# ALMOST AUTOMORPHIC AND ALMOST PERIODIC DYNAMICS IN SKEW-PRODUCT SEMIFLOWS

Dedicated to Professor R. Ellis on the Occasion of His 70th Birthday

**Part I. Almost Automorphy and Almost Periodicity**

by

Yingfei Yi

**Part II. Skew-product Semiflows**

by

Wenxian Shen and Yingfei Yi

**Part III. Applications to Differential Equations**

by

Wenxian Shen and Yingfei Yi

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## ABSTRACT

The current series of papers, which consists of three parts, are devoted to the study of almost automorphic dynamics in differential equations. By making use of techniques from abstract topological dynamics, we show that almost automorphy, a notion which was introduced by S. Bochner in 1955, is essential and fundamental in the qualitative study of almost periodic differential equations.

Fundamental notions from topological dynamics are introduced in the first part. Harmonic properties of almost automorphic functions such as Fourier series and frequency module are studied. A module containment result is provided.

In the second part, we study lifting dynamics of $\omega$-limit sets and minimal sets of a skew-product semiflow from an almost periodic minimal base flow. Skew-product semiflows with (strongly) order preserving or monotone natures on fibers are given a particular attention. It is proved that a linearly stable minimal set must be almost automorphic and become almost periodic if it is also uniformly stable. Other issues such as flow extensions and the existence of almost periodic global attractors, etc. are also studied.

The third part of the series deals with dynamics of almost periodic differential equations. In this part, we apply the general theory developed in the previous two parts to study almost automorphic and almost periodic dynamics which are lifted from certain coefficient structures (e.g., almost automorphic or almost periodic) of differential equations. It is shown that (harmonic or subharmonic) almost automorphic solutions exist for a large class of almost periodic ordinary, parabolic and delay differential equations.

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**Part I. Almost Automorphy and Almost Periodicity**

Yingfei Yi

# PART I
# ALMOST AUTOMORPHY AND ALMOST PERIODICITY

## 1. Introduction

In this series of papers, we study almost automorphic and almost periodic dynamics in skew-product semiflows which arise in the qualitative study of nonautonomous ordinary, parabolic and functional differential equations. We shall show in the current work that almost automorphy is a fundamental notion in the dynamical study of almost periodic differential equations.

The theory of almost periodicity, since founded in the 1920's, has given a strong impetus to the development of harmonic analysis on groups and to the development of both topological and smooth dynamical systems. Due to the time variation in most physical systems, following the pioneer work of Favard ([16], [17]), a vast amount of research has been directed toward the study of almost periodic differential equations in the past fifty years or so (see [18], [29], [34], [38], [58] for surveys). Recently, motivated by applications, important extensions have been given to the study of almost periodic partial differential equations (see [1], [34], [38] and references therein).

The notion of almost automorphy, as a generalization to almost periodicity, was first introduced by S. Bochner in 1955 in a work of differential geometry ([5]). Fundamental properties of almost automorphic functions on groups and abstract almost automorphic minimal flows were studied by W. A. Veech ([50], [51], [54]) and others (see [20], [22], [41], [48], [49]). Almost automorphic phenomena, indicating somewhat complexity and chaos were found in symbolic dynamics. For example, almost automorphic symbolic minimal flows may admit positive topological entropy in both ergodic [30] and non-ergodic [22] cases. It was shown in [36] that the later case is generic among certain dynamical systems of two symbols.

The study of almost automorphic dynamics in differential equations has been less emphasized, perhaps because the importance of the notion of almost automorphy in differential equations was not clear. An open question had been the existence of an almost periodic differential equation with an almost automorphic solution which was not almost periodic. During the early 80's, several

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examples of almost periodic scalar ODE's were constructed by R. A. Johnson, in which the associated skew-product flows admit non-almost periodic almost automorphic, ergodic or non-ergodic minimal sets (see Part III for details). Almost automorphic dynamics for linear scalar ODE's with almost periodic coefficients were studied in [27], [28] (see also [60] for the case of infinite dimensional linear almost periodic equations). A significance of almost automorphy has been indicated in Johnson's work on almost periodic Floquet theory of two dimensional linear system of ODE's ([26]), in which an almost automorphic strong Perron transformation has to be introduced to transform the original system into a canonical form. Recently, in a series of work of Shen and Yi ([44]-[47]), almost periodic scalar parabolic equations in one space dimension (which particularly include almost periodic scalar ODE's) were systemically investigated. It was shown that all minimal sets in the associated skew-product semiflows are almost automorphic. Other issues such as properties of $\omega$-limit sets, asymptotic behavior of bounded solutions, hyperbolicity and stability, and ergodicity of a minimal set were also studied in [44]-[47] (see also Part III of the current series). The notion of almost automorphy was shown to be essential in these works. First, as far as lifting properties from the coefficient space to the solution space are concerned, the dynamics is generally not closed within the category of almost periodicity but is closed within that of almost automorphy. It turns out that almost automorphic solutions are the right class for almost periodic systems. Second, the appearance of almost automorphic dynamics indicates a major difference between a periodic system and an almost periodic one. For example, in monotone dynamical systems, a 'lifting' of from periodic coefficients can never be almost automorphic. In terms of long time behavior of a bounded solution, almost automorphism of its $\omega$-limit set often reflects a kind of 'non-uniform' asymptotic phenomena. In addition, non-unique ergodicity of a minimal set may imply certain complicated or even chaotic dynamics of the original system.

Another significance of studying almost automorphic dynamics is its connection with the Levitan $N$-almost periodicity. Since an almost automorphic function is essentially $N$-almost periodic as shown in [20],[41], the current study of almost automorphic dynamics ties up closely with the study of $N$-almost periodic ones (see [33], [34]).

In this series of work, we shall extend our previous investigations by showing the existence of almost automorphic dynamics in a large class of almost periodic ordinary, parabolic, and functional differential equations, which further address the importance of almost automorphy in the qualitative study of differential equations. Since almost periodic functions are in particular almost automorphic, though it is not our main concern, the current work also contains some new results on the existence of almost periodic dynamics.

The current part can be viewed as a preliminary for the whole series. In Section 2, we review the concept of Ellis semigroup introduced by R. Ellis ([12]), along with some fundamental notions in the algebraic theory of topological dynamics such as distal, proximal, (uniform) almost periodicity and almost automorphy. In particular, the following properties of minimal sets studied in this section will play important roles in later parts of the series:

1) If a minimal flow $(X, \mathbb{R})$ is one sided distal, then it is two-sided distal;
2) If the proximal relation $P(X)$ on a minimal flow $(X, \mathbb{R})$ is an equivalence relation, then any proximal pair in $X$ is two-sided proximal.

As remarked in [14], abstract topological dynamics usually plays less of a role in the qualitative study of autonomous differential equations, because not only is the differential structure ignored but the topological properties of the reals are not made essential used of. This is however not the case for nonautonomous equations. As seen in our current study, certain differential structures (e.g., linear stability, hyperbolicity, monotonicity) when coupled with topological structures of the coefficient space (e.g., almost automorphy, almost periodicity) often give rise to an essential issue at the the level of topological dynamics.

We study in Section 3 harmonic properties of almost automorphic functions which resemble those of almost periodic ones (see [7], [18], [34], [38]). By introducing a universal object for almost automorphic minimal flows, we define Fourier series, frequency module for an almost automorphic function based on the original work of Veech ([51],[52],[54]) on abstract almost automorphic or minimal functions. For an almost automorphic function $f$, by 'restricting' its Fourier series (not unique in general) on its compact hull $H(f)$, we show that its frequency module $\mathcal{M}(f)$ is isomorphic to the character group $Y_f'$ of the maximal almost periodic factor $Y_f$ of $H(f)$. As a consequence, we generalize a classical module containment result of Favard ([17]) for almost periodic functions. Roughly speaking, if $f$ and $g$ are two almost automorphic functions, then $\mathcal{M}(g) \subset \mathcal{M}(f)$ if and only if $f$ is 'returning' by a sequence implies that $g$ is also 'returning' by the same sequence.

Due to our applications to differential equations, we consider only real flows and functions of real variables. However, most of our results also hold for general transformation groups and for abstract functions defined on a locally compact abelian group. For more complete abstract theory of topological dynamics, we refer the readers to [3], [14], [55] and references therein.

## 2. Topological Dynamics

### 2.1. Minimal Flows and the Ellis Semigroup.

One of the objectives of topological dynamics is to study 'long term' behavior of actions of a topological group on a topological space. The natural formulation in this context is that of a transformation group or a flow.

Definition 2.1.

1) Let $X$ be a $T_2$ space, called the phase space. A (real) *flow* $(X, \mathbb{R})$ is a continuous mapping $\Pi : X \times \mathbb{R} \to X$, where $\mathbb{R}$ is the additive group of reals, which

satisfies the following properties:

i) $\Pi(x,0) = x$ $(x \in X)$;

ii) $\Pi(\Pi(x,s),t) = \Pi(x,s+t)$ $(x \in X, s,t \in \mathbb{R})$.

2) A flow $(X,\mathbb{R})$ is a *compact flow* if the phase space is compact, is a *point flow* if there is a $x_0 \in X$ with dense orbit $\{\Pi(x_0,t)|t \in \mathbb{R}\}$ (which will also be denoted by $(X,x_0,\mathbb{R})$).

For convenience, we sometimes denote $\Pi(x,t)$ by $\Pi_t(x)$ or simply by $x \cdot t$.

Definition 2.2. Let $(X,\mathbb{R})$ be a flow.

1) A subset $M \subset X$ is said to be *invariant* if for each $x \in M$, its orbit $\{\Pi(x,t)|t \in \mathbb{R}\}$ lies in $M$.
2) A non-empty compact invariant set $M \subset X$ is *minimal* if it contains no non-empty, proper, closed invariant subset. $(X,\mathbb{R})$ is *minimal* if $X$ itself is a minimal set.
3) Let $x_0 \in X$ be such that $\{\Pi(x_0,t)|t \geq t_0\}$ or $\{\Pi(x_0,t)|t \leq -t_0\}$ is relatively compact for a $t_0 \geq 0$. The following set

$$\omega(x_0) = \bigcap_{\tau \geq t_0} cl\{\Pi(x_0,t+\tau)|t \geq 0\}, \tag{2.1}$$

or

$$\alpha(x_0) = \bigcap_{\tau \leq -t_0} cl\{\Pi(x_0,t+\tau)|t \leq 0\} \tag{2.2}$$

is called the *$\omega$-limit set* or the *$\alpha$-limit set* of $x_0$ respectively.

It is well known that both $\omega$-limit sets and $\alpha$-limit sets in a flow are compact invariant, and, a flow is minimal if and only if each orbit is dense. Moreover, as a consequence of the Zorn's lemma, a compact flow always contains a minimal set.

An algebraic way to study the nature of a compact flow was introduced by R. Ellis ([12], [14]). One basic idea of the algebraic theory is to associate a semigroup, the so called *Ellis semigroup* or *enveloping semigroup*, to a compact flow. The notion of Ellis semigroup allows one to study the dynamics of a compact flow by looking into the algebraic property associated to it.

Let $(X,\mathbb{R})$ be a compact flow. The space $X^X$ of self maps of $X$, when furnished with the point open topology, is a compact $T_2$ space by the Tychonoff theorem, and, composition of maps provides a natural semigroup structure on $X^X$. For each $t \in \mathbb{R}$, we note that $\Pi_t : X \to X$, $x \mapsto x \cdot t$ defines a homeomorphism, hence an element of $X^X$.

Definition 2.3. The *Ellis semigroup* $E(X)$ associated to a compact flow $(X,\mathbb{R})$ is the closure of $\{\Pi_t|t \in \mathbb{R}\}$ in $X^X$.

Clearly, $E(X)$ is a sub-semigroup of $X^X$ with identity $e = \Pi_0$, and the composition $\Pi_t\gamma \equiv \gamma \cdot t$ $(\gamma \in E(X), t \in \mathbb{R})$ defines a compact point flow $(E(X),e,\mathbb{R})$.

Definition 2.4.
1) A (*left*) *ideal* in $E(X)$ is a non-empty subset $I$ in $E(X)$ with $E(X)I \subset I$. A (left) ideal $I$ in $E(X)$ is said to be *minimal* if it contains no non-empty proper (left) subideal in $E(X)$.
2) An *idempotent point* $u \in E(X)$ is such that $u^2 = u$.

It is observed in [14] that $I$ is an (left) (minimal) ideal in $E(X)$ if and only if $I$ is an invariant (minimal) subset of the compact flow $(E(X), \mathbb{R})$. It follows that a minimal (left) ideal in $E(X)$ always exists. The structure of a minimal (left) ideal is as follows.

Theorem 2.1. (Ellis [14]) *Let $I$ be a minimal* (*left*) *ideal in $E(X)$ and $J(I)$ be the set of idempotent points of $E(X)$ in $I$. Then the following holds:*

1) $J(I) \neq \emptyset$;
2) *For each $u \in J(I)$, $uI$ is a group with identity $u$ and the family $\{uI\}_{u\in J(I)}$ forms a partition of $I$.*

Minimal flows can be characterized by using idempotent points of $E(X)$.

Theorem 2.2. (Ellis [14]) *A compact point flow $(X, x_0, \mathbb{R})$ is minimal if and only if there is an idempotent point $u \in E(X)$ with $ux_0 = x_0$.*

One of the important subjects in topological dynamics is to study dynamical relations between different flows. This leads to the following definition.

Definition 2.5. Consider flows $(X, \mathbb{R})$, $(Y, \mathbb{R})$. A *flow homomorphism* $\phi$ : $(X, \mathbb{R}) \to (Y, \mathbb{R})$ is a continuous map $\phi : X \to Y$ which preserves flows, that is, $\phi(x \cdot t) = \phi(x) \cdot t$ $(x \in X, t \in \mathbb{R})$. An onto flow homomorphism is called a *flow epimorphism* and an one to one flow epimorphism is referred to as a *flow isomorphism*. If $\phi$ is an epimorphism, then $(Y, \mathbb{R})$ is called a *factor* of $(X, \mathbb{R})$, $(X, \mathbb{R})$ is called an *extension* of $(Y, \mathbb{R})$.

The above concepts play an important role in the abstract study of topological dynamics. For example, the universal treatment of minimal flows (see Ellis [14]) and various structure theorems for flows (see [3], [15], [21], [53]). In the local study of topological dynamics, often, two flows $(X, \mathbb{R})$, $(Y, \mathbb{R})$ are fixed and a flow homomorphism $\phi : (X, \mathbb{R}) \to (Y, \mathbb{R})$ is given. One is then interested in dynamics of $(X, \mathbb{R})$ which are 'lifted' from $(Y, \mathbb{R})$. We note by minimality that a homomorphism of minimal flows is already an epimorphism.

### 2.2. Proximal and Distal.

Definition 2.6. Let $(X, \mathbb{R})$ be a compact flow.

1) Points $x_1, x_2 \in X$ are said to be (*positively, negatively*) *distal* if there is a pseudo-metric $d$ on $X$ such that

$$\inf_{t\in\mathbb{R}(t\in\mathbb{R}^+, t\in\mathbb{R}^-)} d(x_1 \cdot t, x_2 \cdot t) > 0. \tag{2.3}$$

$x_1$, $x_2$ are said to be (*positively, negatively*) *proximal* if they are not (positively, negatively) distal.

2) $x \in X$ is a *distal point* if it is only proximal to itself. $(X, \mathbb{R})$ is called a *point distal flow* if there is a distal point in $X$ with dense orbit. $(X, \mathbb{R})$ is a *distal flow* if every point in $X$ is a distal point.

*Remark 2.1.* We note that $x_1, x_2 \in X$ are (positively, negatively) distal if and only if

$$cl\{(x_1 \cdot t, x_2 \cdot t) | t \in \mathbb{R}(t \in \mathbb{R}^+, t \in \mathbb{R}^-)\} \cap \Delta = \emptyset, \tag{2.4}$$

where $\Delta$ is the diagonal of $X \times X$.

We now state some properties concerning with point distal and distal flows.

