i(y))' &= A(y \cdot t)\Phi(y, t)P_i(y) = A(y \cdot t)P_i(y \cdot t)\Phi(y, t),
\end{aligned}$$

we have

$$(P_i(y \cdot t))' = A(y \cdot t)P_i(y \cdot t) - P_i(y \cdot t)A(y \cdot t).$$

By (7),

$$\begin{aligned}
(P_i(y \cdot t)w)' &= A(y \cdot t)P_i(y \cdot t)w - P_i(y \cdot t)A(y \cdot t)w + P_i(y \cdot t)w' \\
(8) \quad &= A(y \cdot t)P_i(y \cdot t)w + P_i(y \cdot t)B(y \cdot t)w + O(\epsilon_0 P_i(y \cdot t)w).
\end{aligned}$$

Lemma 5. *Let w, B be as above. Then for $T > 0$, there is a $C(T) > 0$ such that*

$$(9) \quad |P_s(y \cdot t)B(y \cdot t)w(t)| \leq C(T)\epsilon_0|w(t)|,$$

$$(10) \quad |B(y \cdot t)P_s(y \cdot t)w(t)| \leq C(T)\epsilon_0|w(t)|,$$

$$(11) \quad |P_{cu}(y \cdot t)B(y \cdot t)w(t)| \leq C(T)|w(t)|$$

for all $y \in Y$ and $t \in [0, T]$.

Proof. We only prove (9). Let $y \in Y, t \in [0, T]$ and let $w = w(t), \bar{z} = \bar{z}(t)$ be as above. We note that

$$\begin{aligned}
|P_s(y \cdot t)B(y \cdot t)w(t)| &\leq |\langle \nabla(\chi_1(\eta(\bar{z}))), w \rangle \chi_2(\xi(\bar{z})) \xi(\bar{z}) P_s v(\bar{z})| \\
&\quad + |(1 - \chi_1(\eta(\bar{z}))) \langle \nabla(\chi_2(\xi(\bar{z}))), w \rangle \xi(\bar{z}) P_s v(\bar{z})| \\
&\quad + |(1 - \chi_1(\eta(\bar{z}))) \chi_2(\xi(\bar{z})) \langle \nabla \xi(\bar{z}), w \rangle P_s v(\bar{z})| \\
&\quad + |(1 - \chi_1(\eta(\bar{z}))) \chi_2(\xi(\bar{z})) \xi(\bar{z}) P_s Jv(\bar{z}) w|.
\end{aligned}$$

By Lemma 1 and the definitions of χ_1, χ_2 , there is a $C > 0$ such that $|\nabla \xi(\bar{z})| \leq C$, $|\nabla(\chi_1(\eta(\bar{z})))| \leq 4\epsilon_0^{-1}$, $|\chi_2(\xi(\bar{z})) \xi(\bar{z})| \leq C\epsilon_0$, $|\nabla \eta(\bar{z}) - v(\bar{z})| \leq C\epsilon_0$, $|\nabla \chi_2(\xi(\bar{z}))| \leq C$, $|\xi(\bar{z})| \leq C\epsilon_0$, and $|P_s Jv(\bar{z}) v_j(\bar{z})| \leq C$ for $j = 1, \dots, n$. Since $v(\bar{z})$ is tangent to $\hat{M}_{cu}(Y)$, we also have $|P_s v(\bar{z})| \leq C_0(T)\epsilon_0$ for a constant $C_0(T) > 0$. It follows that there is a constant $C_1(T) > 0$ such that all terms above are bounded by $C_1(T)\epsilon_0|w(t)|$. \square

Proof of Proposition 1. Below, for simplicity, we denote all constants which depend only on T as $C(T)$. Let w, B be as above and let $t \in [0, T], y = Q(p_1)$.

To prove (a), let $\tilde{z}(t)$ be such that

$$|q_2(t) - q_1(t)| = |\pi_s z_2(t) - \pi_s z_1(t)| = |D\pi_s(\tilde{z}(t))(z_2(t) - z_1(t))|.$$

By Lemma 3, $|\tilde{z}(t) - y \cdot t| \leq C(T)\epsilon_0$. Therefore,

$$\begin{aligned}
|q_2(t) - q_1(t)| &= |\pi_s z_2(t) - \pi_s z_1(t)| \\
(12) \quad &\leq |P_s(y \cdot t)w(t)| + C(T)\epsilon_0|w(t)|.
\end{aligned}$$

Applying the variation of constant formula to (8), we have

$$\begin{aligned}
P_s(y \cdot t)w(t) &= \Phi(y, t)P_s(y)w(0) \\
&\quad + \int_0^t \Phi(y, t)\Phi^{-1}(y, s)P_s(y \cdot s)B(y \cdot t)w(s)ds \\
(13) \quad &\quad + \int_0^t \Phi(y, t)\Phi^{-1}(y, s)O(\epsilon_0 P_s(y \cdot s)w(s))ds.
\end{aligned}$$

By (9),

$$\begin{aligned}
|P_s(y \cdot t)w(t)| &\leq e^{\alpha t}|P_s(y)w(0)| + \int_0^t e^{\alpha(t-s)}C(T)\epsilon_0|P_s(y \cdot s)w(s)|ds \\
&\quad + \int_0^t e^{\alpha(t-s)}C(T)\epsilon_0|P_{cu}(y \cdot s)w(s)|ds.
\end{aligned}$$

It follows from Lemma 2 that

$$(14) \quad |P_s(y \cdot t)w(t)| \leq e^{\alpha t + C(T)\epsilon_0} \left(|P_s(y)w(0)| + C(T)\epsilon_0 \int_0^t e^{\alpha(t-s)}|P_{cu}(y \cdot s)w(s)|ds \right).$$

Similarly,

$$(15) \quad |P_{cu}(y \cdot t)w(t)| \leq e^{\gamma t + C(T)} \left(|P_{cu}(y)w(0)| + C(T) \int_0^t e^{\alpha(t-s)}|P_s(y \cdot s)w(s)|ds \right),$$

where $\gamma > 0$ is such that $|\Phi(y_*, t)| \leq Ce^{\gamma t}$ ($y_* \in Y$) for some $C > 0$.