THEOREM 2.3. *Let $(X, \mathbb{R})$ be a compact point flow.*

1) (Veech [53]) *$x_0 \in X$ is a distal point if and only if $ux_0 = x_0$ for every idempotent point $u$ in $E(X)$.*
2) (Ellis [13]) *If $(X, \mathbb{R})$ is point distal with metric phase space $X$, then the set of distal points in $X$ is residual.*

*Remark 2.2.* By Theorems 2.2 and 2.3 1), we see that a point distal flow is necessarily minimal.

THEOREM 2.4. (Ellis [14]) *A compact flow $(X, \mathbb{R})$ is distal if and only if $E(X)$ is a group.*

*Remark 2.3.* If $E(X)$ is a group, by Theorem 2.1, then $E(X)$ is the only minimal left ideal in $E(X)$, that is, $(E(X), \mathbb{R})$ is minimal.

By the definition of distality, if $(X, \mathbb{R})$ is distal, then it must be both positively and negatively distal. One also has the following.

COROLLARY 2.5. (Sacker-Sell [42]) *If a compact flow $(X, \mathbb{R})$ is either positively or negatively distal, then it is distal.*

*Proof.* We only prove the case when $(X, \mathbb{R})$ is negatively distal. Let $e$ be the identity in $E(X)$ and denote $E_-(X)$ as the $\alpha$-limit set of $e$ in $(E(X), \mathbb{R})$. Since $E_-(X)$ is compact invariant, it contains a minimal set $I$, or equivalently, a minimal (left) ideal $I$ in $E(X)$. Let $u$ be an idempotent point in $I$. Then for any $x \in X$, $x^* = ux$ satisfies $(x^*, x^*) = (ux, ux^*) = u(x, x^*) \in cl\{(x \cdot t, x^* \cdot t) | t \leq 0\}$, that is, $x, x^*$ are negatively proximal. Thus $x = x^* = ux$. Since $x$ is arbitrary, $u = e$, that is, $I = eI$ is a group (Theorem 2.1). Now, $E(X) = E(X)e \subset E(X)I \subset I$. Hence $E(X) \equiv I$ is a group. By Theorem 2.4, $(X, \mathbb{R})$ is distal. □

THEOREM 2.6. (Auslander [3], Furstenberg [21]) *Let* $p : (X, \mathbb{R}) \to (Y, \mathbb{R})$ *be a homomorphism of distal minimal flows. Then* $p : X \to Y$ *is an open map.*

DEFINITION 2.7. Let $(X, \mathbb{R})$ be a compact flow. The (*positive, negative*) *proximal relation* $P(X)$ ($P_+(X)$, $P_-(X)$) is a subset of $X \times X$ defined as follows:

$$P(X)(P_+(X), P_-(X)) = \{(x_1, x_2) \in X \times X | x_1, x_2 \text{ are (positively, negatively) proximal}\}. \tag{2.5}$$

$P(X)$ is clearly invariant, reflexive and symmetric but not transitive in general.

THEOREM 2.7. (Ellis [14]) *Let* $(X, \mathbb{R})$ *be a compact flow. Then the following holds.*

1) $(x_1, x_2) \in P(X)$ *if and only if there exists a minimal* (*left*) *ideal* $I$ *in* $E(X)$ *such that* $\gamma x_1 = \gamma x_2$ $(\gamma \in I)$.
2) $P(X)$ *is an equivalence relation if and only if there is only one minimal* (*left*) *ideal in* $E(X)$.
3) *If* $P(X)$ *is closed, then it is an equivalence relation.*

COROLLARY 2.8. *Let* $(X, \mathbb{R})$ *be a compact flow. Suppose that* $P(X)$ *is an equivalence relation. Then* $P(X) = P_+(X) = P_-(X)$.

*Proof.* Clearly $P_\pm(X) \subset P(X)$. By Theorem 2.7 2), there is a unique minimal (left) ideal $I$ in $E(X)$, that is, $I$ is the unique minimal set of $(E(X), \mathbb{R})$. It follows that $I \subset \alpha(e) \cap \omega(e)$, here $e$ denotes the identity of $E(X)$. Now let $(x_1, x_2) \in P(X)$. Since by Theorem 2.7 1), $\gamma x_1 = \gamma x_2$ $(\gamma \in I)$, we have $(x_1, x_2) \in P_+(X) \cap P_-(X)$. Thus $P(X) \subset P_+(X) \cap P_-(X)$, that is, $P(X) = P_+(X) = P_-(X)$. □

### 2.3. Almost Periodicity.

DEFINITION 2.8. A compact flow $(X, \mathbb{R})$ is *almost periodic* if for any $\epsilon > 0$ and any pseudo-metric $d$ on $X$, there is a syndetic subset $A \subset \mathbb{R}$ (that is $\mathbb{R} = A + K$ for some compact subset $K$) such that $d(x \cdot t, x) < \epsilon$ $(x \in X, t \in A)$.

THEOREM 2.9. (Ellis [14]) *The following are equivalent.*

1) $(X, \mathbb{R})$ *is almost periodic;*
2) $(X, \mathbb{R})$ *is* equicontinuous*, that is,* $\{\Pi_t | t \in \mathbb{R}\} \subset X^X$ *forms an equicontinuous family;*
3) $E(X)$ *is a group of continuous maps in* $X^X$.

By Theorems 2.4 and 3) above, we see that an almost periodic flow is necessarily distal.

Corollary 2.10.

1) *A compact flow* $(X, \mathbb{R})$ *is almost periodic if and only if* $E(X)$ *is an abelian topological group.*
2) (Ellis [14]) *If* $(X, \mathbb{R})$ *is an almost periodic minimal flow, then for any* $x_0 \in X$, $\theta_{x_0} : (E(X), \mathbb{R}) \to (X, \mathbb{R})$, $\theta_{x_0}(\gamma) = \gamma x_0$ *is a flow isomorphism* (*that is,* $X$ *is essentially a compact abelian topological group with identity* $x_0$).

*Proof.* 1) If $(X, \mathbb{R})$ is almost periodic, then $E(X)$ is a group which consists of continuous maps of $X$. It follows that the group action on $E(X)$ is separately continuous. By [11], $E(X)$ is a topological group. Let $\gamma, \eta \in E(X)$ and let $\{t_\alpha\}$ be a net in $\mathbb{R}$ such that $\Pi_{t_\alpha} \to \eta$ in $E(X)$. Since $\Pi_{t_\alpha}\gamma = \gamma\Pi_{t_\alpha}$ for all $\alpha$, by continuity of $\gamma$, $\eta\gamma = \gamma\eta$. Thus $E(X)$ is abelian.

Conversly, suppose that $E(X)$ is an abelian topological group. By Theorem 2.4, $(X, \mathbb{R})$ is distal. Let $\gamma \in E(X)$, $x \in X$ and $\{x_\alpha : \alpha \in \Lambda\}$ be a net in $X$ such that $x_\alpha \to x$. Since $X$ laminates into minimal sets ([14]), without loss of generality, we assume that there is a net $\{\gamma_\alpha : \alpha \in \Lambda\}$ in $E(X)$ with $\gamma_\alpha x = x_\alpha$ and $\{\gamma_\alpha\}$ converges, say to $\gamma_0$ in $E(X)$. Thus $\gamma_0 x = x$. It follows that $\gamma x_\alpha = \gamma\gamma_\alpha x = \gamma_\alpha\gamma x \to \gamma_0\gamma x = \gamma\gamma_0 x = \gamma x$, that is, $\gamma$ is continuous. By Theorem 2.9 3), $(X, \mathbb{R})$ is almost periodic.

2) The map $\theta_{x_0}$ is clearly a flow epimorphism. Let $H = \{\gamma | \theta_{x_0}\gamma = x_0\}$. Then $H$ is a closed subgroup of the compact abelian topological group $E(X)$, and $\theta_{x_0}$ induces an isomorphism of $(E(X)/H, \mathbb{R})$ to $(X, \mathbb{R})$. Take $\gamma \in H$, that is, $\gamma x_0 = x_0$. Since $\Pi_t \gamma x_0 = \gamma(x_0 \cdot t) = \Pi_t x_0 = x_0 \cdot t$ ($t \in \mathbb{R}$), by the continuity of $\gamma$ (Theorem 2.9) and the minimality of $X$, one has $\gamma x = x$ ($x \in X$), that is, $\gamma = e$, the identity of $E(X)$. Hence $H = \{e\}$. $\square$

Definition 2.9. Let $(X, \mathbb{R})$ be a compact flow. For $x \in X$ and a net $\alpha = \{t_n\}$ in $\mathbb{R}$, the *generalized translation* is defined as $T_\alpha x = \lim_n x \cdot t_n$, provided that the limit exists.

We now give a version of Bochner's theorem ([6]) in the case of flows.

Corollary 2.11. *Let* $(X, \mathbb{R})$ *be a compact flow. The following are equivalent.*

1) $(X, \mathbb{R})$ *is almost periodic;*
2) *Any nets* $\alpha'$, $\beta'$ *in* $\mathbb{R}$ *have subnets* $\alpha$, $\beta$ *such that* $T_\alpha T_\beta x$, $T_{\alpha+\beta} x$ *exist and*

$$T_\alpha T_\beta x = T_{\alpha+\beta} x \tag{2.6}$$

*for all* $x \in X$, *where* $\alpha + \beta = \{t_n + s_n\}$ *if* $\alpha = \{t_n\}$, $\beta = \{s_n\}$.

*Proof.* 1) $\Longrightarrow$ 2). By Corollary 2.10, $E(X)$ is an abelian topological group. Let $\alpha'$, $\beta'$ be nets in $\mathbb{R}$. Then there are subnets $\alpha = \{t_n\}$, $\beta = \{s_n\}$ such that

$\{\Pi_{t_n}\}$, $\{\Pi_{s_n}\}$ converge in $E(X)$, say to $\gamma$ and $\eta$ respectively. It follows from the joint continuity that $\Pi_{t_n+s_n} \to \gamma\eta$ in $E(X)$. Thus (2.6) holds for all $x \in X$.

2) $\Longrightarrow$ 1). Let $\gamma \in E(X)$ and $\alpha = \{t_n\}$ be a net in $\mathbb{R}$ such that $\Pi_{t_n} \to \gamma$ in $E(X)$. By taking a subnet, we assume that $\Pi_{-t_n} \to \eta$ in $E(X)$. Denote $\beta = \{-t_n\}$. By (2.6), one has $\gamma\eta x = x$ $(x \in X)$, similarly, $\eta\gamma x = x$ $(x \in X)$, that is, $\gamma$ has an inverse in $E(X)$. Thus $E(X)$ is a group. By (2.6) again, $E(X)$ is in fact an abelian group. It follows that the multiplication on $E(X)$ is separately continuous, that is, $E(X)$ is a topological group ([11]). By Corollary 2.10, $(X, \mathbb{R})$ is almost periodic. $\square$

We end this section with a lifting property of compact almost periodic flows. Consider an epimorphism of compact flows $p : (X, \mathbb{R}) \to (Y, \mathbb{R})$. It is clear that if $(X, \mathbb{R})$ is almost periodic, then so is $(Y, \mathbb{R})$. The converse is obviously not true in general. However, we have the following.

THEOREM 2.12. (Sacker-Sell [42]) *Let* $p : (X, \mathbb{R}) \to (Y, \mathbb{R})$ *be a homomorphism of compact distal flows, where* $X$, $Y$ *are metric spaces and* $(Y, \mathbb{R})$ *is minimal. If there is* $y_0 \in Y$ *with* $card p^{-1}(y_0) = N$*, then the following holds.*

1) *$X$ is an $N$-fold covering of $Y$;*
2) *$(X, \mathbb{R})$ is almost periodic if and only if $(Y, \mathbb{R})$ is.*

*Remark 2.4.*

1) Let $(X, \mathbb{R})$, $(Y, \mathbb{R})$ be as in the above theorem and assume that $(X, \mathbb{R})$ is an almost periodic minimal flow. By Corollary 2.10, $X$, $Y$ are both compact abelian groups. Denote $X'$, $Y'$ as their corresponding character groups. Then the group $(X')^N \equiv \{\chi^N \in X' | \chi \in X'\}$ is isomorphic to a subgroup of $Y'$.

To see this, we let $e_X$, $e_Y$ be identities of groups $X$, $Y$ respectively and denote $H = p^{-1}(e_Y)$. Clearly, $H$ is a closed subgroup of $X$ and $X/H \simeq Y$. Note that $p$ induces an epimorphism $p : X \to Y$ of compact abelian groups, the adjoint homomorphism $p^* : Y' \to X'$ is then one to one, and moreover $Imp^* = (X', H) = \{\chi \in X' | \chi(H) = 1\}$, the annihilator of $H$ in $X'$ ([40]). Since $cardH = N$, for any $x \in H$, $x^N = e_X$ holds. Now, if $\chi \in X'$, $x \in H$, then $\chi^N(x) = \chi(x^N) = \chi(e_X) = 1$. It follows that $(X')^N \subset Imp^* \simeq Y'$.

2) The metrizability of $X$, $Y$ in Theorem 2.12 is not essential by the proof in [42]. In fact, if $(X, \mathbb{R})$, $(Y, \mathbb{R})$ are as in 1), then $X$ is metrizable if and only if $Y$ is. This follows from 1) and the fact that a compact abelian group is metrizable if and only if its character group is at most countable ([24]).

### 2.4. Almost Automorphy.

DEFINITION 2.10. Let $(X, \mathbb{R})$ be a compact flow. A point $x \in X$ is an *almost automorphic point* if any net $\alpha'$ in $\mathbb{R}$ has a subnet $\alpha = \{t_n\}$ such that $T_\alpha x$, $T_{-\alpha}T_\alpha x$ exist and $T_{-\alpha}T_\alpha x = x$, where $-\alpha = \{-t_n\}$. A flow $(X, \mathbb{R})$ is *almost automorphic* if there is an almost automorphic point $x_0 \in X$ with dense orbit.

PROPOSITION 2.13. *An almost automorphic flow is minimal.*

*Proof.* Let $(X,\mathbb{R})$ be an almost automorphic flow and $x_0 \in X$ be an almost automorphic point. For fixed idempotent point $u \in E(X)$, we assume that $\alpha = \{t_n\}$ is a net in $\mathbb{R}$ such that $\Pi_{t_n} \to u$, $\Pi_{-t_n} \to v$ for some $v \in E(X)$. Denote $x^* = ux_0$. Since $ux^* = u^2 x_0 = ux_0 = x^*$, by Theorem 2.2, $X^* = cl\{x^* \cdot t\}$ is a minimal set. Now, $vx^* = vux_0 = T_{-\alpha}T_\alpha x_0 = x_0$, that is, $x_0 \in X^*$. It follows that $X = X^*$. □

*Remark 2.5.*

1) By the above proposition, one sees that an almost automorphic flow is in fact an *almost automorphic minimal flow*, that is, a minimal flow which contains an almost automorphic point.

2) Let $\beta = -\alpha$ in (2.6). It is clear that an almost periodic minimal flow $(X,\mathbb{R})$ is necessarily almost automorphic minimal and $X$ consists of almost automorphic points.

DEFINITION 2.11. Let $p : (X,\mathbb{R}) \to (Y,\mathbb{R})$ be a homomorphism of minimal flows. $(X,\mathbb{R})$ is said to be an *almost automorphic extension* of $(Y,\mathbb{R})$ if there is a $y_0 \in Y$ such that $card p^{-1}(y_0) = 1$.

The next theorem gives a characterization of almost automorphic minimal flows.

THEOREM 2.14. (Veech [51]) $(X,\mathbb{R})$ *is an almost automorphic minimal flow if and only if it is an almost automorphic extension of an almost periodic minimal flow* $(Y,\mathbb{R})$.

DEFINITION 2.12. Let $(X,\mathbb{R})$ be an almost automorphic minimal flow. An almost periodic factor $(Y,\mathbb{R})$ of $(X,\mathbb{R})$ is said to be *maximal* if $(X,\mathbb{R})$ is an almost automorphic extension of $(Y,\mathbb{R})$.

COROLLARY 2.15. *Let* $(X,\mathbb{R})$ *be an almost automorphic minimal flow and* $(Y,\mathbb{R})$ *be a maximal almost periodic factor of* $(X,\mathbb{R})$. *Denote* $p : (X,\mathbb{R}) \to (Y,\mathbb{R})$ *as the flow homomorphism. Then the following holds:*

1) $(X,\mathbb{R})$ *is point distal and*

$$\begin{aligned}\{x \in X | x \text{ is a distal point}\} &= \{x \in X | x \text{ is an almost automorphic point}\} \\ &= \{x \in X | p^{-1}p(x) = \{x\}\}.\end{aligned}$$

2) *The proximal relation* $P(X)$ *is a closed (hence an equivalence) relation.*

3) $(Y,\mathbb{R})$ *is isomorphic to* $(X/P(X),\mathbb{R})$.

*Proof.* We first observe that any $x_1, x_2 \in X$ with $px_1 = px_2$ are proximal. To see this, let $y_0 \in Y$ be such that $p^{-1}(y_0)$ is a singleton, say $\{x_0\}$, and let

$px_1 = px_2 = y$. Since $(Y, \mathbb{R})$ is minimal, there is a net $\{t_n\} \subset \mathbb{R}$ such that $y \cdot t_n \to y_0$. It follows that $x_1 \cdot t_n \to x_0$ and $x_2 \cdot t_n \to x_0$, that is, $x_1$, $x_2$ are proximal.

Since $(Y, \mathbb{R})$ is almost periodic (hence distal), we have shown that a) $x_1, x_2$ are proximal if and only if $p(x_1) = p(x_2)$, that is, they are on a same fiber; b) $x \in X$ is distal if and only if $p^{-1}px = \{x\}$. Results 2), 3) are easily followed.

By definition, if $x$ is such that $p^{-1}p(x) = \{x\}$, then it is an almost automorphic point. Now, let $x \in X$ be an almost automorphic point, and denote $y = p(x)$. By Theorem 2.14, there is a $y_0 \in Y$ such that $p^{-1}(y_0)$ is a singleton $\{x_0\}$. Let $x^* \in p^{-1}(y)$. Since $(X, \mathbb{R})$ is minimal, there is a net $\{t_n\} \subset \mathbb{R}$ with $x_0 \cdot t_n \to x^*$. It follows that $y_0 \cdot t_n \to y$. Using the almost periodicity of $(Y, \mathbb{R})$, one also has $y \cdot (-t_n) \to y_0$. Hence $x \cdot (-t_n) \to x_0$. But $x$ is an almost automorphic point, that is, $x_0 \cdot t_n \to x$. Thus $x = x^*$. $\square$

*Remark 2.6.*

1) Let $(X_i, \mathbb{R})$ $(i = 1, 2)$ be almost automorphic minimal flows with maximal almost periodic factors $(Y_i, \mathbb{R})$ $(i = 1, 2)$ respectively. Denote $p_i : (X_i, \mathbb{R}) \to (Y_i, \mathbb{R})$ $(i = 1, 2)$. It can be shown that if $\phi : (X_1, \mathbb{R}) \to (X_2, \mathbb{R})$ is a flow homomorphism, then there is a unique flow homomorphism $\bar{\phi} : (Y_1, \mathbb{R}) \to (Y_2, \mathbb{R})$ for which the following diagram

$$\begin{array}{ccc} (X_1, \mathbb{R}) & \xrightarrow{\phi} & (X_2, \mathbb{R}) \\ \downarrow p_1 & & \downarrow p_2 \\ (Y_1, \mathbb{R}) & \xrightarrow{\bar{\phi}} & (Y_2, \mathbb{R}) \end{array} \tag{2.7}$$

commutes.

2) By Corollary 2.15 or 1) above, maximal almost periodic factor of an almost automorphic minimal flow is unique up to flow isomorphism.
3) If $(X, \mathbb{R})$ is an almost automorphic minimal flow with metric phase space, by Theorem 2.3 2) and Corollary 2.15, then its almost automorphic points form a residual subset (this was already shown in [53]).
4) Combining Remark 2.5 2) and Corollary 2.15, one sees that a minimal flow $(X, \mathbb{R})$ is almost periodic if and only if it is almost automorphic and each $x \in X$ is an almost automorphic point. This gives another characterization of almost periodic minimal flows (see [51]).

LEMMA 2.16. *Let $p : (X, \mathbb{R}) \to (Y, \mathbb{R})$ be an epimorphism of flows, where $X$, $Y$ are compact metric spaces. Then the set*

$$\begin{aligned} Y_0 = \{y_0 \in Y |\ & \textit{for any } x_0 \in p^{-1}(y_0), y \in Y \textit{ and any sequence} \\ & \{t_i\} \subset \mathbb{R} \textit{ with } y \cdot t_i \to y_0, \textit{ there is a sequence} \\ & \{x_i\} \subset p^{-1}(y) \textit{ such that } x_i \cdot t_i \to x_0\} \end{aligned} \tag{2.8}$$

*is residual and invariant.*

*Proof.* Consider the set valued map $h : Y \to 2^X$, $y \mapsto p^{-1}(y)$, where $2^X$ is furnished with the Hausdorff metric. By the continuity of $p$, the map $h$ is upper semicontinuous. It follows that the set $Y^*$ of points at which $h$ is continuous is residual ([9]). In particular, $h$ is lower semicontinuous on $Y^*$, that is, for any $y_0 \in Y^*$ and $x_0 \in p^{-1}(y_0)$, if $\{y_i\}$ is a sequence in $Y$ with $y_i \to y_0$, then there is a sequence $\{x_i^*\} \subset X$, $x_i^* \in p^{-1}(y_i)$, such that $x_i^* \to x_0$. Now let $y \in Y$ and $\{t_i\} \subset \mathbb{R}$ be such that $y_i \equiv y \cdot t_i \to y_0$ and denote $x_i = x_i^* \cdot (-t_i)$. Then $x_i \in p^{-1}(y)$ and $x_i \cdot t_i \to x_0$. This shows that $y_0 \in Y_0$. Hence $Y^* \subset Y_0$ and $Y_0$ is residual. $Y_0$ is clearly invariant. □

We note that, if $(X, \mathbb{R})$ in Lemma 2.16 is minimal and distal, then by Theorem 2.6, $Y = Y^* = Y_0$ in the above.

COROLLARY 2.17. *Let $(X, \mathbb{R})$ be an almost automorphic minimal flow with metric phase space and $(Y, \mathbb{R})$ be its maximal almost periodic factor. Denote $p : (X, \mathbb{R}) \to (Y, \mathbb{R})$ as the almost automorphic extension. Then the following are equivalent.*

1) *$x \in X$ is an almost automorphic point;*
2) *$p(x) \in Y_0$, where $Y_0$ is defined by* (2.8);
3) *$y = p(x)$ is a point of continuity of the set valued map $h : y \to p^{-1}(y)$.*

*Proof.* We note that, as a continuous Hausdorff image of a compact metric space, $Y$ is metrizable ([37]).

1) $\Longrightarrow$ 3) Let $y = px$. By Corollary 2.15, then $p^{-1}(y) = \{x\}$, a singleton. Clearly, $h$ is continuous at $y$.

3) $\Longrightarrow$ 2) This is immediate by the proof of Lemma 2.16.

2) $\Longrightarrow$ 1) By Corollary 2.15, it is sufficient to show that $p^{-1}(px) = \{x\}$. Let $y = p(x) \in Y_0$ and fix a $x_0$ with $y_0 \equiv px_0$ and $p^{-1}(y_0) = \{x_0\}$ (Theorem 2.14). If $\{t_n\}$ is any sequence with $y_0 \cdot t_n \to y$, then $x_0 \cdot t_n \to x$. It follows that $p^{-1}(y) = \{x\}$. □

*Remark 2.7.* From the above, we see that a compact flow $(X, \mathbb{R})$ with metric phase space is an almost automorphic minimal flow if and only if it is an *almost* 1-1 *extension* of its maximal almost periodic factor $(Y, \mathbb{R})$, that is, $p : (X, \mathbb{R}) \to (Y, \mathbb{R})$, and $Y_0 = \{y \in Y | card p^{-1}(y) = 1\}$ is a residual subset of $Y$.

## 3. Harmonics of Almost Automorphic Functions

### 3.1. Almost Automorphic and Almost Periodic Functions.

Let $V$ be a finite dimensional vector space over the complex field $\mathbb{C}$.

DEFINITION 3.1. A function $f \in C(\mathbb{R}^n \times \mathbb{R}, V)$ is said to be *admissible* if for any compact set $K \subset \mathbb{R}^n$, $f$ is bounded and uniformly continuous on $K \times \mathbb{R}$. $f$

is $C^r$ $(r \geq 1)$ *admissible* if $f$ is $C^r$ in $z \in \mathbb{R}^n$ and Lipschitz in $t$, and $f$ as well as its partial derivatives up to order $r$ are admissible.

DEFINITION 3.2. A function $f \in C(\mathbb{R}, V)$ is *almost automorphic* if for any sequence $\{t'_n\} \subset \mathbb{R}$, there is a subsequence $\{t_n\}$ and a function $g : \mathbb{R} \to V$ such that

$$f(t + t_n) \to g(t) \quad \text{and} \quad g(t - t_n) \to f(t) \tag{3.1}$$

hold pointwise. $f$ is *almost periodic* if for any sequence $\{t'_n\}$ there is a subsequence $\{t_n\}$ such that $\{f(t + t_n)\}$ converges uniformly.

*Remark 3.1.*

1) It is easy to see that an almost automorphic function is always bounded.

2) By definition, an almost periodic function is necessarily almost automorphic. But the converse is not true. It is observed in [54] that the function

$$f(t) = \frac{2 + \exp(it) + \exp(i\sqrt{2}t)}{|2 + \exp(it) + \exp(i\sqrt{2}t)|} \tag{3.2}$$

is almost automorphic but not almost periodic.

3) Let $f$ be an almost automorphic function. Since for each sequence $\{t_n\}$, the convergence in (3.1) is only pointwise, the limiting function $g$ in (3.1) need not be continuous in general. If for each sequence $\{t_n\}$ the corresponding limiting function $g$ in (3.1) is continuous, then $f$ is called a *continuous almost automorphic function*. By [51], a continuous almost automorphic function is uniformly continuous. Clearly, each almost periodic function is a continuous almost automorphic function.

4) Definition 3.2 was given by S. Bochner. In history, there also exist Bohr versions of almost automorphy and almost periodicity. Equivalence of the Bochner and Bohr almost periodicity is well known (see [18], [38]). It can be shown that the class of continuous almost automorphic functions agrees with that of Bohr almost automorphic functions (see [51] for the case of scalar valued functions).

DEFINITION 3.3. $f \in C(\mathbb{R}^n \times \mathbb{R}, V)$ is *uniformly almost automorphic* (*almost periodic*) if $f$ is admissible and almost automorphic (almost periodic) in $t \in \mathbb{R}$.

Since we are interested in differential equations and their solutions, almost automorphic functions to be considered are within either the class of continuous almost automorphic functions or uniform almost automorphic functions. Since a continuous almost automorphic function is a special case of uniform almost automorphic functions (Remark 3.1 1), 3)), we will limit our discussions to uniform almost automorphic functions for the rest of sections.

Let $f \in C(\mathbb{R}^n \times \mathbb{R}, V)$ be an admissible function. The function $f$ generates a family $\{f_\tau | \tau \in \mathbb{R}\}$ in $C(\mathbb{R}^n \times \mathbb{R}, V)$, where $f_\tau(z, t) \equiv f(z, t + \tau)$ $(\tau \in \mathbb{R})$ denotes the time translation. Let $H(f)$, the *hull of* $f$, be the closure of $\{f_\tau | \tau \in \mathbb{R}\}$ in the compact open topology. By the Ascoli's theorem, $H(f)$ is compact and in fact is metrizable ([43]). Moreover, the time translation $g \cdot t \equiv g_t$ ($g \in H(f)$) induces a nature flow $(H(f), \mathbb{R})$ ([43]).

THEOREM 3.1. *Consider an admissible function $f \in C(\mathbb{R}^n \times \mathbb{R}, V)$. The following holds.*

1) *Each $g \in H(f)$ is also admissible ($C^r$ admissible if $f$ is).*
2) *There is a unique $F \in C(\mathbb{R}^n \times H(f), V)$ such that $F(z, f \cdot t) \equiv f(z,t)$. Moreover, if $f$ is $C^r$ admissible, then $F$ is $C^r$ in $x \in \mathbb{R}^n$ and Lipschitz in $g \in H(f)$.*
3) *$(H(f), \mathbb{R})$ is almost automorphic (almost periodic) minimal if $f$ is uniformly almost automorphic (almost periodic).*

*Proof.* 1), 2) follow from arguments in [25], [56], [57]. We only note that the function $F$ is defined as

$$F(z,g) = g(z,0) \quad (z \in R^n, g \in H(f)).$$

3) follows from Proposition 2.13 and Remark 2.5 2). □

**3.2. Universal Almost Automorphic Flow.**

In this section, we construct a *universal* almost automorphic minimal flow $(X_0, \mathbb{R})$, that is, any almost automorphic minimal flow is a factor of $(X_0, \mathbb{R})$.

THEOREM 3.2. *There is a universal almost automorphic minimal flow $(X_0, \mathbb{R})$ and an almost automorphic point $x_0 \in X_0$ such that each uniform almost automorphic function $f \in C(\mathbb{R}^n \times \mathbb{R}, V)$ can be extended uniquely to a function $\tilde{f} \in C(\mathbb{R}^n \times X_0, V)$, and moreover $\tilde{f}(z, x_0 \cdot t) \equiv f(z,t)$ $(z \in \mathbb{R}^n, t \in \mathbb{R})$.*

*Proof.* Let $A_c = \{f \in C(\mathbb{R}, \mathbb{C}) | f$ is a continuous complex valued almost automorphic function$\}$. Then $A_c$ is a sub-algebra of the algebra $C_*(\mathbb{R}, \mathbb{C})$ of bounded continuous complex valued functions ([51]). Denote $X_0$ as the space of maximal ideals of $A_c$ (which is a compact $T_2$ space). By [8], for each $x \in X_0$, there is a $q$ in the Stone-Céch compactification $\beta\mathbb{R}$ of $\mathbb{R}$, such that

$$x = M_q = \{f \in A_c | f_\beta(q) = 0\},$$

where $f_\beta \in C(\beta\mathbb{R}, \mathbb{C})$ denotes the unique extension of $f \in C_*(\mathbb{R}, \mathbb{C})$. By [14], there is a well defined *universal minimal flow* $(\beta\mathbb{R}, \mathbb{R})$, and if $e \in \beta\mathbb{R}$ corresponds to $0 \in \mathbb{R}$, then $f_\beta(e \cdot t) \equiv f(t)$, $(t \in \mathbb{R}, f \in C_*(\mathbb{R}, \mathbb{C}))$. For $x = M_q \in X_0$ and $t \in \mathbb{R}$, let

$$x \cdot t = \{f_t | f \in x\} = \{f \in A_c | f_\beta(q \cdot t) = 0\}. \tag{3.3}$$

Clearly, (3.3) defines a flow $(X_0, \mathbb{R})$ and $p : (\beta\mathbb{R}, \mathbb{R}) \to (X_0, \mathbb{R}) : q \mapsto M_q$ is a flow homomorphism. Denote

$$x_0 = \{f \in A_c | f_\beta(e) = 0\} = \{f \in A_c | f(0) = 0\}. \tag{3.4}$$

For any $f \in A_c$, define $\tilde{f} \in C(X_0, \mathbb{C}) : \tilde{f}(pq) \equiv f_\beta(q)$. Then $\tilde{f}$ is a unique extension of $f$ in $C(X_0, \mathbb{C})$, and $\tilde{f}(x_0 \cdot t) \equiv f_\beta(e \cdot t) \equiv f(t)$. It follows that $x_0$ is an almost automorphic point and $X_0 = cl\{x_0 \cdot t | t \in \mathbb{R}\}$, that is, $(X_0, \mathbb{R})$ is an almost automorphic minimal flow.