Substituting (15) into (14) and applying Lemma 2, we have

$$(16) \quad |P_s(y \cdot t)w(t)| \leq e^{\alpha t + C(T)\epsilon_0} (|P_s(y)w(0)| + C(T)\epsilon_0|P_{cu}(y)w(0)|).$$

Substituting (16) into (15) and applying Lemma 2 again, we have

$$(17) \quad |P_{cu}(y \cdot t)w(t)| \leq e^{\gamma t + C(T)\epsilon_0}|P_{cu}(y)w(0)| + C(T)|P_s(y)w(0)|.$$

Finally, applying (11), (16) and (17) to (13), we conclude that

$$|q_2(T) - q_1(T)| \leq \left(e^{C(T)\epsilon_0}|\Phi_s(y, T)| + C(T)\epsilon_0 \right) |q_2 - q_1| + C(T)\epsilon_0|p_2 - p_1|.$$

This completes the proof of (a).

Next we prove (b). Similarly as in the proof of (a), we have

$$\begin{aligned}
|p_2(t) - p_1(t)| &= |\pi_{cu}z_2(t) - \pi_{cu}z_1(t)| \\
&= |D\pi_{cu}(\tilde{z}(t))(z_2(t) - z_1(t))| \\
(18) \quad &\geq |P_{cu}(y \cdot t)w(t)| - C(T)\epsilon_0|w(t)|,
\end{aligned}$$

where $\tilde{z}(t)$ is such that $|\tilde{z}(t) - y \cdot t| \leq C(T)\epsilon_0$. It remains to estimate $|P_{cu}(y \cdot t)w(t)|$.

By (7), (9) and (10),

$$\begin{aligned}
(P_{cu}(y \cdot t)w)' &= A(y \cdot t)P_{cu}(y \cdot t)w + P_{cu}(y \cdot t)B(y \cdot t)w + O(\epsilon_0 P_{cu}(y \cdot t)w) \\
&= (A(y \cdot t) + B(y \cdot t))P_{cu}(y \cdot t)w + P_{cu}(y \cdot t)B(y \cdot t)P_s(y \cdot t)w \\
&\quad - P_s(y \cdot t)B(y \cdot t)P_{cu}(y \cdot t)w + O(\epsilon_0 P_{cu}(y \cdot t)w) \\
&= (A(y \cdot t) + B(y \cdot t))P_{cu}(y \cdot t)w \\
&\quad + O(\epsilon_0 P_{cu}(y \cdot t)w) + O(\epsilon_0 P_s(y \cdot t)w).
\end{aligned}$$

A direct computation yields

$$(19) \quad \left(\frac{1}{2} |P_{cu}(y \cdot t)w|^2 \right)' = \langle A(y \cdot t)P_{cu}(y \cdot t)w, P_{cu}(y \cdot t)w \rangle + \langle B(y \cdot t)P_{cu}(y \cdot t)w, P_{cu}(y \cdot t)w \rangle + O(\epsilon_0 |P_{cu}(y \cdot t)w|^2).$$

We **claim** that the following inequality holds:

$$(20) \quad \left(\frac{1}{2} |P_{cu}(y \cdot t)w|^2 \right)' \geq \min\{\langle A(y \cdot t)P_{cu}(y \cdot t)w, P_{cu}(y \cdot t)w \rangle, 0\} + O(\epsilon_0 |P_{cu}(y \cdot t)w|^2).$$

Let $\bar{z} = \bar{z}(t)$ be as in (8).

Case 1: $\xi(\bar{z}) = 0$.

In this case,

$$\langle B(y \cdot t)P_{cu}(y \cdot t)w, P_{cu}(y \cdot t)w \rangle = -(1 - \chi_1(\eta(\bar{z})) \langle \nabla \xi(\bar{z}), P_{cu}(y \cdot t)w \rangle \langle P_{cu}v(\bar{z}), P_{cu}w \rangle).$$

By expressing $P_{cu}(y \cdot t)w$ as a linear combination of $P_{cu}(y \cdot t)v(\bar{z})$ and its orthogonal complement, we see that either (i) $P_{cu}(y \cdot t)w \perp P_{cu}(y \cdot t)v(\bar{z})$ or (ii) $P_{cu}w = \lambda P_{cu}v$ for some λ .

If (i) holds, then $\langle B(y \cdot t)P_{cu}w, P_{cu}w \rangle = 0$ and (20) is clearly true.

If (ii) holds, then by (6),

$$\begin{aligned} \langle \nabla \xi(\bar{z}), P_{cu}w \rangle &= \langle A^\top(\bar{z})v(\bar{z}), P_{cu}w \rangle + \sum_{j=2}^n \xi_j(\bar{z}) \langle (Jv_j(\bar{z}))^\top v(\bar{z}), P_{cu}w \rangle \\ &= \langle A(\bar{z})P_{cu}w, v(\bar{z}) \rangle - \sum_{j=2}^n \xi_j(\bar{z}) \langle (Jv(\bar{z}))^\top v_j(\bar{z}), P_{cu}w \rangle, \end{aligned}$$

where $Jv(\bar{z})P_{cu}w = \lambda Jv(\bar{z})P_{cu}v(\bar{z}) = O(\epsilon_0)$. It follows that

$$\begin{aligned} &\langle (A(y \cdot t) + B(y \cdot t))P_{cu}(y \cdot t)w, P_{cu}(y \cdot t)w \rangle \\ &= \chi_1(\eta(\bar{z})) \langle A(y \cdot t)P_{cu}(y \cdot t)w, P_{cu}(y \cdot t)w \rangle + O(\epsilon_0 |P_{cu}(y \cdot t)w|^2). \end{aligned}$$

If $\langle A(y \cdot t)P_{cu}(y \cdot t)w, P_{cu}(y \cdot t)w \rangle > 0$, then the first term above is nonnegative. If $\langle A(y \cdot t)P_{cu}(y \cdot t)w, P_{cu}(y \cdot t)w \rangle \leq 0$, then the first term above is greater than or equals to $\langle A(y \cdot t)P_{cu}w, P_{cu}w \rangle$. Hence, (20) holds.

Case 2: $\xi(\bar{z}) > 0$.