To show the universality, we let $(X, \mathbb{R})$ be any almost automorphic minimal flow and $\bar{x}_0 \in X$ be an almost automorphic point. For any $f \in C(X, \mathbb{C})$, $h(t) \equiv f(\bar{x}_0 \cdot t)$ is a continuous almost automorphic function. Thus, there exists a unique $\tilde{h}_f \in C(X_0, \mathbb{C})$ with

$$\tilde{h}_f(x_0 \cdot t) \equiv h(t) \equiv f(\bar{x}_0 \cdot t). \tag{3.5}$$

Let $x \in X_0$ and let $\{t_n\}$ be a net in $\mathbb{R}$ such that

$$x_0 \cdot t_n \to x. \tag{3.6}$$

Since (3.5) holds for arbitrary $f \in C(X, \mathbb{C})$, $\{\bar{x}_0 \cdot t_n\}$ converges, say to $\bar{x}$. Define $p_{\bar{x}_0} : X_0 \to X$, $p_{\bar{x}_0} x = \bar{x}$. By (3.5), it is easy to see that $p_{\bar{x}_0}$ is well defined, continuous, and $p_{\bar{x}_0}(x \cdot t) = (p_{\bar{x}_0} x) \cdot t$, that is, it is a homomorphism of minimal flows (which depends on the choice of $\bar{x}_0$).

Now let $\phi_f : (X_0, \mathbb{R}) \to (H(f), \mathbb{R})$ be the flow homomorphism with $\phi_f x_0 = f$ and let $F \in C(\mathbb{R}^n \times H(f), V)$ be the extension of $f \in C(\mathbb{R}^n \times \mathbb{R}, V)$ according to Theorem 3.1. The function $\tilde{f} \in C(\mathbb{R}^n \times X_0, V)$ is then given by

$$\tilde{f}(z, x) = F(z, \phi_f x)$$

$(z \in \mathbb{R}^n, x \in X_0)$. □

*Remark 3.2.*

1) $X_0$ can be also obtained by the commutative Gelfand-Naimark theorem ([10]). In fact, since $A_c$ is a commutative $C^*$-algebra with identity ([51]), it is isometrically isomorphic with $C(X_0, \mathbb{C})$.

2) The universal object $X_0$ is clearly a Hausdorff compactification of $\mathbb{R}$. Let $(Y_0, \mathbb{R})$ be its maximal almost periodic factor and denote $p_0 : (X_0, \mathbb{R}) \to (Y_0, \mathbb{R})$. By Corollary 2.10 2), $Y_0$ can be given a structure of compact abelian topological group. In fact, it is not difficult to see that $Y_0$ is a Bohr compactification of $\mathbb{R}$. It follows that any scalar valued almost periodic function $f$ admits a unique extension $\tilde{f} \in C(Y_0, \mathbb{C})$, and $f(t) \equiv \tilde{f}(p_0 x_0 \cdot t)$. In particular, any character $\chi_\lambda = e^{i\lambda t}$ of $\mathbb{R}$ extends uniquely to a continuous character $\tilde{\chi}_\lambda$ of $Y_0$ with $\tilde{\chi}_\lambda(p_0 x_0 \cdot t) \equiv e^{i\lambda t}$.

3) Let $(X,\mathbb{R})$ be an almost automorphic minimal flow and $(Y,\mathbb{R})$ be its maximal almost periodic factor. Denote $p:(X,\mathbb{R})\to(Y,\mathbb{R})$ as the flow homomorphism and let $X_0, Y_0, p_0$ be as in 1). By Remark 2.6 1), for each almost automorphic point $\bar{x}_0\in X$, the homomorphism $\phi_{\bar{x}_0}:(X_0,\mathbb{R})\to(X,\mathbb{R})$ defined in Theorem 3.2 induces a unique homomorphism $\phi_{p\bar{x}_0}:(Y_0,\mathbb{R})\to(Y,\mathbb{R})$ for which the following diagram

$$\begin{array}{ccc} (X_0,\mathbb{R}) & \xrightarrow{\phi_{\bar{x}_0}} & (X,\mathbb{R}) \\ \downarrow p_0 & & \downarrow p \\ (Y_0,\mathbb{R}) & \xrightarrow{\phi_{p\bar{x}_0}} & (Y,\mathbb{R}) \end{array} \tag{3.7}$$

commutes. In particular, if $(X,\mathbb{R})$, $(Y,\mathbb{R})$ are also universal objects, then $\phi_{\bar{x}_0}$ and $\phi_{p\bar{x}_0}$ become isomorphisms.

### 3.3. Fourier Analysis.

Let $X_0, x_0$ be as in Theorem 3.2 and $Y_0, p_0, \tilde{\chi}_\lambda$ be as in Remark 3.2 2). For a uniform almost automorphic function $f\in C(\mathbb{R}^n\times\mathbb{R},V)$, $\tilde{f}\in C(\mathbb{R}^n\times X_0,V)$ shall be denoted as its unique extension according to Theorem 3.2.

For a fixed invariant probability measure $\nu$ of $(X_0,\mathbb{R})$, let the *Fourier coefficient* of $f$ associated to each $\lambda\in\mathbb{R}$ be defined as follows:

$$a_\lambda^\nu(f)(z)=\int_{X_0}\tilde{f}(z,x)\overline{\tilde{\chi}_\lambda(p_0x)}\nu(dx). \tag{3.8}$$

We note by Remark 3.2 3) that Fourier coefficients (3.8) can be equivalently defined with respect to any choice of universal objects $X_0, Y_0$.

Definition 3.4.

1) The set

$$\mathcal{S}_\nu(f)=\{\lambda\in\mathbb{R}\mid a_\lambda^\nu(f)(z)\not\equiv 0\} \tag{3.9}$$

is called the *spectrum of* $f$.

2) The *frequency module of* $f$ is the smallest subgroup $\mathcal{M}(f)$ of $\mathbb{R}$ containing $S_\nu(f)$.

3) The series

$$f(z,t)\sim\sum_{\lambda\in\mathcal{M}(f)}a_\lambda^\nu(f)(z)e^{i\lambda t} \tag{3.10}$$

is referred to as a *Fourier series* of $f$.

*Remark 3.3.* If $f\in C(\mathbb{R}^n\times\mathbb{R},V)$ is uniformly almost periodic, then Definition 3.4 gives rise to the usual definition of Fourier series, spectrum and frequency module ([18], [34], [38]) since $\tilde{f}\in C(\mathbb{R}^n\times Y_0,V)$ and $(Y_0,\mathbb{R})$, as an almost periodic minimal flow, is uniquely ergodic.

Definition 3.5. Let $(X, \mathbb{R})$ be an almost automorphic minimal flow, $(Y, \mathbb{R})$ be a maximal almost periodic factor of $(X, \mathbb{R})$, and $\mu$ be an invariant probability measure of $(X, \mathbb{R})$. $(X, Y, \mu)$ is referred to as a *Fourier triple* of $f$ if $f$ can be extended to a function $\tilde{f} \in C(\mathbb{R}^n \times X, V)$ and $f(z,t) \equiv \tilde{f}(z, \bar{x}_0 \cdot t)$ for an almost automorphic point $\bar{x}_0 \in X$.

Let $(X_0, \mathbb{R})$ be the universal almost automorphic minimal flow and let $(X_f, \mathbb{R})$ be the (time) translated flow on $X_f = H(f)$. Denote $(Y_0, \mathbb{R})$, $(Y_f, \mathbb{R})$ as maximal almost periodic factors of $(X_0, \mathbb{R})$, $(X_f, \mathbb{R})$ respectively. Then for invariant probability measures $\nu$, $\mu$ of $(X_0, \mathbb{R})$, $(X_f, \mathbb{R})$ respectively, $(X_0, Y_0, \nu)$ and $(X_f, Y_f, \mu))$ defines the 'largest' and 'smallest' Fourier triples respectively. In fact, if $(X, \mathbb{R})$ is any almost automorphic minimal flow which extends $(X_f, \mathbb{R})$ (that is, there exists a flow homomorphism $(X, \mathbb{R}) \to (X_f, \mathbb{R})$), then it is easy to see that $(X, Y, \mu)$ is a Fourier triple of $f$, where $Y, \mu$ are as in the above definition for the current $(X, \mathbb{R})$.

Using ideas of [51], the Fourier series (3.10) can be summed to the function $f$ by the well known Bochner-Fejer summation procedure. In fact, such a summation can be carried over with respect to any fixed Fourier triple. Let $(X, Y, \mu)$ be a Fourier triple of $f$ and $\tilde{f} \in C(\mathbb{R}^n \times X, V)$ be the extension of $f$ with $f(z,t) \equiv \tilde{f}(z, \bar{x}_0 \cdot t)$ for an almost automorphic point $\bar{x}_0 \in X$. Denote $p : (X, \mathbb{R}) \to (Y, \mathbb{R})$ as the flow homomorphism and $Y'$ as the character group of the compact abelian group $Y$ with identity $y_0 = px_0$. Recall that the family of *Bochner-Fejer kernels* ([18], [34], [38], [51]) is a net $K_\alpha$ of trigonometric polynomials of $\{\bar{\chi} \mid \chi \in Y'\}$ satisfying the following properties:

i) $K_\alpha(y) \geq 0$ $(y \in Y)$;
ii) For any neighborhood $U$ of $y_0$, $lim_\alpha sup_{y \in Y \setminus U} K_\alpha(y) = 0$;
iii) $\int_Y K_\alpha(y)\mu_0(dy) = 1$, where $\mu_0$ is the Haar measure on $Y$.

Define the family of *Bochner-Fejer polynomials* $\{S^\mu_\alpha(f)\}$ as follows:

$$S^\mu_\alpha(f)(z,t) = \int_X \tilde{f}(z, x \cdot t) K_\alpha(px)\mu(dx) = \int_X \tilde{f}(z, x) K_\alpha(px \cdot -t)\mu(dx). \tag{3.11}$$

Clearly, $\{S^\mu_\alpha(f)\}$ is a family of trigonometric polynomials of continuous characters of $\mathbb{R}$. Moreover, by arguments of [51], one has the following.

Theorem 3.3. *Let $\{S^\mu_\alpha(f)\}$ be as above. Then there is a sequence $S^\mu_m(f)$ of $\{S^\mu_\alpha(f)\}$ such that* i) *$\{S^\mu_m(f)\}$ is jointly uniformly almost automorphic;* ii) *$S^\mu_m(f) \to f$ as $m \to \infty$ uniformly on compact sets.*

The following result is originally due to Veech ([51]) for continuous scalar almost automorphic functions.

Corollary 3.4.

1) *A Fourier series of $f$ determines $f$ uniquely;*
2) *A Fourier series of $f$ converges to $f$* jointly uniformly almost automorphically*, that is, for each invariant probability measure $\nu$ of $(X_0, \mathbb{R})$, there is a sequence of jointly uniform almost automorphic* weighted partial sums $S_m^\nu(f)$ of the Fourier series *which converges to $f$ uniformly on compact sets.*

*Proof.* 2) is a direct application of Theorem 3.3 to the 'largest' Fourier triple $(X_0, Y_0, \nu)$. We note by (3.11) that the Bochner-Fejer polynomials $S_m^\mu(f)$ given in Theorem 3.3 are really weighted Fourier partial sums of the corresponding Fourier series and the associated Bochner-Fejer kernels $K_m$ depends only on the frequency module $\mathcal{M}(f)$. Now, 1) follows easily from 2) since $a_\lambda^\nu(f)(z) \equiv 0$ $(\lambda \in \mathcal{M}(f))$ implies that $S_m^\nu(f) \equiv 0$ $(m = 1, 2, \cdots)$, hence $f \equiv 0$. □

Denote $X_f = H(f)$ and let $(Y_f, \mathbb{R})$ be a maximal almost periodic factor of the natural flow $(X_f, \mathbb{R})$. Thus, $Y_f$ is an abelian solenoidal group and $y_f \equiv p_f f$ can be viewed as its identity. Let $Y_f'$ denotes the character group of $Y$. Since $X_f$ is metrizable, so is $Y_f$ ([37]). It follows that $Y_f'$ is isomorphic to an at most countable subgroup $\Lambda$ of the discrete group $\mathbb{R}_d$ ([24]), that is, for each $\lambda \in \Lambda$, $e^{i\lambda t}$ admits a unique extension to a continuous character $\hat{\chi}_\lambda \in Y_f'$, and $e^{i\lambda t} \equiv \hat{\chi}_\lambda(y_f \cdot t)$. Now let $p_f$, $\phi_f$, $\phi_{pf}$ be as in (3.7) for the current $X_f$ and $Y_f$. We first note that the homomorphism $\phi_f : (X_0, \mathbb{R}) \to (X_f, \mathbb{R})$ induces an onto mapping $\phi_f : M(X_0) \to M(X_f)$, $\nu \to \mu$ of invariant probability measures by the following identity:

$$\int_{X_f} h d\mu = \int_{X_0} h \circ \phi_f d\nu \tag{3.12}$$

$(h \in C(X_f))$. Let $\nu$ be an invariant probability measure on $(X_0, \mathbb{R})$ and denote $\mu = \phi_f \nu$. For each $\lambda \in \Lambda$, define

$$\hat{a}_\lambda^\mu(f)(z) = \int_{X_f} F(z, g) \overline{\hat{\chi}_\lambda(p_f g)} \mu(dg), \tag{3.13}$$

where $F$ is the extension of $f$ on $C(\mathbb{R}^n \times X_f, V)$ as in Theorem 3.1. By (3.7) and (3.11), the Fourier coefficients $a_\lambda^\nu(f)(z)$ in (3.8) equals to $\hat{a}_\lambda^\mu(f)(z)$ if $\lambda \in \Lambda$.

Lemma 3.5. *For each invariant probability measure $\nu$ on $(X_0, \mathbb{R})$,*

$$\mathcal{S}_\nu(f) = \{\lambda \in \Lambda \mid \hat{a}_\lambda^\mu(f)(z) \not\equiv 0\}, \tag{3.14}$$

*where $\mu = \phi_f \nu$.*

*Proof.* We denote the right hand side of (3.14) by $\mathcal{S}_\mu(f)$. Clearly, $\mathcal{S}_\mu(f) \subset \mathcal{S}_\nu(f)$. Now let $\lambda \in \mathcal{S}_\nu(f)$, that is, there exists a $z_0 \in \mathbb{R}^n$ such that $a_\lambda^\nu(f)(z_0) \neq 0$.

By applying Corollary 3.4 to the Fourier triple $(X_f, Y_f, \mu)$, we obtain a sequence $\{S^\mu_m(f)\}$ of Bochner-Fejer polynomials which converges to $f$ jointly uniformly almost automorphically. Note that for each $m$, $S^\mu_m(f)$ is a uniform almost periodic function. By (3.11), (3.13), it is easy to see that $\mathcal{S}_\nu(S^\mu_m(f)) \subset \mathcal{S}_\mu(f)$. Since $\{S^\mu_m(f)\}$ is jointly uniformly almost automorphic, $a^\nu_\lambda(S^\mu_m(f))(z) \to a^\nu_\lambda(f)(z)$ $(z \in \mathbb{R}^n)$, in particular, there is a $m_0$ such that $a^\nu_\lambda(S^\mu_{m_0}(f))(z_0) \neq 0$. Above all, $\mathcal{S}_\nu(f) \subset \cup \mathcal{S}_\nu(S^\mu_m(f)) \subset \mathcal{S}_\mu(f)$. □

*Remark 3.4.* By the above lemma, (3.13) coincides with the Fourier coefficient (3.8). Consequently, the Fourier series (3.10) has the alternative form

$$f(z,t) \sim \sum_{\lambda \in \mathcal{M}(f)} \hat{a}^\mu_\lambda(f)(z) e^{i\lambda t}. \tag{3.15}$$

Therefore, although a Fourier series of $f$ need not be unique in general, it only depends on the choice of an invariant measure of $(X_f, \mathbb{R})$. By the above lemma, since $\Lambda$ is at most countable, so are $\mathcal{S}_\nu(f)$ and $\mathcal{M}(f)$.

We now show that $\mathcal{M}(f)$ is however uniquely defined.