In this case, $\chi_2(\xi(\bar{z})) = 1$. Therefore,

$$\begin{aligned} \langle B(y \cdot t)P_{cu}(y \cdot t)w, P_{cu}(y \cdot t)w \rangle &= \langle \nabla(\chi_1(\eta(\bar{z}))), P_{cu}w \rangle \xi(\bar{z}) \langle P_{cu}w, P_{cu}v(\bar{z}) \rangle \\ &\quad - (1 - \chi_1(\eta(\bar{z}))) \langle \nabla \xi(\bar{z}), P_{cu}w \rangle \langle P_{cu}w, P_{cu}v(\bar{z}) \rangle \\ &\quad - (1 - \chi_1(\eta(\bar{z}))) \xi(\bar{z}) \langle P_{cu}Jv(\bar{z})P_{cu}w, P_{cu}w \rangle \\ &\quad + \langle P_{cu}(y \cdot t)JF(\bar{z}, y \cdot t)P_{cu}w, P_{cu}w \rangle. \end{aligned}$$

By Lemma 1 and the fact that $\nabla(\chi_1(\eta(\bar{z}))) = \chi_1'(\eta(\bar{z}))\nabla\eta(\bar{z})$, the first term above is a sum of a nonnegative term and a term of order ϵ_0 . The second term above is of order ϵ_0 as in (ii) above. By Lemma 1 (i), the third term above is positive. The last term is of higher order of ϵ_0 . Therefore,

$$\begin{aligned} &\langle (A(y \cdot t) + B(y \cdot t))P_{cu}(y \cdot t)w, P_{cu}(y \cdot t)w \rangle \\ &\geq \min\{\langle A(y \cdot t)P_{cu}(y \cdot t)w, P_{cu}(y \cdot t)w \rangle, 0\} + O(\epsilon_0 |P_{cu}(y \cdot t)w|^2), \end{aligned}$$

that is, (20) holds.

Case 3: $\xi(\bar{z}) < 0$.

In this case, $|\chi_2(\xi(\bar{z}))\xi(\bar{z})| = O(\epsilon_0^2)$. Therefore,

$$\begin{aligned} & |\langle \nabla(\chi_1(\eta(\bar{z}))), P_{cu}w \rangle \chi_2(\xi(\bar{z}))\xi(\bar{z}) \langle P_{cu}(y \cdot t)v(\bar{z}), P_{cu}w \rangle| \\ & \leq C(T)(\epsilon_0)^{-1}\epsilon_0^2|P_{cu}w|^2 = C(T)\epsilon_0|P_{cu}w|^2, \end{aligned}$$

and

$$\begin{aligned} & |(1 - \chi_1(\eta(\bar{z})))\chi_2(\xi(\bar{z}))\xi(\bar{z}) \langle P_{cu}(y \cdot t)Jv(\bar{z})P_{cu}w, P_{cu}w \rangle| \\ & \leq C(T)\epsilon_0^2(\epsilon_0)^{-1}|P_{cu}w|^2 = C(T)\epsilon_0|P_{cu}w|^2. \end{aligned}$$

Thus,

$$\begin{aligned} & \langle (A(y \cdot t) + B(y \cdot t))P_{cu}(y \cdot t)w, P_{cu}(y \cdot t)w \rangle \\ & = \langle A(y \cdot t)P_{cu}(y \cdot t)w, P_{cu}(y \cdot t)w \rangle + O(\epsilon_0|P_{cu}(y \cdot t)w|^2). \end{aligned}$$

Therefore, (20) holds in any cases.

Now let $\theta(t), C(t)$ be nonnegative continuous functions in $t \in [0, T]$ such that

$$(21) \quad \left(\frac{1}{2}|P_{cu}w|^2 \right)' = \min\{\langle A(y \cdot t)P_{cu}(y \cdot t)w, P_{cu}(y \cdot t)w \rangle, 0\} + (\theta(t) + C(t)\epsilon_0)|P_{cu}(y \cdot t)w|^2.$$

Denote

$$\mathcal{I} = \{t \in [0, T] : \langle A(y \cdot t)P_{cu}w(t), P_{cu}w(t) \rangle < 0\}$$

and $\mathcal{I}^c = [0, T] \setminus \mathcal{I}$. Since \mathcal{I} is open, we can express \mathcal{I} as a countable union of non-overlapping intervals, say, $\mathcal{I} = \cup_{j=1}^{+\infty} (\underline{t}_j, \bar{t}^j)$. It can be verified directly that, for any $t \in (\underline{t}_j, \bar{t}^j)$,

$$\exp\left(\int_{\underline{t}_j}^t (\theta(\tau) + C(\tau)\epsilon_0)d\tau\right) \Phi(y \cdot \underline{t}_j, t - \underline{t}_j)P_{cu}(y \cdot \underline{t}_j)w(\underline{t}_j)$$

is a solution of (21). Since $P_{cu}(y \cdot t)w(t)$ is also a solution, we have

$$\begin{aligned} & |P_{cu}(y \cdot t)w(t)| \\ & \geq \exp\left(\int_{\underline{t}_j}^t (\theta(\tau) + C(\tau)\epsilon_0)d\tau\right) |\Phi(y \cdot \underline{t}_j, t - \underline{t}_j)P_{cu}(y \cdot \underline{t}_j)w(\underline{t}_j)| \\ & \geq e^{C(t)\epsilon_0} |\Phi(y \cdot \underline{t}_j, t - \underline{t}_j)P_{cu}(y \cdot \underline{t}_j)w(\underline{t}_j)| \\ (22) \quad & \geq e^{C(t)\epsilon_0} m(\Phi_{cu}(y \cdot \underline{t}_j, t - \underline{t}_j)) |w(\underline{t}_j)|. \end{aligned}$$

By (21), for $[s, t] \in \mathcal{I}^c$ we have

$$(23) \quad |P_{cu}(y \cdot t)w(t)| \geq e^{O(\epsilon)(t-s)} |P_{cu}(y \cdot s)w(s)|.$$

Applying (22) and (23), and also using **(H2)** and Lemma 1, we have

$$\begin{aligned} |P_{cu}(y \cdot t)w(t)| & \geq e^{C(T)\epsilon_0} m(\Phi_{cu}(y \cdot \underline{t}_n, t - \underline{t}_n)) \cdots m(\Phi_{cu}(y, \underline{t}_1)) |p_2 - p_1| \\ & \quad - C(T)\epsilon_0(|q_2 - q_1| + |p_2 - p_1|) \\ & \geq e^{C(T)\epsilon_0} \|\Phi_s(y \cdot \underline{t}_n, t - \underline{t}_n)\|^\beta \cdots \|\Phi_s(y, \underline{t}_1)\|^\beta |p_2 - p_1| \\ & \quad - C(T)\epsilon_0(|q_2 - q_1| + |p_2 - p_1|) \\ & \geq (e^{C(T)\epsilon_0} \|\Phi_s(y, t)\|^\beta - C(t)\epsilon_0) |p_2 - p_1| - C(T)\epsilon_0 |q_2 - q_1|. \end{aligned}$$

This completes the proof. \square

5. PROOF OF MAIN THEOREM

5.1. Existence of center-unstable manifold. Using the bundle structure constructed in Section 3, we first choose a function space to which the graph transform will apply.

Define

$$\Gamma := \{h : U_{\epsilon_0} \rightarrow N_{\epsilon_0}; h(p) \in S_{\epsilon_0}(p), \forall p \in U_{\epsilon_0}, |h|_{C^0} < \infty\}$$

and

$$(24) \quad \Gamma_\rho := \{h \in \Gamma : \text{Lip}(h) \leq \rho\},$$

where $\rho > 0$,

$$\text{Lip}(h) = \sup_{p \in U_{\epsilon_0}} \text{Lip}_p(h), \text{ and } \text{Lip}_p(h) = \limsup_{p' \rightarrow p, p' \in U_{\epsilon_0}} \frac{|h(p') - h(p)|}{|p' - p|}.$$

Lemma 6. Γ_ρ is closed in Γ .

Proof. Suppose $h_n \in \Gamma_\rho$ and $h_n \rightarrow h$ in C^0 -norm. We shall show that $\text{Lip}_p(h) \leq \rho$ for any $p \in U_{\epsilon_0}$. For $p', p'' \in U_{\epsilon_0}$, let $\tilde{d}(p', p'')$ denote the induced Euclidean distance between p' and p'' in U_{ϵ_0} (i.e., the length of the shortest path in U_{ϵ_0} joining p' and p''). Then for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$(1 - \epsilon)\tilde{d}(p', p'') \leq |p' - p''| \leq \tilde{d}(p', p'')$$

whenever $|p' - p''| \leq \delta$.

For any $p_1 \in U_{\epsilon_0}$ with $|p_1 - p| \leq \delta$, we let γ be the shortest path in U_{ϵ_0} from p_1 to p . Then for any $p' \in \gamma$, there exists $\delta(p') > 0$ such that

$$|h_n(p'') - h_n(p')| \leq (\rho + \epsilon)|p'' - p'|$$

for all n and $p'' \in \gamma$ with $|p'' - p'| \leq \delta(p')$. Since $\cup_{p' \in \gamma} (B_{\delta(p')/2}(p') \cap \gamma)$ covers γ , there exist $p_2, p_3, \dots, p_m = p \in \gamma$, ordered from p_1 to p , such that $\cup_{i=1}^m (B_{\delta(p_i)/2}(p_i) \cap \gamma)$ covers γ . By choosing δ sufficiently small, we may assume that

$$p_i \notin B_{\delta(p_{i+2})/2}(p_{i+2}) \cup B_{\delta(p_{i-2})/2}(p_{i-2}) \text{ for } i = 3, \dots, m-2.$$

Thus,

$$|p_i - p_{i+1}| < \delta(p_i)/2 + \delta(p_{i+1})/2 < \max\{\delta(p_i), \delta(p_{i+1})\},$$

which implies that either $p_i \in B_{\delta(p_{i+1})}(p_{i+1})$ or $p_{i+1} \in B_{\delta(p_i)}(p_i)$ ($i = 1, \dots, m-1$).

In any case,

$$|h_n(p_i) - h_n(p_{i+1})| \leq (\rho + \epsilon)|p_i - p_{i+1}|$$

for all n and $i = 1, \dots, m-1$. Therefore,

$$\begin{aligned} |h_n(p_1) - h_n(p)| &\leq \sum_{i=1}^m |h_n(p_i) - h_n(p_{i+1})| \\ &\leq (\rho + \epsilon) \sum_{i=1}^m |p_i - p_{i+1}| \\ &\leq (\rho + \epsilon) \sum_{i=1}^m \tilde{d}(p_i, p_{i+1}) \\ &= (\rho + \epsilon) \tilde{d}(p_1, p) \\ &\leq \frac{\rho + \epsilon}{1 - \epsilon} |p_1 - p|. \end{aligned}$$

By taking $n \rightarrow +\infty$ in the above, we have

$$|h(p_1) - h(p)| \leq \frac{\rho + \epsilon}{1 - \epsilon} |p_1 - p|.$$

Since ϵ is arbitrary, $\text{Lip}_p(h) \leq \rho$. \square

For each $h \in \Gamma_\rho$, the graph, $\text{graph}(h) := \{(p, h(p)) : p \in U_{\epsilon_0}\}$ of h is a section of the fiber bundle N_{ϵ_0} with base space U_{ϵ_0} . We now define the graph transform.

Let Γ_ρ ($0 < \rho \leq 1$) be as in (24) for a prescribed $\epsilon_0 > 0$ and let ϕ^T ($T > 0$) denote the time T map of the modified flow (3). Define the **graph transform**

$$\phi^* : \Gamma_\rho \rightarrow \Gamma_\rho \text{ by } \phi^* h = H, \text{ where } \text{graph}(H) = \phi^T(\text{graph}(h)) \cap N_{\epsilon_0}.$$

Remark 3. Let $\sigma_h(p) = \pi_{cu} \cdot \phi^T \cdot (id, h)(p)$ for $p \in U_{\epsilon_0}$. Then $\phi^*(h) = \pi_s \cdot \phi^T \cdot (id, h) \cdot \sigma_h^{-1}$ provided σ_h maps U_{ϵ_0} to itself and is invertible.

The existence of a center-unstable manifold for the modified system (3) is an immediate consequence of the following proposition.

Proposition 2. *Fix $0 < \rho \leq 1$. There exist $T > 0$, $\epsilon_0 = \epsilon_0(T) > 0$ such that the following holds.*

- (i) $\phi^* : \Gamma_\rho \rightarrow \Gamma_\rho$ is well-defined;
- (ii) ϕ^* is a contraction with respect to the C^0 -norm.
- (iii) Let $h \in \Gamma_\rho$ be the fixed point of ϕ^* . Then $M_{cu}(Y) := \text{graph}(h)$ is an invariant Lipschitz manifold of (3).

Proof. First, by **(H2)**, we can fix a $T > 0$ such that

$$\|\Phi_s(y, T)\|^{1-\beta} < \frac{1}{16}, \quad \|\Phi_s(y, T)\|^\beta < \frac{1}{8}.$$

Let $C(T)$ be as in Proposition 1. We choose an $\epsilon_0 = \epsilon_0(T)$ such that

$$0 < \epsilon_0 < \frac{\|\Phi_s(y, T)\|^\beta}{2C(T)}.$$

Let $h \in \Gamma_\rho$. We wish to show that $\phi^* h \in \Gamma_\rho$; that is, $\phi^T(\text{graph}(h)) \cap N_{\epsilon_0}$ is a function H with $\text{Lip}(H) \leq \rho$. Equivalently, we shall show that for any $(p_0, q_0) \in \phi^T(\text{graph}(h)) \cap N_{\epsilon_0}$, there exists $\delta > 0$ such that if $(p, q) \in \phi^T(\text{graph}(h)) \cap N_{\epsilon_0}$ and $|(p, q) - (p_0, q_0)| \leq \delta$, then $\frac{|q - q_0|}{|p - p_0|} \leq \rho$.