THEOREM 3.6. *$\mathcal{M}(f) \simeq Y'_f$ for any invariant measure $\nu$ on $(X_0, \mathbb{R})$.*

*Proof.* By Lemma 3.5, it is sufficient to show that $\mathcal{M}(f)$ separates points of $Y_f$ ([24], [40]), that is, for any $y \in Y_f$ with $y \neq y_f$ (the identity of $Y_f$), there is a $\lambda \in \mathcal{M}(f)$ such that $\hat{\chi}_\lambda(y) \neq 1$, where $\hat{\chi}_\lambda$ is the unique extension of $e^{i\lambda t}$ on $Y'_f$. Suppose not, then there is a $y_1 \in Y_f$ with $y_1 \neq y_f$ such that $\hat{\chi}_\lambda(y_1) \equiv 1$ for all $\lambda \in \mathcal{M}(f)$. Let $\{t_i\}$ be a sequence such that $y_f \cdot t_i \to y_1$ and $f \cdot t_i = f_{t_i} \to g$, where $g$ is some point on $p^{-1}(y_1)$. Clearly, $g \neq f$. Note that $f$ is uniformly almost automorphic, so are $\hat{f}_i = f_{t_i} - f$ for all $i$. Let $\mu = \phi_f \nu$ and $\{S^\mu_m(f)\}$ be as in the proof of Lemma 3.5. Then each $S^\mu_m(f)$ is a weighted partial sum of the Fourier series (3.15). Since for any $z_0 \in \mathbb{R}^n$ and $\lambda \in \mathcal{M}(f)$,

$$\hat{a}^\mu_\lambda(\hat{f}_i)(z_0) = \hat{a}^\mu_\lambda(f)(z_0)(\hat{\chi}_\lambda(y_f \cdot t_i) - 1),$$

we have $\hat{a}^\mu_\lambda(\hat{f}_i)(z_0) \to 0$ as $i \to \infty$. It follows that for any $t_0 \in \mathbb{R}$, $S^\mu_m(\hat{f}_i)(z_0, t_0) \to 0$ as $i \to \infty$. In fact, by the joint almost automorphism of $\{S^\mu_m(f)\}$, one can assume that the above convergence is also uniform for $m = 1, 2, \cdots$. Since $S^\mu_m(\hat{f}_i)(z_0, t_0) \to \hat{f}_i(z_0, t_0)$ as $m \to \infty$, $\lim_{i\to\infty} \hat{f}_i(z_0, t_0) = g(z_0, t_0) - f(z_0, t_0) = 0$. Now, $z_0, t_0$ are arbitrary, we then have $g \equiv f$, a contradiction. □

COROLLARY 3.7. *Let $f$ be a uniform almost automorphic function. Then for any uniform almost automorphic function $g \in H(f)$ (there are residually many by* Remark 2.5 2)*), $\mathcal{M}(g) = \mathcal{M}(f)$.*

*Proof.* This is because $H(g) = H(f)$. □

*Remark 3.5.* As shown in [20], [41], almost automorphic functions are $N$-almost periodic in the sense of Levitan (see [33], [34]). For continuous almost automorphic functions, this can be seen easily from Theorem 3.6 above. We first note that by [34], a function $f(t)$ is $N$-almost periodic if and only if there is an at most countable module $\mathcal{M} \subset \mathbb{R}$ such that if $\{t_n\}$ is a sequence with $e^{i\lambda t_n} \to 1$ $(\lambda \in \mathcal{M})$, then $f(t+t_n) \to f(t)$ uniformly on every finite interval. We now let $f(t)$ be a continuous almost automorphic function. By Remark 3.4, $\mathcal{M} = \mathcal{M}(f)$ is at most countable. Let $\{t_n\}$ be a sequence such that $e^{i\lambda t_n} \to 1$ $(\lambda \in \mathcal{M})$. It follows from Theorem 3.6 that $\chi_\lambda(y_f \cdot t_n) \to 1$ for all $\chi_\lambda \in Y_f'$, here $y_f = p_f f$ denotes the identity of $Y_f$, that is, $y_f \cdot t_n \to y_f$. Thus, $f \cdot t_n \to f$ in $(H(f), \mathbb{R})$, that is, $f(t+t_n) \to f(t)$ uniformly on every finite interval.

### 3.4. Module Containment.

Definition 3.6. Let $f \in C(\mathbb{R}^n \times \mathbb{R}, V)$ be a uniform almost automorphic function and $\alpha = \{t_n\} \subset \mathbb{R}$ be a sequence. The *generalized translation* $T_\alpha f$ is the limit of $f_{t_n}$ in the compact open topology, provided that the limit exists.

We now prove a theorem which generalizes the classical result of module containment for almost periodic functions (see [17]). Let $f \in C(\mathbb{R}^n \times \mathbb{R}, V)$, $g \in C(\mathbb{R}^m \times \mathbb{R}, W)$ be two uniform almost automorphic functions, where $V$, $W$ are two finite dimensional vector spaces. Consider $X_f = H(f)$, $X_g = H(g)$ and denote $p_f : (X_f, \mathbb{R}) \to (Y_f, \mathbb{R})$, $p_g : (X_g, \mathbb{R}) \to (Y_g, \mathbb{R})$ as flow homomorphisms, where $(Y_f, \mathbb{R})$ and $(Y_g, \mathbb{R})$ are maximal almost periodic factors of the natural flows $(X_f, \mathbb{R})$ and $(X_g, \mathbb{R})$ respectively.

Theorem 3.8. *The following are equivalent.*

1) *$\mathcal{M}(g) \subset \mathcal{M}(f)$;*
2) *Whenever $T_\alpha f = f$ for a sequence $\alpha$, then $T_\alpha g = g$;*
3) *There is a flow homomorphism $\phi : (Y_f, \mathbb{R}) \to (Y_g, \mathbb{R})$ with $\phi p_f f = p_g g$.*

*Proof.* We note that $Y_f$, $Y_g$ are compact abelian topological groups, and $y_f \equiv p_f f$, $y_g \equiv p_g g$ can be viewed as the identities of $Y_f$, $Y_g$ respectively.

1) $\Longrightarrow$ 3): By Theorem 3.6, $\mathcal{M}(f) \simeq Y_f'$, $\mathcal{M}(g) \simeq Y_g'$. Since $\mathcal{M}(g) \subset \mathcal{M}(f)$, $Y_g'$ is isomorphic to a subgroup of $Y_f'$. Let $\tilde{\phi} : Y_g' \to Y_f'$ be the embedding. Then the adjoint map $\phi : Y_f'' \to Y_g''$ is an epimorphism ([40]), that is, $\phi$ induces an epimorphism $\phi : Y_f \to Y_g$ of compact abelian groups by the Pontryagin duality theorem ([40]). Now, $\phi(y \cdot t) = \phi(y)\phi(y_f \cdot t) = \phi(y) y_g \cdot t = \phi(y) \cdot t$ $(y \in Y_f, t \in \mathbb{R})$, that is, $\phi : (Y_f, \mathbb{R}) \to (Y_g, \mathbb{R})$ is a flow homomorphism.

3) $\Longrightarrow$ 1) Since $\phi : (Y_f, \mathbb{R}) \to (Y_g, \mathbb{R})$ is a homomorphism of almost periodic minimal flows, $\phi : Y_f \to Y_g$ induces a group epimorphism. Thus, by [40], its adjoint homomorphism $\phi^* : Y_g' \to Y_f'$ is one to one, that is, $Y_g'$ is isomorphic to a subgroup of $Y_f'$. By Theorem 3.6, $\mathcal{M}(g) \subset \mathcal{M}(f)$.

3) $\Longrightarrow$ 2): If $T_\alpha f = f$ for some sequence $\alpha = \{t_n\} \subset \mathbb{R}$, that is, $f \cdot t_n \to f$ in $X_f$, then $y_f \cdot t_n \to y_f$ in $Y_f$. It follows that $y_g \cdot t_n = \phi(y_f) \cdot t_n \to \phi(y_f) = y_g$ in $Y_g$. Since $p_g^{-1} y_g = \{g\}$ is a singleton, $g \cdot t_n \to g$ in $X_g$, that is, $T_\alpha g = g$.

2) $\Longrightarrow$ 3): We use the same notation $T_\alpha$ as a generalized translation for flows (see Definition 2.9).

For any $y \in Y_f$, let $\beta = \{t_n\}$ be a sequence such that $y_f \cdot t_n \to y$, that is, $T_\beta y_f = y$. By taking a subsequence, we assume that $T_\beta y_g$ exists and denote the corresponding limit by $y_*$. Define $\phi : Y_f \to Y_g$ by $\phi(y) = y_*$. We first check that $\phi$ is well defined. Let $\beta' = \{t'_n\}$ be another sequence with $y_f \cdot t'_n \to y$, that is, $T_{\beta'} y_f = y$, and assume without loss of generality that $T_{\beta'} y = y_{**}$. For the sequence $\alpha = \beta' - \beta = \{t_n - t'_n\}$, since $T_\alpha y_f = T_{-\beta} T_{\beta'} y_f = y_f$ (Corollary 2.11) and $p_f^{-1} y_f = \{f\}$, one has $T_\alpha f = f$, hence $T_\alpha g = g$, that is, $T_\alpha y_g = y_g$. By Corollary 2.11, $y_{**} = T_{\beta'} y_g = T_{\beta+\alpha} y_g = T_\beta T_\alpha y_g = T_\beta y_g = y_*$. Thus, $\phi$ is well defined and $\phi(y_f) = y_g$ by the above arguments. Above all, we have shown that for any sequence $\beta \subset \mathbb{R}$,

$$T_\beta y_f = y \quad \text{implies} \quad T_\beta y_g = \phi(y). \tag{3.16}$$

To show the continuity of $\phi$, we fix a $y \in Y_f$ and let $y_n \to y$ in $Y_f$. Without loss of generality, assume that $\phi(y_n) \to y^*$ in $Y_g$. By minimality, for each $n$, there is a sequence $\beta_n$ such that $T_{\beta_n} y_f = y_n$. It follows that $T_{\beta_n} y_g = \phi(y_n)$, that is,

$$T_{\beta_n}(y_f, y_g) = (y_n, \phi(y_n)) \to (y, y^*).$$

Therefore, by the standard diagonal process, one can extract a sequence $\beta = \{t_k\}$ in $\cup_n \beta_n$ such that $T_\beta(y_f, y_g) = (T_\beta y_f, T_\beta y_g) = (y, y^*)$. By (3.16), $y^* = \phi(y)$. Thus, $\phi$ is continuous.

Now, for any $y \in Y_f$, $t \in \mathbb{R}$, we let $\beta$ be a sequence such that $T_\beta y_f = y$. Since $T_{\beta+t} y_f = y \cdot t$, $T_{\beta+t} y_f = \phi(y \cdot t)$. But $T_{\beta+t} y_g = (T_\beta y_g) \cdot t = \phi(y) \cdot t$. Thus, $\phi(y \cdot t) = \phi(y) \cdot t$ ($y \in Y_f$, $t \in \mathbb{R}$), that is, $\phi$ is a homomorphism of minimal flows.

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**Part II. Skew-product Semiflows**

Wenxian Shen and Yingfei Yi

# PART II
# SKEW-PRODUCT SEMIFLOWS

## 1. Introduction

The study of skew-product flows originated in ergodic theory of discrete dynamical systems (see [1], [4]). Continuous skew-product flows, since applied in works of Miller ([15]) and Sell ([27], [28]), has provided a unified topological way to study dynamics of nonautonomous, in particular almost periodic ordinary differential equations (see e.g., [10], [11], [23]-[29], [39]). In the current part, we shall consider skew-product semiflows which can be generated by a large class of almost periodic ordinary, partial and delay differential equations. Due to the fact that solutions of a differential equation need not exist globally in time, a notion of local (skew-product) semiflow might be more appropriate. Since we are interested in the long time behavior of solutions of differential equations in which the associated local semiflow when restricted to an $\omega$-limit set becomes a global one, we shall not give a precise definition of local semiflow. Instead, let us just say that a local semiflow is the one that locally behaves like a semiflow.

In the current part, we shall focus on lifting dynamics of a skew-product semiflow from an almost periodic base. Since our aim is to analyze the effect of almost periodic coefficients to the behavior of solutions in a differential equation, we would like to pay our particular attentions to skew-product semiflows having relatively simple structures, for example, those of order preserving or monotone natures. Strongly order preserving or strongly monotone semiflows (flows, mappings) were first studied by M. Hirsch ([6]-[9]) and H. Matano ([12], [13]) independently. Extended studies have been made in [5], [19]-[22], [33]-[37], etc. In the current work, by considering certain strongly order preserving natures on fibers, we shall address fundamental roles played by almost automorphic dynamics in skew-product semiflows with almost periodic base. Our main results are as follows.

Let $\Pi : X \times Y \times \mathbb{R}^+ \to X \times Y$ be a skew-product semiflow (see Section 2), where $X$ is a Banach space, $Y$ is a compact metric space with an almost periodic minimal flow $(Y, \mathbb{R})$. We denote $p : X \times Y \to Y$ as the natural projection.

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Dedicated to Professor R. Ellis on the Occasion of His 70th Birthday

Theorem A. (Lifting properties) *Suppose that* $X$ *is strongly ordered and* $\Pi$ *is strongly monotone* (*see* Section 4). *Let* $E \subset X \times Y$ *be a minimal set of* $\Pi$ *which admits a flow extension. Then the following holds.*

1) (*Almost automorphy*) *If* $E$ *is linearly stable* (*see* Section 4), *then* $(E, \mathbb{R})$ *is almost automorphic. More precisely, there is an almost periodic minimal flow* $(\tilde{Y}, \mathbb{R})$ *such that*

$$(E, \mathbb{R}) \xrightarrow{\tilde{p}} (\tilde{Y}, \mathbb{R}) \xrightarrow{p_*} (Y, \mathbb{R})$$

*holds, where* $p = p_* \circ \tilde{p}$, $p_*$ *is an* $N$-1 *extension, that is,* $card p_*^{-1}(y) \cap E = N$ $(y \in Y)$, $\tilde{p}$ *is an almost* 1-1 (*or almost automorphic*) *extension.*

2) (*Almost periodicity*) *If* $E$ *is also uniformly stable* (*see* Section 2), *then* $(E, \mathbb{R})$ *is almost periodic and moreover*

$$(E, \mathbb{R}) \xrightarrow{p} (Y, \mathbb{R})$$

*is an* $N$-1 $(N \geq 1)$ *extension.*

Theorem B. (Global Attractor) *Suppose that* $\Pi$ *is strictly contracting* (*see* Section 2) *and admits a relatively compact forward orbit. Then* $\Pi$ *has a unique minimal set* $E$ *which admits a flow extension, and*

$$(E, \mathbb{R}) \xrightarrow{p} (Y, \mathbb{R})$$

*is a* 1-1 *extension* (*hence* $(E, \mathbb{R})$ *is almost periodic*). *Moreover, any relatively compact orbit of* $\Pi$ *is asymptotic to a unique almost periodic orbit in* $E$.

The result 1) of Theorem A is an extension of a result of Poláčik and Tereščák ([21]) for strongly monotone maps. The result 2) of Theorem A resembles a result of Sacker-Sell ([26]) in general skew-product flows, in which the minimal set ($\omega$-limit set) was assumed to be both asymptotically and uniformly stable. Theorem B is an extension to our earlier results in scalar, 1-dimensional, almost periodic parabolic equations ([30], [31]). Two special properties of a strongly monotone skew-product semiflow are crucial in the proofs of the above results, namely, a non-ordering principle and a continuous separation theorem which generalize those of Hirsch ([6], [7]) and Poláčik and Tereščák ([22]), Mierczyński ([14]) respectively. In general, 'almost automorphy' in Theorem A can not be replaced by 'almost periodicity' (see examples in Part III). This in fact reflects a nature of almost periodic dependence, because, if $(Y, \mathbb{R})$ in Theorem A is either trivial or periodic, by introducing a Poincaré map, then one sees easily that an almost automorphic lifting never occurs. There are essential differences between an almost periodic dependence and a periodic one even in skew-product semiflows with monotone natures. In a strongly monotone skew-product semiflow with periodic base (which is equivalent to a strongly monotone map), although an unstable $\omega$-limit or minimal set can be chaotic (see [17], [18], [32]), stable ones are all periodic ([21]). But in almost periodic case, a stable minimal set, being almost automorphic by Theorem A, can well be complicated (see Example 3.5 of Part III). Furthermore, the generic convergence property preserved by a periodic monotone system (see [21] and references therein) will however fail for an almost periodic one even within the category of almost automorphy (see Part III, Remark 5.1 4) for details).