Fix a $(p_0, q_0) \in \phi^T(\text{graph}(h)) \cap N_{\epsilon_0}$ and define $(p^*, q^*) = \phi^{-T}(p, q)$ for $(p, q) \in \phi^T(\text{graph}(h)) \cap N_{\epsilon_0}$. Clearly, p^*, q^* are continuous functions of (p, q) . It follows that there exists a $\delta > 0$ such that if $|(p, q) - (p_0, q_0)| \leq \delta$, then $|(p^*, q^*) - (p_0^*, q_0^*)| \leq \epsilon_0$. Now let $(p, q) \in \phi^T(\text{graph}(h)) \cap N_{\epsilon_0}$ with $|(p, q) - (p_0, q_0)| \leq \delta$ and denote $(p_1, q_1) = (p_0^*, q_0^*)$, $(p_2, q_2) = (p^*, q^*)$. Due to the overflowing property of the modified flow, we have $\phi^t(p_i, q_i) \in N_{\epsilon_0}$ ($i = 1, 2$) for all $0 \leq t \leq T$. By (4) and (5),

$$\begin{aligned} \frac{|q - q_0|}{|p - p_0|} &= \frac{|q_2(T) - q_1(T)|}{|p_2(T) - p_1(T)|} \\ &\leq \frac{(e^{C(T)\epsilon_0} \|\Phi_s(y, T)\| + C(T)\epsilon_0) |q_2 - q_1| + C(T)\epsilon_0 |p_2 - p_1|}{(e^{C(T)\epsilon_0} \|\Phi_s(y, T)\|^\beta - C(T)\epsilon_0) |p_2 - p_1| - C(T)\epsilon_0 |q_2 - q_1|} \\ &\leq \left(e^{C(T)\epsilon_0} \|\Phi_s(y, T)\|^{1-\beta} + C(T)\epsilon_0 \right) \rho \leq \rho. \end{aligned}$$

This proves (i).

We now prove (ii). Let $h_1, h_2 \in \Gamma_\rho$ and denote $H_i = \phi^*(h_i)$, for $i = 1, 2$. For any $p \in U_{\epsilon_0}$, we let $p_1, p_2 \in U_{\epsilon_0}$ be such that $\pi_{cu} \cdot \phi^T(p_i, h_i(p_i)) = p$ ($i = 1, 2$). By Proposition 1, $|p_2 - p_1| = O(\epsilon_0)$. Denote $y = Q(p_1)$. Then,

$$\begin{aligned} |H_2(p) - H_1(p)| &= |\pi_s \phi^T(p_2, h_2(p_2)) - \pi_s \phi^T(p_1, h_1(p_1))| \\ &\leq |\pi_s \phi^T(p_1, h_1(p_1)) - \pi_s \phi^T(p_1, h_2(p_1))| \\ (25) \quad &\quad + |\pi_s \phi^T(p_1, h_2(p_1)) - \pi_s \phi^T(p_2, h_2(p_2))|. \end{aligned}$$

By (4),

$$\begin{aligned} &|\pi_s \phi^T(p_1, h_1(p_1)) - \pi_s \phi^T(p_1, h_2(p_1))| \\ &\leq \left(e^{C(T)\epsilon_0} \|\Phi_s(y, T)\| + C(T)\epsilon_0 \right) |h_2(p_1) - h_1(p_1)| \\ (26) \quad &\leq \left(e^{C(T)\epsilon_0} \|\Phi_s(y, T)\| + C(T)\epsilon_0 \right) |h_2 - h_1|, \end{aligned}$$

and

$$\begin{aligned} &|\pi_s \phi^T(p_1, h_2(p_1)) - \pi_s \phi^T(p_2, h_2(p_2))| \\ &\leq \left(e^{C(T)\epsilon_0} \|\Phi_s(y, T)\| + C(T)\epsilon_0 \right) |h_2(p_1) - h_2(p_2)| + C(T)\epsilon_0 |p_2 - p_1| \\ (27) \quad &\leq \left(e^{C(T)\epsilon_0} \|\Phi_s(y, T)\| \rho + C(T)\epsilon_0 \right) |p_2 - p_1|. \end{aligned}$$

Since

$$\begin{aligned} &\pi_{cu} \phi^T(p_2, h_2(p_2)) - \pi_{cu} \phi^T(p_1, h_2(p_1)) + \pi_{cu} \phi^T(p_1, h_2(p_1)) - \pi_{cu} \phi^T(p_1, h_1(p_1)) \\ &= \pi_{cu} \phi^T(p_2, h_2(p_2)) - \pi_{cu} \phi^T(p_1, h_1(p_1)) = 0, \end{aligned}$$

we have

$$(28) \quad |\pi_{cu} \phi^T(p_2, h_2(p_2)) - \pi_{cu} \phi^T(p_1, h_2(p_1))| = |\pi_{cu} \phi^T(p_1, h_2(p_1)) - \pi_{cu} \phi^T(p_1, h_1(p_1))|.$$

By (5),

$$(29) \quad \begin{aligned} &|\pi_{cu} \phi^T(p_2, h_2(p_2)) - \pi_{cu} \phi^T(p_1, h_2(p_1))| \\ &\geq \left(e^{C(T)\epsilon_0} \|\Phi_s(y, T)\|^\beta - C(T)\epsilon_0 \right) |p_2 - p_1|. \end{aligned}$$

By (28), (29) and (5),

$$\begin{aligned} |p_2 - p_1| &\leq \frac{|\pi_{cu} \phi^T(p_2, h_2(p_2)) - \pi_{cu} \phi^T(p_1, h_2(p_1))|}{e^{C(T)\epsilon_0} \|\Phi_s(y, T)\|^\beta - C(T)\epsilon_0} \\ &= \frac{|\pi_{cu} \phi^T(p_1, h_2(p_1)) - \pi_{cu} \phi^T(p_1, h_1(p_1))|}{e^{C(T)\epsilon_0} \|\Phi_s(y, T)\|^\beta - C(T)\epsilon_0} \\ (30) \quad &\leq \frac{C(T)\epsilon_0 |h_2(p_1) - h_1(p_1)|}{e^{C(T)\epsilon_0} \|\Phi_s(y, T)\|^\beta - C(T)\epsilon_0}. \end{aligned}$$

Substituting (26), (27) and (30) into (25), we see that

$$\begin{aligned} |H_2(p) - H_1(p)| &\leq \left(e^{C(T)\epsilon_0} \|\Phi_s(y, T)\| + C(T)\epsilon_0 \right) \\ &\quad \cdot \left(1 + \frac{C(T)\epsilon_0}{e^{C(T)\epsilon_0} \|\Phi_s(y, T)\|^\beta - C(T)\epsilon_0} \right) |h_2 - h_1|. \end{aligned}$$

Let λ denote the coefficient of $|h_2 - h_1|$ in the above. By our choice of T and ϵ_0 , it is clear that $0 < \lambda < 1$. This proves (ii).

To prove (iii), we fix a small $\tau_0 > 0$ such that $\phi^\tau M_{cu}(Y)$ is a graph for $0 \leq \tau \leq \tau_0$. Since

$$\phi^T(\phi^\tau M_{cu}(Y)) = \phi^\tau(\phi^T M_{cu}(Y)) = \phi^\tau M_{cu}(Y),$$

by the uniqueness of the fixed point ϕ^* , $\phi^\tau M_{cu}(Y) = M_{cu}(Y)$ for $0 \leq \tau \leq \tau_0$. Now for any t , we write $t = [\frac{t}{\tau_0}] + \tau$, where $0 \leq \tau < \tau_0$. Then

$$\phi^t M_{cu}(Y) = \phi^{[\frac{t}{\tau_0}] + \tau} M_{cu}(Y) = \phi^\tau(\phi^{[\frac{t}{\tau_0}]} M_{cu}(Y)) = \phi^\tau M_{cu}(Y) = M_{cu}(Y),$$

that is, $M_{cu}(Y)$ is invariant to (3). \square

Hereafter, $M_{cu}(Y)$ will be referred to as a *center-unstable manifold* of Y .

5.2. Smoothness of center-unstable manifold. We now discuss the smoothness of $M_{cu}(Y)$ following the ideas of [18], [26], [47]. Namely, we first show that $M_{cu}(Y)$ is C^1 following the arguments of [18] and then use the C^r Section Theorem in [26], [47] to improve the smoothness inductively. For other approaches to prove the smoothness of an invariant manifold (for example, the application of Henry's Lemma and the use of a scale of Banach spaces), we refer the readers to [13], [41], [53], [51], [56].

Recall, for $p \in U_{\epsilon_0}$,

$$h(\sigma_h(p)) = \pi_s \cdot \phi^T \cdot (id, h)(p),$$

where $\sigma_h = \pi_{cu} \cdot \phi^T \cdot (id, h)$.

Let D_p denote the differential operator with respect to p . Formally, we obtain

$$D_p h(\sigma_h(p)) D_p \sigma_h(p) = D_z(\pi_s \phi^T)(p, h(p))(id, D_p h)(p),$$

or

$$D_p h(\sigma_h(p)) = D_z(\pi_s \phi^T)(p, h(p))(id, D_p h)(p) [D_p(\pi_{cu} \phi^T)(p, h(p))(id, D_p h)(p)]^{-1}.$$

Motivated by the above formula, we define $\mathcal{L}_h : T_p U_{\epsilon_0} \rightarrow T_{(\sigma_h(p), h(\sigma_h(p)))} S_{\epsilon_0}(\sigma_h(p))$ as

$$(\mathcal{L}_h l)(\sigma_h(p)) = D_z(\pi_s \phi^T)(p, h(p))(id, l(p)) [D_p(\pi_{cu} \phi^T)(p, h(p))(id, l(p))]^{-1}.$$

Let E be the vector bundle over U_{ϵ_0} for which the fiber at $p \in U_{\epsilon_0}$ is the space of bounded linear maps from $T_p U_{\epsilon_0}$ to $T_{(p, h(p))} S_{\epsilon_0}(p)$. Define the bundle map $F_h : E \rightarrow E$:

$$(p, l(p)) \rightarrow (\sigma_h(p), (\mathcal{L}_h l)(\sigma_h(p))).$$

Then the following diagram

$$\begin{array}{ccccc} & & F_h & & \\ & E & \rightarrow & E & \\ \Pi & \downarrow & & \downarrow & \Pi \\ & U_{\epsilon_0} & \xrightarrow{\sigma_h} & U_{\epsilon_0} & \end{array}$$

commutes. Denote L_0 as the zero section of E (that is, $L_0 : U_{\epsilon_0} \rightarrow E$ is such that $L_0(p) = 0$ for all $p \in U_{\epsilon_0}$) and let $L_{n+1}(p) = \mathcal{L}_h L_n(p)$, $n = 0, 1, \dots$. Define the norm of a section L by $|L| = \sup_{p \in U_{\epsilon_0}} |L(p)|$ where $|L(p)|$ is the operator norm of $L(p)$.

Lemma 7. $|L_n| \leq \frac{1}{5}$ for all $n \geq 0$.

Proof. Clearly, $|L_0| = 0 \leq \frac{1}{5}$. Assume $|L_n| \leq \frac{1}{5}$ for some $n > 0$. If T and ϵ_0 are as in Proposition 1, then

$$\begin{aligned} |L_{n+1}(\sigma_h(p))| &= |(\mathcal{L}_h L_n)(\sigma_h(p))| \\ &\leq |D_z(\pi_s \phi^T)(p, h(p))(id, L_n(p))| |D_p(\pi_{cu} \phi^T)(p, h(p))(id, L_n(p))|^{-1} \\ &\leq \frac{(1 + \frac{1}{5})(e^{C(T)\epsilon_0} \|\Phi_s(y, T)\| + C(T)\epsilon_0)}{(1 - \frac{1}{5})(e^{C(T)\epsilon_0} \|\Phi_s(y, T)\|^\beta - C(T)\epsilon_0)} \leq \frac{1}{5} \end{aligned}$$

for all $p \in U_{\epsilon_0}$. The proof is then complete by induction. \square

Lemma 8. $\{L_n\}_{n=0}^{+\infty}$ is a Cauchy sequence.