The abstract theory of topological dynamics plays an important role in the current study. For example, in the case of linear stability of Theorem A, it is shown that the proximal relation coincides with the order relation on $E$, and therefore both relations on $E$ are invariant, closed and equivalence. Also, similar to [26], distality is implied by the uniform stability in the case of Theorem A 2), and by a contracting property in the case of Theorem B.

The current part is organized as follows. In Section 2, we introduce skew-product semiflows and discuss their fundamental properties such as flow extensions of an $\omega$-limit set, lifting properties related to a uniform stability, and the contracting dynamics stated in Theorem B. Section 3 deals with general properties (such as the non-ordering principle) of strongly order preserving skew-product semiflows. Strongly monotone skew-product semiflows are investigated in Section 4. The continuous separation result and Theorem A are proved in this section.

## 2. Flow Extension

### 2.1. Skew-product Semiflow.

Definition 2.1. Let $Z$ be a complete metric space. Denote $\mathbb{R}^+ = \{t \in \mathbb{R} \mid t \geq 0\}$. A *semiflow* $(Z, \mathbb{R}^+)$ is a continuous mapping $\Pi : Z \times \mathbb{R}^+ \to Z$ satisfying the following properties:

i) $\Pi(z, 0) = z$ $(z \in Z)$;
ii) $\Pi(\Pi(z, s), t) = \Pi(z, s + t)$ $(z \in Z, s, t \in \mathbb{R}^+)$.

We refer an orbit in a semiflow as a *forward orbit*.

Definition 2.2. Let $X$, $Y$ be metric spaces and $(Y, \mathbb{R})$ be a compact flow (called the *base flow*). A *skew-product semiflow* $\Pi : X \times Y \times \mathbb{R}^+ \to X \times Y$ is a semiflow of the following form

$$\Pi(x, y, t) = (u(x, y, t), y \cdot t). \tag{2.1}$$

Let $(Z, \mathbb{R}^+)$ be a semiflow and let $z_0 \in Z$ be such that the orbit $\{\Pi(z_0, t) | t \geq t_0\}$ is relatively compact for some $t_0 \geq 0$. The $\omega$-limit set $\omega(z_0)$ can be similarly defined as in Definition 2.2 of Part I. It is easily seen that, an $\omega$-limit set $\omega(z_0)$ is *positively invariant*, that is, the original semiflow, when restricted to $\omega(z_0)$, is a sub-semiflow. To apply topological dynamics techniques to the current study of semiflows, it is important to know whether the semiflow restricted to a compact, positively invariant subset can be actually extended to a flow. We usually only require such an extension on an $\omega$-limit set because, in a semiflow which is generated by a differential equation such as a parabolic equation, the whole semiflow does not admit a flow extension in general, but an $\omega$-limit set often does (see Part III for details). We now give some general discussions.

Definition 2.3. A *flow extension* of a semiflow $\Pi = (Z, \mathbb{R}^+)$ is a flow $\tilde{\Pi} = (Z, \mathbb{R})$ such that $\tilde{\Pi}(z,t) = \Pi(z,t)$ $(z \in Z, t \in \mathbb{R}^+)$. A compact, positively invariant set of $\Pi$ is said to admit a *flow extension* if the semiflow restricted to it does.

A least requirement for a semiflow to admit a flow extension is that each forward orbit can be extended backward in 'time'. This leads to the following definition.

Definition 2.4. Given a semiflow $\Pi = (Z, \mathbb{R}^+)$ and a point $z \in Z$, a *backward orbit* (*entire orbit*) *of* $z$ is a continuous function $\psi : \mathbb{R}^- = \{t \in \mathbb{R} | t \le 0\} \to Z$ $(\psi : \mathbb{R} \to Z)$ such that $\psi(0) = z$, and, for any $s \le 0$ $(s \in \mathbb{R})$, $\Pi(\psi(s), t) = \psi(s+t)$ holds for $0 \le t \le -s$ $(0 \le t)$.

*Remark 2.1.* Clearly, if $\psi$ is a backward orbit of $z$, then $\psi$ can be extended to an entire orbit

$$\bar{\psi}(s) = \begin{cases} \psi(s), & s \le 0, \\ \Pi(z,s), & s \ge 0. \end{cases} \tag{2.2}$$

Conversely, any entire orbit $\psi$ of $z$ when restricted to $\mathbb{R}^-$ is a backward orbit of $z$.

Proposition 2.1. *Let* $\Pi = (Z, \mathbb{R}^+)$ *be a semiflow and* $z_0 \in Z$ *be such that* $\{\Pi(z_0, t) \mid t \ge t_0 \ge 0\}$ *is relatively compact. Then any* $z \in \omega(z_0)$ *admits a backward orbit in* $\omega(z_0)$.

*Proof.* Let $z \in \omega(z_0)$. Then there is an increasing sequence $\{t_n\} \subset \mathbb{R}^+$ such that $\Pi(z_0, t_n) \to z$. For any $k \in \mathbb{Z}^-$, we note that $\{\Pi(z_0, k + t_n) \mid n \gg 1\}$ is relatively compact. Therefore, by taking a subsequence, $\lim_n \Pi(z_0, k + t_n)$ exists. Let $x_k = \lim_n \Pi(z_0, k+t_n)$, and $\psi(s) = \Pi(x_k, s-k)$ for $k \le s \le k+1$, $k = -1, -2, \cdots$. Clearly, $\psi$ defines a backward orbit of $z$, and $\{\psi(s) | s \le 0\} \subset \omega(z_0)$. $\square$

Next, we observe the following.

Proposition 2.2. *Let* $\Pi = (Z, \mathbb{R}^+)$ *be a semiflow which admits a flow extension* $\tilde{\Pi} = (Z, \mathbb{R})$. *Then every point in* $Z$ *must have a unique backward orbit.*

*Proof.* First, any $z \in Z$ clearly admits a backward orbit $\psi : \mathbb{R}^- \to Z$, $\psi(s) = \tilde{\Pi}(z,s)$ $(s \in \mathbb{R}^-)$. Suppose that $z \in Z$ is a point which admits two backward orbits $\psi_1$ and $\psi_2$, that is, there is a $s_0 < 0$ such that $\psi_1(s_0) \neq \psi_2(s_0)$. Then $\tilde{\Pi}(\psi_i(s_0), -s_0) = \Pi(\psi_i(s_0), -s_0) = \psi_i(s_0 - s_0) = \psi_i(0) = z_0$, $i = 1, 2$. This is impossible since $\tilde{\Pi}$ is a flow. □

By the above proposition, we see already that a flow $(X, \mathbb{R})$, when viewed as a semiflow $(X, \mathbb{R}^+)$ admits no other flow extension except the flow itself.

Definition 2.5. We say that a semiflow $\Pi = (Z, \mathbb{R}^+)$ admits a *unique backward extension* (a *backward extension*) if each $z \in Z$ has a unique (at least one) backward orbit. A compact, positively invariant set of $\Pi$ admits a *unique backward extension* (a *backward extension*) if the semiflow restricted to it does.

We now show that the concepts of flow extension and unique backward extension are essentially the same.

Theorem 2.3. *Consider a semiflow* $\Pi = (Z, \mathbb{R}^+)$*, where* $Z$ *is locally compact. Then the semiflow has a flow extension if and only if it admits a unique backward extension.*

*Proof.* The 'only if' part is stated in Proposition 2.2. We now prove the 'if' part. Since each point in $Z$ admits a unique backward orbit, there is a function $\psi : Z \times \mathbb{R}^- \to Z$ such that for fixed $s \leq 0$, $\psi(\cdot, s) : Z \to Z$ is one to one and for fixed $z \in Z$, $\Pi(\psi(z, s), t) = \psi(z, s + t)$ for $0 \leq t \leq -s$. Define $\tilde{\Pi} : Z \times \mathbb{R} \to Z$ as follows:

$$\tilde{\Pi}(z, t) = \begin{cases} \Pi(z, t) & \text{if } t \geq 0, \\ \psi(z, t) & \text{if } t \leq 0. \end{cases} \tag{2.3}$$

We only need to check the continuity of $\tilde{\Pi}$.

Let $z_0 \in Z$. $\tilde{\Pi}$ is clearly continuous at $(z_0, t_0)$ if $t_0 > 0$. Note that for any $t < 0$, the unique backward extension implies that $\tilde{\Pi}(\cdot, t)$ is one to one in $z$, and $\tilde{\Pi}^{-1}(\cdot, t) = \tilde{\Pi}(\cdot, -t) = \Pi(\cdot, -t)$ is continuous in both $z$ and $t$. By the inverse function theorem ([16]), $\tilde{\Pi}$ is continuous at $(z_0, t_0)$ if $t_0 < 0$. For $\tau_0 \neq 0$, note that $\tilde{\Pi}(z, \tau_0 + t) = \tilde{\Pi}(\tilde{\Pi}(z, \tau_0), t)$ $(z \in Z, t \in \mathbb{R})$. We now let $t \to -\tau_0$, $z \to z_0$ to conclude that $\lim_{z \to z_0, s \to 0} \tilde{\Pi}(z, s) = z$, that is, $\tilde{\Pi}$ is also continuous at $(z_0, 0)$. □

Definition 2.6. A compact, positively invariant set $E$ of a semiflow $(Z, \mathbb{R}^+)$ is *minimal* if it contains no non-empty, closed, proper positively invariant subset. If $Z$ itself is minimal, then $(Z, \mathbb{R}^+)$ is called a *minimal semiflow*.

Proposition 2.4. *Suppose that a minimal semiflow* $(Z, \mathbb{R}^+)$ *admits a flow extension* $(Z, \mathbb{R})$*. Then* $(Z, \mathbb{R})$ *is a minimal flow.*

*Proof.* It is easy to see that $(Z, \mathbb{R}^+)$ is minimal semiflow if and only if each forward orbit $\{z \cdot t \mid t \geq 0\}$ $(z \in Z)$ is dense. It follows that each orbit $\{z \cdot t \mid t \in \mathbb{R}\}$ $(z \in Z)$ in $(Z, \mathbb{R})$ is dense, that is, $(Z, \mathbb{R})$ is minimal. □

Definition 2.7. Let $E$ be a compact positively invariant set of (2.1) which admits a flow extension. $E$ is said to be of *positive* (*negative*) *fiber distal type* if for any $y \in Y$, any two points on $E \cap p^{-1}(y)$ are positively (negatively) distal, where $p : X \times Y \to Y$ denotes the natural projection.

We note that if a compact, positively invariant set $E$ of (2.1) is of positive fiber distal type, then for any $(z, y) \in E$, it has at most one backward orbit. To see this, suppose that there is a point $(z_0, y_0) \in E$ which has two backward orbits $\psi_1(s)$, $\psi_2(s)$ $(s \leq 0)$, that is, there is a $s_0 < 0$ with $\psi_1(s_0) \neq \psi_2(s_0)$. Since $\Pi(\psi_i(s_0), t) = \psi_i(s_0 + t)$, $i = 1, 2$, for $0 \leq t \leq -s_0$, in particular, one has $\Pi(\psi_i(s_0), -s_0) = \psi_i(0) = z_0$, $i = 1, 2$. On the other hand, $\psi_1(s_0)$, $\psi_2(s_0)$ are positively distal, that is,

$$\inf_{t \geq 0} d(\Pi(\psi_1(s_0), t), \Pi(\psi_2(s_0), t)) > 0, \tag{2.4}$$

where $d$ denotes the metric on $E$, a contradiction. By the above discussion and Proposition 2.1, we have the following.

Proposition 2.5.

1) *If* $(x_0, y_0) \in X \times Y$ *is such that* $\{\Pi(x_0, y_0, t) \mid t \geq \delta_0\}$ *for some* $\delta_0 > 0$ *is relatively compact and* $(\omega(x_0, y_0), \mathbb{R}^+)$ *is of positive fiber distal type, then* $\omega(x_0, y_0)$ *admits a flow extension.*

2) *If* $z_0 \in Z$ *is such that* $\{\Pi(z_0, t) | t \geq \delta_0 > 0\}$ *is relatively compact and* $(\omega(z_0), \mathbb{R}^+)$ *is positively distal, then* $\omega(z_0)$ *admits a flow extension.*

*Proof.* 2) is an immediate consequence of 1) by taking $(Y, \mathbb{R})$ as the trivial flow. □

### 2.2. Lifting Flow Associated to a Semiflow.

The positive distality of a compact, positively invariant set $E$ of a semiflow can sometimes be implied by a stability condition. To illustrate the idea, following [29], let us introduce the concept of the lifting flow associated to a semiflow.

Consider a compact semiflow $\Pi = (Z, \mathbb{R}^+)$ and denote $d$ as the metric on $Z$. We assume that every $z \in Z$ admits a backward orbit hence an entire orbit (e.g., $Z$ is an $\omega$-limit set of a semiflow, see Proposition 2.1). Let $Z^*$ be the set of entire orbits of $(Z, \mathbb{R}^+)$, that is,

$$Z^* = \{\psi \in C(\mathbb{R}, Z) | \Pi(\psi(s), t) = \psi(t + s), t \geq 0\}. \tag{2.5}$$

We note that $Z^*$ is compact with respect to the compact open topology on $C(\mathbb{R}, Z)$. Note also that the compact open topology on $C(\mathbb{R}, Z)$ is metrizable, for instance, a metric $d^*$ on $Z^*$ can be defined as follows: For any $\psi_1, \psi_2 \in Z^*$,

$$d^*(\psi_1, \psi_2) = \sum_{n=1}^{\infty} \frac{d_n(\psi_1, \psi_2)}{2^n}, \tag{2.6}$$

where $d_n(\psi_1, \psi_2) = \max_{-n \leq s \leq n} d(\psi_1(s), \psi_2(s))$.

DEFINITION 2.8. The flow $\Pi^* = (Z^*, \mathbb{R})$: $\Pi^*(\psi, t)(s) \equiv \psi(t+s)$ is called the *lifting flow* associated to a compact semiflow $(Z, \mathbb{R}^+)$.

It is easy to verify that $\Pi^*$ such defined is indeed a flow.

Now, consider the mapping $p^* : Z^* \to Z$,

$$p^*(\psi) = \psi(0). \tag{2.7}$$

$p^*$ is clearly continuous, onto and semiflow preserving, that is,

$$p^*(\Pi^*(\psi, t)) = \Pi(p^*(\psi), t) \tag{2.8}$$

$(\psi \in Z^*, t \geq 0)$.

PROPOSITION 2.6. *Let $(Z^*, \mathbb{R})$, $(Z, \mathbb{R}^+)$, $p^*$ be as above. Then $(Z, \mathbb{R}^+)$ admits a flow extension $(\tilde{Z}, \mathbb{R})$ if and only if $p^*$ is* 1-1.

*Proof.* By Theorem 2.3, $(Z, \mathbb{R}^+)$ has a flow extension if and only if it admits a unique backward extension, which is equivalent to say that each $z \in Z$ has a unique entire orbit (Remark 2.1), that is, $p^*$ is 1-1. □

*Remark 2.2.* We note that if $(Z, \mathbb{R}^+)$ admits a flow extension $(Z, \mathbb{R})$, then there is a flow isomorphism between $(Z, \mathbb{R})$ and the lifting flow $(Z^*, \mathbb{R})$. Thus, $(Z^*, \mathbb{R})$ is essentially the flow extension $(Z, \mathbb{R})$ in this case.

### 2.3 Uniform Stability.

Following ideas from [26], we now discuss implications of the uniform stability on flow extensions. Let us consider the skew-product semiflow $\Pi = (X \times Y, \mathbb{R}^+)$ defined in (2.1). Denote $d$ as the metric on $X \times Y$.

DEFINITION 2.9. A forward orbit $\Pi(x_0, y_0, t)$ in $(X \times Y, \mathbb{R}^+)$ is said to be *uniformly stable* if for every $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$, called the *modulus of uniform stability*, such that $d(\Pi(x_0, y_0, t+\tau), \Pi(x, y_0, t+\tau)) \leq \epsilon$ $(t \geq 0)$ whenever $\tau \geq 0$ and $d(\Pi(x_0, y_0, \tau), \Pi(x, y_0, \tau)) \leq \delta(\epsilon)$.