Proof. Let $p \in U_{\epsilon_0}$. For each $n = 1, 2, \dots$, we have

$$\begin{aligned} L_{n+1}(\sigma_h(p)) - L_n(\sigma_h(p)) &= (\mathcal{L}_h L_n)(p) - (\mathcal{L}_h L_{n-1})(p) \\ &= D_z(\pi_s \phi^T)(p, h(p))(id, L_n(p)) [D_p(\pi_{cu} \phi^T)(p, h(p))(id, L_n(p))]^{-1} \\ &\quad - D_z(\pi_s \phi^T)(p, h(p))(id, L_{n-1}(p)) [D_p(\pi_{cu} \phi^T)(p, h(p))(id, L_{n-1}(p))]^{-1} \\ &= D_z(\pi_s \phi^T)(p, h(p))(0, L_n(p) - L_{n-1}(p)) [D_p(\pi_{cu} \phi^T)(p, h(p))(id, L_n(p))]^{-1} \\ &\quad + D_z(\pi_s \phi^T)(p, h(p))(id, L_{n-1}(p)) ([D_p(\pi_{cu} \phi^T)(p, h(p))(id, L_n(p))]^{-1} \\ &\quad - [D_p(\pi_{cu} \phi^T)(p, h(p))(id, L_{n-1}(p))]^{-1}). \end{aligned}$$

Let $y = Q(p)$. Then

$$\begin{aligned} &|D_z(\pi_s \phi^T)(p, h(p))(0, L_n(p) - L_{n-1}(p)) [D_p(\pi_{cu} \phi^T)(p, h(p))(id, L_n(p))]^{-1}| \\ &\leq |D_z(\pi_s \phi^T)(p, h(p))| |L_n(p) - L_{n-1}(p)| |D_p(\pi_{cu} \phi^T)(p, h(p))(id, L_n(p))|^{-1} \\ &\leq (e^{C(T)\epsilon_0} \|\Phi_s(y, T)\| + C(T)\epsilon_0) (e^{C(T)\epsilon_0} \|\Phi_s(y, T)\|^\beta \\ &\quad - C(T)\epsilon_0)^{-1} (1 - \frac{1}{5})^{-1} |L_n(p) - L_{n-1}(p)| \\ &\leq \rho (1 - \frac{1}{5})^{-1} |L_n(p) - L_{n-1}(p)|, \end{aligned}$$

and

$$\begin{aligned} &|[D_p(\pi_{cu} \phi^T)(p, h(p))(id, L_n(p))]^{-1} - [D_p(\pi_{cu} \phi^T)(p, h(p))(id, L_{n-1}(p))]^{-1}| \\ &\leq |D_p(\pi_{cu} \phi^T)(p, h(p))|^{-1} |(id, L_n(p))^{-1} - (id, L_{n-1}(p))^{-1}| \\ &\leq |D_p(\pi_{cu} \phi^T)(p, h(p))|^{-1} |(id, L_n(p))|^{-1} |L_n(p) - L_{n-1}(p)| |(id, L_{n-1}(p))|^{-1} \\ &\leq (e^{C(T)\epsilon_0} \|\Phi_s(y, T)\|^\beta - C(T)\epsilon_0)^{-1} \frac{1}{5} (1 - \frac{1}{5})^{-2} |L_n(p) - L_{n-1}(p)|. \end{aligned}$$

Therefore,

$$|L_{n+1} - L_n| \leq \frac{5}{16} \rho |L_n - L_{n-1}| \leq \frac{5}{16} |L_n - L_{n-1}|,$$

and $\{L_n\}$ is a Cauchy sequence. \square

Lemma 9. h is C^1 and $D_p h = L^1$, where $L^1 = \lim_{n \rightarrow \infty} L_n$ and $p \in U_{\epsilon_0}$.

Proof. We need to show that

$$|h(p') - h(p) - L^1(p)v(p', p)| = o(|p' - p|),$$

as $p' \rightarrow p$ in U_{ϵ_0} , where $\exp_p v(p', p) = p'$. We note that $p' - p$ is not necessarily in $T_p U_{\epsilon_0}$, but $p' - p - v(p', p) = O(|p' - p|^2)$ (see [50]).

Following [18], define an increasing function $\gamma : (0, 1) \rightarrow \mathbb{R}$ by

$$\gamma(a) = \sup_{0 < |p' - p| < a} \frac{|h(p') - h(p) - L^1(p)v(p', p)|}{|p' - p|}.$$

Then it suffices to show that $\gamma(a) \rightarrow 0$ as $a \rightarrow 0$.

Note that

$$\begin{aligned} h(\sigma_h(p')) - h(\sigma_h(p)) &= \pi_s \phi^T(p', h(p')) - \pi_s \phi^T(p, h(p)) \\ &= D_z(\pi_s \phi^T)(p, h(p))(p' - p, h(p') - h(p)) \\ &\quad + O(|p' - p|^2), \end{aligned}$$

and

$$\begin{aligned} \sigma_h(p') - \sigma_h(p) &= \pi_{cu} \phi^T(p', h(p')) - \pi_{cu} \phi^T(p, h(p)) \\ &= D_p(\pi_{cu} \phi^T)(p, h(p))(p' - p, h(p') - h(p)) \\ &\quad + O(|p' - p|^2). \end{aligned}$$

Since $L^1(\sigma_h(p)) = D_z(\pi_s \phi^T)(p, h(p))(id, L^1(p))[D_p(\pi_{cu} \phi^T)(p, h(p))(id, L^1(p))]^{-1}$, we have $L^1(\sigma_h(p))v(\sigma_h(p'), \sigma_h(p)) = D_z(\pi_s \phi^T)(p, h(p))(id, L^1(p))v(p', p)$ and

$$\begin{aligned} &|h(\sigma_h(p')) - h(\sigma_h(p)) - L^1(\sigma_h(p))v(\sigma_h(p'), \sigma_h(p))| \\ &= |D_z(\pi_s \phi^T)(p, h(p))(p' - p - v(p', p), h(p') - h(p) - L^1(p)v(p', p))| \\ &\quad + O(|p' - p|^2) \\ &\leq \left(e^{C(T)\epsilon_0} \|\Phi_s(y, T)\| + C(T)\epsilon_0 \right) |h(p') - h(p) - L^1(p)v(p', p)| \\ &\quad + O(|p' - p|^2). \end{aligned}$$

But by Proposition 1,

$$\text{Lip}_p(\sigma_h) \geq e^{C(T)\epsilon_0} \|\Phi_s(y, T)\|^\beta - C(T)\epsilon_0.$$