An $\omega$-limit set $\omega(x_0, y_0)$ is *uniformly stable* if $\Pi(x_0, y_0, t)$ $(t \geq 0)$ is.

LEMMA 2.7. (Sell [28]) *Let $\Pi(x_0, y_0, t)$ be a forward orbit of* (2.1) *which is uniformly stable and relatively compact for $t \geq t_0 \geq 0$. Then for every $(x_*, y_*) \in \omega(x_0, y_0)$, $\Pi(x_*, y_*, t)$ is uniformly stable with the same modulus of uniform stability as that of $\Pi(x_0, y_0, t)$.*

THEOREM 2.8. *Consider the skew-product semiflow* (2.1) *and assume that $(Y, \mathbb{R})$ is minimal and distal. If an $\omega$-limit set $\omega(x_0, y_0)$ of $\Pi$ is uniformly stable, then it admits a flow extension which is minimal and distal.*

*Proof.* Denote $Z = \omega(x_0, y_0)$. By Proposition 2.1, $(Z, \mathbb{R}^+)$ has a backward extension. Let $(Z^*, \mathbb{R})$ be the lifting flow of the compact semiflow $(Z, \mathbb{R}^+)$.

Denote $d^*$ as the metric on $Z^*$ defined in (2.6), $p^* : Z^* \to Z$ as the projection in (2.7), and $p : X \times Y \to Y$ as the natural projection. We first show that $(Z^*, \mathbb{R})$ is negatively distal hence distal by Corollary 2.5 in Part I. Let $p_0 = p \circ p^*$. It is clear that $p_0 : (Z^*, \mathbb{R}) \to (Y, \mathbb{R})$ is a flow homomorphism. Since $(Y, \mathbb{R})$ is distal, to show $(Z^*, \mathbb{R})$ is negatively distal, it suffices to show that $(Z^*, \mathbb{R})$ is of negative fiber distal type. Suppose not, then there are $\psi_1, \psi_2 \in Z^*$ with $p_0\psi_1 = p_0\psi_2$ such that

$$\inf_{t<0} d^*(\Pi^*(\psi_1, t), \Pi^*(\psi_2, t)) = 0. \tag{2.9}$$

Since $\psi_1 \neq \psi_2$, without loss of generality, we may assume that $\psi_1(0) \neq \psi_2(0)$. Let $\epsilon_0 = \frac{1}{2} d(\psi_1(0), \psi_2(0))$ and let $\delta_0 \equiv \delta_0(\epsilon_0)$ be the modulus of uniform stability on $Z$ (see Lemma 2.7). By (2.9), there is a $\tau_0 < 0$ such that

$$d^*(\Pi^*(\psi_1, \tau_0), \Pi^*(\psi_2, \tau_0)) = \sum_{n=1}^{\infty} \frac{d_n}{2^n} < \delta_0, \tag{2.10}$$

where

$$\begin{aligned} d_n &= \max_{-n \le s \le n} d(\Pi^*(\psi_1, \tau_0)(s), \Pi^*(\psi_2, \tau_0)(s)) \\ &= \max_{-n \le s \le n} d(\psi_1(s + \tau_0), \psi_2(s + \tau_0)). \end{aligned} \tag{2.11}$$

It follows that there is a $n_0 \geq 1$ such that

$$d(\psi_1(\tau_0), \psi_2(\tau_0)) \le d_{n_0} < \delta_0. \tag{2.12}$$

But the uniform stability implies that

$$2\epsilon_0 = d(\psi_1(0), \psi_2(0)) = d(\Pi(\psi_1(\tau_0), -\tau_0), \Pi(\psi_2(\tau_0), -\tau_0)) < \epsilon_0,$$

a contradiction. Thus, $(Z^*, \mathbb{R})$ is distal (negatively distal in particular). Since $p^* : (Z^*, \mathbb{R}^+) \to (Z, \mathbb{R}^+)$ preserves semiflows, $(Z, \mathbb{R}^+)$ is positively distal. By Proposition 2.5 2), $(Z, \mathbb{R}^+)$ admits a flow extension $(Z, \mathbb{R})$ and by Corollary 2.5 in Part I, $(Z, \mathbb{R})$ is distal. The minimality of $(Z, \mathbb{R})$ follows from arguments of [26]. □

## 2.4. Contracting Semiflows.

We now consider a different type of stability which has a strong implication on dynamics of a skew-product semiflow.

Again, let $\Pi = (X \times Y, \mathbb{R}^+)$ be the skew-product semiflow (2.1) on $Z = X \times Y$. Denote

$$\hat{Z} = \{((x_1, y), (x_2, y)) | x_1, x_2 \in X, y \in Y\} \subset Z \times Z. \tag{2.13}$$

DEFINITION 2.10. The skew-product semiflow (2.1) is said to be (*strictly*) *contracting* if there is a continuous function $L : \hat{Z} \to \mathbb{R}^+$, called a *Lyapunov function*, which satisfies the following properties:

1) $L((x_1, y), (x_2, y)) = 0$ if and only if $(x_1, y) = (x_2, y)$;
2) If $(x_1, y) \neq (x_2, y)$, then $L(\Pi(x_1, y, t), \Pi(x_2, y, t))(<) \leq L((x_1, y), (x_2, y))$ $(t > 0)$.

A strictly contracting skew-product semiflow has the following properties.

THEOREM 2.9. *Let* $\Pi = (X \times Y, \mathbb{R}^+)$ *be a strictly contracting skew-product semiflow. Assume that* a) $(Y, \mathbb{R})$ *is distal;* b) *there is a point* $(x_0, y_0) \in X \times Y$ *with relatively compact forward orbit. Then the following holds.*

1) $\Pi$ *has a unique compact, positively invariant subset* $E$*;*
2) $(E, \mathbb{R}^+)$ *admits a flow extension* $(E, \mathbb{R})$*;*
3) $p : (E, \mathbb{R}) \to (Y, \mathbb{R})$ *is a* 1-1 *extension, where* $p : X \times Y \to Y$ *denotes the natural projection;*
4) $E$ *is the* global attractor *of* $\Pi$*, that is, if* $(x, y) \in X \times Y$ *has a relatively compact forward orbit, then* $d(\Pi(x, y, t), \Pi(x^*, y, t)) \to 0$ *as* $t \to +\infty$*, where* $(x^*, y) = E \cap p^{-1}(y)$*,* $d$ *is the metric on* $X \times Y$*.*

*Remark 2.3.* If $(Y, \mathbb{R})$ is almost periodic, then the conclusion 4) of Theorem 2.9 simply says that every relatively compact forward orbit is asymptotically almost periodic since $(E, \mathbb{R})$ is almost periodic.

LEMMA 2.10. *Assume that* $\Pi = (X \times Y, \mathbb{R}^+)$ *is a contracting skew-product semiflow with* $(Y, \mathbb{R})$ *being distal and let* $E$ *be a compact, positively invariant set of* $\Pi$ *which admits a backward extension. Then the following holds.*

1) $E$ *has a flow extension* (*i.e.,* $E$ *has a unique backward extension*)*;*
2) $(E, \mathbb{R})$ *is distal;*
3) *If* $\Pi$ *is strictly contracting, then* $p : (E, \mathbb{R}) \to (Y, \mathbb{R})$ *is a* 1-1 *extension.*

*Proof.* We let $\Pi^* = (E^*, \mathbb{R})$ be the lifting flow of $(E, \mathbb{R}^+)$ defined in Definition 2.8. Let

$$\mathcal{E}^* = \{(\psi_1, \psi_2) | \psi_1, \psi_2 \in E^*, pp^*\psi_1 = pp^*\psi_2\}, \tag{2.14}$$

where $p^* : E^* \to E$ is the projection $p^*\psi = \psi(0)$. Define $L^* : \mathcal{E}^* \to \mathbb{R}^+$:

$$L^*(\psi_1, \psi_2) = L(\psi_1(0), \psi_2(0)), \tag{2.15}$$

where $L$ is the Lyapunov function on $\tilde{E} = \{((x_1, y), (x_2, y)) | (x_1, y), (x_2, y) \in E\}$. It is easy to see that $L^*$ is also continuous, and moreover,

$$L^*(\Pi^*(\psi_1, t), \Pi^*(\psi_2, t)) \leq L^*(\psi_1, \psi_2) \tag{2.16}$$

for any $(\psi_1, \psi_2) \in \mathcal{E}^*$ and $t > 0$. Now, let $d^*$ be the metric on $\mathcal{E}^*$ defined in (2.6) and $\psi_1, \psi_2 \in \mathcal{E}^*$ with $\psi_1 \neq \psi_2$. Then $\inf_{t\leq 0} d^*(\Pi^*(\psi_1, t), \Pi^*(\psi_2, t)) > 0$. For otherwise, there is a sequence $t_n \to -\infty$ such that $d^*(\Pi^*(\psi_1, t_n), \Pi^*(\psi_2, t_n)) \to 0$. By continuity, $L^*(\Pi^*(\psi_1, t_n), \Pi^*(\psi_2, t_n)) \to 0$. But (2.16) implies that, as $n \gg 1$, $L^*(\Pi^*(\psi_1, t_n), \Pi^*(\psi_2, t_n)) \geq L^*(\Pi^*(\psi_1, t), \Pi^*(\psi_2, t)))$ for any $t \in \mathbb{R}$. Therefore $\psi_1(t) = \psi_2(t)$, a contradiction. This together with the distality of $(Y, \mathbb{R})$ shows that $(E^*, \mathbb{R})$ is negatively distal, hence distal by Part I, Corollary 2.5. It follows that $(E, \mathbb{R}^+)$ is positively distal, and therefore, $(E, \mathbb{R}^+)$ admits a flow extension $(E, \mathbb{R})$. By Part I, Corollary 2.5 again, $(E, \mathbb{R})$ is distal. This proves 1) and 2).

To prove 3), suppose for contradiction that there is a $y_0 \in Y$ such that $E \cap p^{-1}(y_0)$ contains two distinct points $(x_1, y_0)$, $(x_2, y_0)$. Let $\hat{E} = cl\{((x_1, y_0) \cdot t, (x_2, y_0) \cdot t) \mid t \in \mathbb{R}\}$. Clearly, as a subflow of the product flow $(E \times E, \mathbb{R})$, $(\hat{E}, \mathbb{R})$ is distal. Therefore, the Ellis semigroup $E(\hat{E})$ is a group (Part I, Theorem 2.4). By Part I, Remark 2.3, $(E(\hat{E}), \mathbb{R})$ is minimal. Denote $\hat{\Pi}$ as the flow on $\hat{E}$ and $e$ as the identity in $E(\hat{E})$. Then there is a net $\{t_n\} \subset \mathbb{R}_-$ such that $\hat{\Pi}(\cdot, t_n) \to e$ in $E(\hat{E})$. It follows that $(x_i, y_0) \cdot t_n \to (x_i, y_0)$ $(i = 1, 2)$. Fix a $n_0$. By the strictly contracting property, $L((x_1, y_0), (x_2, y_0)) = \lim_{n\to\infty} L((x_1, y_0) \cdot t_n, (x_2, y_0) \cdot t_n) \geq L((x_1, y_0) \cdot t_{n_0}, (x_2, y_0) \cdot t_{n_0}) > L((x_1, y_0), (x_2, y_0))$, a contradiction. □

*Proof of Theorem 2.9.* Let $E = \omega(x_0, y_0)$. By Proposition 2.1, $E$ admits a backward extension. 2) and 3) then follow from Lemma 2.10 immediately.

Now, suppose $\Pi$ has two compact, positively invariant subsets $E_1$, $E_2$. Note that both $(E_1, \mathbb{R})$ and $(E_2, \mathbb{R})$ are 1-1 extensions of $(Y, \mathbb{R})$. For fixed $y_0 \in Y$, denote $(x_i, y_0) = E_i \cap p^{-1}(y_0)$. Then $(x_1, y_0) \neq (x_2, y_0)$ and $(x_1, y_0)$, $(x_2, y_0)$ are distal. Using exactly the same arguments as in the proof of Lemma 2.10 3), one obtains a contradiction. Therefore, 1) is also true.

To show 4), we only note that if $(x, y) \in X \times Y$ has a relatively compact forward orbit and if $d(\Pi(x, y, t), \Pi(x^*, y, t)) \not\to 0$ as $t \to \infty$, then $\omega(x, y) \neq \omega(x^*, y)$. It follows that $\Pi$ has at least two compact, positively invariant subsets. This is a contradiction to 1). □

## 3. Strongly Order Preserving Dynamics

### 3.1. Ordering on Fibers.

Consider a skew-product semiflow $\Pi = (X \times Y, \mathbb{R}^+)$,

$$\Pi(x, y, t) = (u(x, y, t), y \cdot t) \quad t \geq 0, \tag{3.1}$$

where $(Y, \mathbb{R})$ is minimal. We denote $p : X \times Y \to Y$ as the natural projection.

Assume that each fiber $p^{-1}(y) (y \in Y)$ is an ordered metric space and denote by '$\geq_y$' the partial ordering on $p^{-1}(y)$ $(y \in Y)$. We say $(x_1, y) >_y (x_2, y)$ if $(x_1, y) \geq_y (x_2, y)$ and $x_1 \neq x_2$.

Below, $E$ denotes a compact, positively invariant subset of $\Pi$ which admits a flow extension. Recall that by Part I, Lemma 2.16, the set

$$\begin{aligned} Y_0 =\{y_0 \in Y \mid \ & \text{for any } x_0 \in p^{-1}(y_0) \cap E, y \in Y \text{ and } \{t_n\} \text{ with} \\ & y \cdot t_n \to y_0, \text{ there is a } \{x_n\} \subset E \cap p^{-1}(y) \text{ such that } x_n \cdot t_n \to x_0\} \end{aligned} \tag{3.2}$$

is residual.

Definition 3.1. For each $y \in Y$, a *fiberwise strong ordering* '$\gg_y$' on $p^{-1}(y) \cap E$ is defined as follows: $(x_1, y) \gg_y (x_2, y)$ if and only if there are neighborhoods $\mathcal{N}_1$, $\mathcal{N}_2 \subset p^{-1}(y) \cap E$ of $(x_1, y)$, $(x_2, y)$ respectively such that $(x_1^*, y) >_y (x_2^*, y)$ for all $(x_i^*, y) \in \mathcal{N}_i$ $(i = 1, 2)$.

Definition 3.2. $(x_0^1, y_0), (x_0^2, y_0) \in E$ form a *strongly order preserving pair* if $(x_0^1, y_0)$, $(x_0^2, y_0)$ are fiberwise strongly ordered, say $(x_0^1, y_0) \gg_{y_0} (x_0^2, y_0)$, and there are neighborhoods $U_1, U_2$ of $(x_0^1, y_0)$, $(x_0^2, y_0)$ in $E$ respectively such that whenever $(x_1, y_0), (x_2, y_0) \in p^{-1}(y_0) \cap E$, $(x_1, y_0) \neq (x_2, y_0)$, and $\Pi(x_i, y_0, T) \in U_i$ $(i = 1, 2)$ for some $T < 0$, then $(x_1, y_0) \gg_{y_0} (x_2, y_0)$.

Theorem 3.1. *Let $K$ be a minimal set of* (3.1) *having flow extension and $Y_0$ be as in* (3.2) *for $E := K$. Then for any $y \in Y_0$, $K \cap p^{-1}(y)$ admits no strongly order preserving pair.*

*Proof.* Fix a $y_0 \in Y_0$. If $card K \cap p^{-1}(y_0) = 1$, then the theorem is proved since by (3.2), $card K \cap p^{-1}(y) = 1$ for all $y \in Y_0$. We now assume that $card K \cap p^{-1}(y_0) > 1$ and $K \cap p^{-1}(y_0)$ contains a strongly order preserving pair $(x_0^1, y_0)$, $(x_0^2, y_0)$. Without loss of generality, we assume that $(x_0^1, y_0) \gg_{y_0} (x_0^2, y_0)$. Let $(x, y_0) \in K \cap p^{-1}(y_0)$. By the minimality of $K$ and (3.2), there are sequences $t_n \to -\infty$ and $\{(x_n, y_0)\} \subset K \cap p^{-1}(y_0)$ such that

$$(x_n, y_0) \cdot t_n \to (x_0^1, y_0),$$

$$(x, y_0) \cdot t_n \to (x_0^2, y_0).$$

It follows that there is a $n_0$ sufficiently large such that $(x_{n_0}, y_0) \cdot t_{n_0} \in U_1$ and $(x, y_0) \cdot t_{n_0} \in U_2$. Thus $(x_{n_0}, y_0) \gg_{y_0} (x, y_0)$, that is, the set

$$A(x, y_0) = \{(x_0, y_0) \in K \cap p^{-1}(y_0) \mid (x_0, y_0) \gg_{y_0} (x, y_0)\} \tag{3.3}$$

is non-empty for any $(x, y_0) \in K \cap p^{-1}(y_0)$.