Therefore,

$$\begin{aligned} &\frac{|h(\sigma_h(p')) - h(\sigma_h(p)) - L^1(\sigma_h(p))v(\sigma_h(p'), \sigma_h(p))|}{|\sigma_h(p') - \sigma_h(p)|} \\ &\leq \left(e^{C(T)\epsilon_0} \|\Phi_s(y, T)\| + C(T)\epsilon_0 \right) \frac{|h(p') - h(p) - L^1(p)v(p', p)| + O(|p' - p|^2)}{|\sigma_h(p') - \sigma_h(p)|} \\ &\leq \frac{e^{C(T)\epsilon_0} \|\Phi_s(y, T)\| + C(T)\epsilon_0}{e^{C(T)\epsilon_0} \|\Phi_s(y, T)\|^\beta - C(T)\epsilon_0} \frac{|h(p') - h(p) - L^1(p)v(p', p)|}{|p' - p|} + O(|p' - p|) \\ &\leq \rho \frac{|h(p') - h(p) - L^1(p)v(p', p)|}{|p' - p|} + O(|p' - p|). \end{aligned}$$

That is, there is a $\delta > 0$ and $\beta(a) = O(a)$ such that

$$(31) \quad \gamma(a) \leq \rho\gamma(\delta a) + \beta(a).$$

Now, if $\delta \leq 1$, then $\gamma(a) \leq (1 - \rho)^{-1}\beta(a) \rightarrow 0$ as $a \rightarrow 0$. If $\delta > 1$, by (30), then

$$\begin{aligned} \gamma(a\delta^{-m}) &\leq \rho^m \gamma(a) + \beta(a\delta^{-m}) + \dots + \rho^{m-1} \beta(a\delta^{-1}) \\ &\leq \rho^m \gamma(a) + (1 - \rho)^{-1} \beta(a\delta^{-1}). \end{aligned}$$

It follows that $\gamma(a) \rightarrow 0$ as $a \rightarrow 0$. This completes the proof. \square

The following lemma is an immediate consequence of the general C^r **Section Theorem** (see [26], [47]).

Lemma 10. *Let B be the disc bundle of radius δ in E . Suppose $\mathcal{L}_h : B_p \rightarrow B_{\sigma_h(p)}$ is Lipschitz with constant k , σ_h and F are C^s , $D^j F$ and $D^j \sigma_h^{-1}$ are bounded for $1 \leq j \leq s$. If $k\mu^s < 1$, where $\mu = \text{Lip}(\sigma_h^{-1})$, then L^1 is C^s , hence h is C^{s+1} .*

Proposition 3. *There exists ϵ_0 such that, if $h \in \Gamma_\rho$ is the fixed point of ϕ^* in Proposition 2. Then h is C^d , where d is the integer defined in Theorem 1.*

Proof. By Lemma 9, h is C^1 . Now assume h is C^s , for $s \leq d-1$. Since $d \leq r$, σ_h and F are C^s . Let T and $\epsilon_0(T)$ be as in Proposition 1, and let $\epsilon_0 \leq \epsilon_0(T)$, we then have

$$k \leq \sup_{y \in Y} \left\{ \frac{e^{C(T)\epsilon_0} \|\Phi_s(y, T)\| + C(T)\epsilon_0}{e^{C(T)\epsilon_0} \|\Phi_s(y, T)\|^\beta - C(T)\epsilon_0} \right\},$$

and

$$\mu \leq \sup_{y \in Y} (e^{C(T)\epsilon_0} \|\Phi_s(y, T)\|^\beta - C(T)\epsilon_0)^{-1}.$$

Hence

$$k\mu^s \leq \frac{e^{C(T)\epsilon_0} \|\Phi_s(y, T)\| + C(T)\epsilon_0}{(e^{C(T)\epsilon_0} \|\Phi_s(y, T)\|^\beta - C(T)\epsilon_0)^{s+1}}.$$

The right hand side above limits to $\|\Phi(y, T)\|^{1-\beta(s+1)}$ as $\epsilon_0 \rightarrow 0$, which is less than 1 because $s+1 \leq d < \frac{1}{\beta}$. So, if we choose ϵ_0 small (smaller than $\epsilon_0(T)$ in general), then $k\mu^s < 1$. It follows from Lemma 10 that h is C^{s+1} . By induction, h is C^d . \square

Remark 4. It is known that, to improve the smoothness, one has to restrict the neighborhood smaller. This is why, in Proposition 3, we need to choose ϵ_0 much smaller than $\epsilon_0(T)$ while the latter is only enough for the existence.

5.3. Proof of Theorem 1.

Proof. Proposition 3 implies that there exists C^d center-unstable manifold $M_{cu}(Y)$ of the modified flow (3). Since, in a small neighborhood of Y , \hat{f} agrees with f , $M_{cu}(Y)$ is also a locally invariant manifold of (1) and $T_y M_{cu}(Y) = V_{cu}(y)$ for $y \in Y$. By reversing the time, we similarly obtain a locally invariant C^d manifold $M_{cs}(Y)$ of (1) in a neighborhood of Y with $T_y M_{cs}(Y) = V_{cs}(y)$ for $y \in Y$. The intersection $M_c(Y) = M_{cu}(Y) \cap M_{cs}(Y)$ therefore gives a locally invariant C^d manifold of (1) with $T_y M_c(Y) = V_c(y)$ for $y \in Y$, which is a desired center manifold.

If $g \in C^r$ is such that $|g - f|_{C^r}$ is small, then so is $|\hat{g} - \hat{f}|_{C^r}$, where \hat{g} is the modification of g in the same fashion as that of f . By arguments of [18] and [6], there exists a locally invariant C^d manifold $M_{cu}(\hat{g})$ for \hat{g} which is C^d close to $M_{cu}(Y)$. Moreover, $M_{cu}(\hat{g})$ is clearly a locally invariant manifold for g when restricting to a small neighborhood of Y . Similarly, there exists a locally invariant C^d manifold $M_{cs}(g)$ for g which is C^d close to $M_{cs}(Y)$. The intersection $M_c(g) = M_{cu}(g) \cap M_{cs}(g)$ is then a C^d invariant manifold for g which is C^d close to $M_c(Y)$. This shows that $M_c(Y)$ is C^d persistent.

Clearly, $M_{cu}(Y)$ carries all the solutions locally bounded in negative time, and $M_{cs}(Y)$ carries all the solutions locally bounded in positive time. We then conclude that $M_c(Y)$ carries all solutions which are locally bounded in \mathbb{R} . The proof of Theorem 1 is now complete. \square

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