Next, we claim that there is a maximum element $(x_M, y_0)$ on $K \cap p^{-1}(y_0)$. By the above discussion, for any $(x_*, y_0) \in K \cap p^{-1}(y_0)$, one can find a $(x_0, y_0) \in K \cap p^{-1}(y_0)$ with $(x_0, y_0) \gg_{y_0} (x_*, y_0)$, that is, there is a neighborhood $B(x_*, y_0)$ of $(x_*, y_0)$ in $K \cap p^{-1}(y_0)$ such that $(x_0, y_0) > (x, y_0)$ for all $(x, y) \in B(x_*, y_0)$. Now, $\cup_{(x_*, y_0) \in K \cap p^{-1}(y_0)} B(x_*, y_0)$ forms an open cover of $K \cap p^{-1}(y_0)$. It then has a finite subcover. Without loss of generality, assume that there is an integer $n_0 \geq 1$ and $\{(x_*^i, y_0)\}_{i=1}^{n_0} \subset K \cap p^{-1}(y_0)$ such that $\cup_{i=1}^{n_0} B(x_*^i, y_0) = K \cap p^{-1}(y_0)$. Let $(x_0^i, y_0)$ $(1 \leq i \leq n_0)$ be associated to $(x_*^i, y_0)$ $(1 \leq i \leq n_0)$ as above. By Zorn's lemma, there is a maximum element $(x_M, y)$ of the finite set $\{(x_0^i, y_0)\}_{i=1}^{n_0}$. We claim that any $(x, y_0) \in K \cap p^{-1}(y_0)$ comparable with $(x_M, y_0)$ satisfies $(x_M, y_0) \gg_{y_0} (x, y_0)$. If not, there is a $(x, y_0) \in K \cap p^{-1}(y_0)$ with $(x, y_0) >_{y_0} (x_M, y_0)$, then $(x, y_0) \in B(x_*^{i_0}, y_0)$ for some $i_0$, that is, $(x_0^{i_0}, y_0) >_{y_0} (x, y_0) >_{y_0} (x_M, y_0)$. This contradicts the fact that $(x_M, y_0)$ is a maximum element of $\{(x_0^i, y_0)\}_{i=1}^{n_0}$. Thus, $(x_M, y_0)$ is a maximum element on $K \cap p^{-1}(y_0)$. But $A(x_M, y_0) = \{(x, y_0) | (x, y_0) \gg_{y_0} (x_M, y_0)\}$ is non-empty, again a contradiction. □

### 3.2. Nonordering Principle.

We consider skew-product semiflow $\Pi$ in (3.1).

DEFINITION 3.3. $X \times Y$ is *strongly ordered* if there is a closed subset

$$O_+(X, Y) \subset \Delta(X, Y) = \{((x_1, y), (x_2, y)) | x_1, x_2 \in X, y \in Y\}$$

with the following properties:

1) $\operatorname{Int} O_+(X, Y) \neq \emptyset$ relative to the subset topology of $\Delta(X, Y)$;
2) $((x, y), (x, y)) \in O_+(X, Y)$ for any $(x, y) \in X \times Y$;
3) If $((x_1, y), (x_2, y))$, $((x_2, y), (x_1, y)) \in O_+(X, Y)$, then $x_1 = x_2$;
4) If $((x_1, y), (x_2, y))$, $((x_2, y), (x_3, y)) \in O_+(X, Y)$, then $((x_1, y), (x_3, y)) \in O_+(X, Y)$.

The set $O_+(X, Y)$ induces a (strong) *partial ordering* '$\geq$' on each fiber $p^{-1}(y)$ $(y \in Y)$ as follows:

$$\begin{aligned}
(x_1, y) \geq (x_2, y) &\iff ((x_1, y), (x_2, y)) \in O_+(X, Y); \\
(x_1, y) > (x_2, y) &\iff (x_1, y) \geq (x_2, y), (x_1, y) \neq (x_2, y); \\
(x_1, y) \gg (x_2, y) &\iff ((x_1, y), (x_2, y)) \in \operatorname{Int} O_+(X, Y).
\end{aligned}$$

Let $O_-(X, Y)$ be the reflection of $O_+(X, Y)$, that is,

$$O_-(X, Y) = \{((x_1, y), (x_2, y)) | ((x_2, y), (x_1, y)) \in O_+(X, Y)\}.$$

DEFINITION 3.4.

1) The set $O(X, Y) = O_+(X, Y) \cup O_-(X, Y)$ is referred to as the *order relation*, that is, $(x_1, y_1)$, $(x_2, y_2)$ are *ordered* if and only if $y_1 = y_2 = y$ and $((x_1, y), (x_2, y)) \in O(X, Y)$.

2) For $K \subset X \times Y$, $O(K) \equiv O(X, Y) \cap (K \times K)$ is called the *order relation on* $K$.

DEFINITION 3.5. $\Pi$ is said to be *strongly order preserving* if $X \times Y$ is strongly ordered and whenever $(x_1, y) > (x_2, y)$, then $\Pi(x_1, y, t) \gg \Pi(x_2, y, t)$ for all $t > 0$.

THEOREM 3.2. *Assume that $\Pi$ is strongly order preserving and let $K$ be a minimal set of $X \times Y$ which admits a flow extension. Then there is a residual set $Y_0 \subset Y$ such that for any $y \in Y_0$, no two elements on $K \cap p^{-1}(y)$ are ordered.*

*Proof.* Let $Y_0$ be the residual sets defined in (3.2) for $E := K$. $Y_0$ is clearly invariant. Suppose for some $y_0 \in Y_0$, there is an ordered pair $(x_0^1, y_0)$, $(x_0^2, y_0)$, say $(x_0^1, y_0) > (x_0^2, y_0)$. By strong order preserving property, if $t_0 > 0$, then $(x_0^1, y_0) \cdot t_0 \gg (x_0^2, y_0) \cdot t_0$. Since it is easy to see that $(x_0^1, y_0) \cdot t_0$, $(x_0^2, y_0) \cdot t_0$ forms a strongly order preserving pair in the sense of Definition 3.2, one has a contradiction to Theorem 3.1. □

A consequence of the above theorem is the following.

COROLLARY 3.3. *Let $K$ be as in* Theorem 3.2.

1) *If $(x_1, y)$, $(x_2, y) \in K$ are ordered, then they are proximal, that is, the order relation implies the proximal relation on $K$.*
2) *If $(K, \mathbb{R})$ is distal, then no two points on a same fiber are ordered.*

*Proof.* 1) Let $Y_0$ be as in Theorem 3.2. If $(x_1, y)$, $(x_2, y) \in K$ are ordered but not proximal, then there is a $\delta_0 > 0$ such that

$$d((x_1, y) \cdot t, (x_2, y) \cdot t) \geq \delta_0 \quad (t \in \mathbb{R}), \tag{3.4}$$

where $d$ denotes the metric on $K$. Now, take $y_0 \in Y_0$ and let $\{t_n\} \to \infty$ be a sequence such that $y \cdot t_n \to y_0$. By taking a subsequence if necessary, one can assume that $(x_1, y) \cdot t_n \to (x_1^*, y_0)$ and $(x_2, y) \cdot t_n \to (x_2^*, y_0)$. By (3.4), $(x_1^*, y_0) \neq (x_2^*, y_0)$. Since $(x_1, y)$, $(x_2, y)$ are ordered, it follows from the strong order preserving property and the closeness of the order relation that $(x_1^*, y_0)$, $(x_2^*, y_0) \in K \cap p^{-1}(y_0)$ are also ordered, which contradicts Theorem 3.2.

2) follows from 1) and the distality of $(K, \mathbb{R})$. □

PROPOSITION 3.4. *Let $\omega(x_0, y_0)$ be an $\omega$-limit set of $\Pi$ which admits a flow extension. Assume that either $(x_0, y_0) \geq (x, y_0)$ or $(x_0, y_0) \leq (x, y_0)$ $((x, y_0) \in \omega(x_0, y_0) \cap p^{-1}(y_0))$. Then there is a residual set $Y_0 \subset Y$ such that for any $y_* \in Y_0$, card$(\omega(x_0, y_0) \cap p^{-1}(y_*)) = 1$. Consequently, $\omega(x_0, y_0)$ contains a unique minimal set $K$. Moreover, $(K, \mathbb{R})$ is an almost* 1-1 *extension of* $(Y, \mathbb{R})$ (*By* Part I, Theorem 2.14, $(K, \mathbb{R})$ is almost automorphic minimal if $(Y, \mathbb{R})$ is almost periodic).

*Proof.* We only prove the case when $(x_0, y_0) \geq (x, y_0)$ for all $(x, y_0) \in \omega(x_0, y_0) \cap p^{-1}(y_0)$. Let $Y_0$ be the residual set defined in (3.2) for $E = \omega(x_0, y_0)$. For any $y_* \in Y_0$ and $(x, y_*), (x_*, y_*) \in \omega(x_0, y_0) \cap p^{-1}(y_*)$, if $(x, y_*) \neq (x_*, y_*)$, then by (3.2), there is a sequence $\{t_n\} \to \infty$ and $\{(x_n, y_0)\} \subset \omega(x_0, y_0) \cap p^{-1}(y_0)$ such that $(x_0, y_0) \cdot (t_n + t) \to (x, y_*) \cdot t$ and $(x_n, y_0) \cdot (t_n + t) \to (x_*, y_*) \cdot t$ as $n \to \infty$

for all $t \in \mathbb{R}$. Fix a $t_0 < 0$. Then $(x_0, y_0) \cdot (t_n + t_0) \gg (x_n, y_0) \cdot (t_n + t_0)$ for $n \gg 1$. Since $O(X, Y)$ is closed, $(x, y_*) \cdot t_0 > (x_*, y_*) \cdot t_0$. This implies that $(x, y_*) \gg (x_*, y_*)$. Similarly, one also has $(x_*, y_*) \gg (x, y_*)$, a contradiction. Thus, $card(\omega(x_0, y_0) \cap p^{-1}(y_*)) = 1$ ($y_* \in Y_0$). Now choose a $y_* \in Y_0$ and denote $\omega(x_0, y_0) \cap p^{-1}(y_*) = (x_*, y_*)$. Clearly, $K = cl\{(x_*, y_*) \cdot t \mid t \in \mathbb{R}\}$ is the only minimal set of $\omega(x_0, y_0)$, and, $(K, \mathbb{R})$ is an almost 1-1 extension of $(Y, \mathbb{R})$. □

*Remark 3.1.* If $\Pi$ is strongly order preserving, then the space $X \times Y$ is in general not totally ordered, that is, $O(X, Y) \neq \Delta(X, Y)$. To see this, let $E$ be a compact, positively invariant set of $\Pi$ which admits a flow extension. Denote the flow $(E, \mathbb{R})$ again by $\Pi$ and denote

$$\tilde{E} = \{((x_1, y), (x_2, y)) \in \Delta(X, Y) \mid (x_1, y), (x_2, y) \in E\},$$

$$\tilde{E}_0 = \{((x_1, y), (x_2, y)) \in E \mid x_1 \neq x_2\}.$$

Then both $\tilde{E}$ and $\tilde{E}_0$ are invariant under the flow $\tilde{\Pi} = \Pi \times \Pi$. Now, for a fixed $t_0 > 0$,

$$\tilde{\Pi}(\cdot, t_0) : \tilde{E}_0 \to \tilde{E}_0$$

is a homeomorphism. Thus $\tilde{E}_0 = \tilde{\Pi}(\tilde{E}_0, t_0)$. If $X \times Y$ is totally ordered and if $\Pi$ is strongly order preserving, then $\tilde{\Pi}(\tilde{E}_0, t_0) \subset \operatorname{Int} \tilde{E}$. It follows that $\tilde{E}_0 = \operatorname{Int} \tilde{E}$. The above discussion implies that for each $y \in Y$ the order topology on $p^{-1}(y) \cap E$ generated by '$\geq$' agrees with its metric topology. In fact, the total orderness is much more restrictive even to spaces where the order topology is the same as the original one.

## 4. Strong Monotonicity

Below, we use the same symbol $\|\cdot\|$ to denote a vector or an operator norm unless specified otherwise.

### 4.1. Linearized Skew-product Semiflow.

Consider a skew-product semiflow $\Pi = (X \times Y, \mathbb{R}^+)$,

$$\Pi(x, y, t) = (u(x, y, t), y \cdot t), \quad t \geq 0, \tag{4.1}$$

where $X$ is a Banach space, $(Y, \mathbb{R})$ is a minimal flow. Assume that $u$ is $C^{1+\alpha}$ $(0 < \alpha \leq 1)$ in $x \in X$, that is, $u$ is $C^1$ in $x$, and $u_x$ is continuous in $y \in Y$, $t > 0$ and is $C^\alpha$ in $x$, moreover, for any $v \in X$,

$$u_x(x, y, t)v \to v \quad as \quad t \to 0+ \tag{4.2}$$

uniformly for $(x, y)$ in compact subsets of $X \times Y$. We note that the Hölder continuity condition above will only be used in Section 4.4.

Throughout the rest of this section, we let $K \subset X \times Y$ be a compact, positively invariant set which admits a flow extension. Define

$$\Phi(x, y, t) = u_x(x, y, t) \tag{4.3}$$

for $(x, y) \in K$, $t \geq 0$. The operator $\Phi$ generates a linear skew-product semiflow $L = (X \times K, \mathbb{R}^+)$, called the *linearized skew-product semiflow of* (4.1) *over* $K$, as follows:

$$L(v, (x, y), t) = (\Phi(x, y, t)v, \Pi(x, y, t)), \quad t \geq 0, \quad (x, y) \in K, \quad v \in X. \tag{4.4}$$

We note that $\Phi$ satisfies the following semi-cocyle property:

$$\Phi(x, y, t + s) = \Phi(\Pi(x, y, s), t)\Phi(x, y, s), \quad s, t \in \mathbb{R}^+, \quad (x, y) \in K. \tag{4.5}$$

Certain hyperbolicity or stability conditions of a compact invariant set $K$ of (4.1) can be characterized in terms of its linearized skew-product semiflow.

Definition 4.1. The linear skew-product semiflow (4.4) is said to have an *exponential dichotomy* (*ED*) over $K$ if there exist $\beta > 0$, $C > 0$ and continuous projections $P(x, y) : X \to X$, such that for any $(x, y) \in K$ the following holds:

1) $\Phi(x, y, t)P(x, y) = P(\Pi(x, y, t))\Phi(x, y, t), \quad t \in \mathbb{R}^+$;
2) $\Phi(x, y, t)|_{R(P(x,y))} : R(P(x, y)) \to R(P(\Pi(x, y, t)))$ is an isomorphism for $t \in \mathbb{R}^+$ (hence $\Phi(x, y, -t) := \Phi^{-1}(\Pi(x, y, -t), t)$, $R(P(x, y)) \to R(P(\Pi(x, y, -t)))$ is well defined for $t \in \mathbb{R}^+$);
3)
$$\|\Phi(x, y, t)(I - P(x, y))\| \leq Ce^{-\beta t}, \quad t \in \mathbb{R}^+,$$
$$\|\Phi(x, y, t)P(x, y)\| \leq Ce^{\beta t}, \quad t \in \mathbb{R}^-.$$

For any given $\lambda \in \mathbb{R}$, we now consider the skew-product semiflow $L_\lambda = (X \times K, \mathbb{R}^+)$,

$$L_\lambda(v, x, y, t) = (e^{-\lambda t}\Phi(x, y, t)v, \Pi(x, y, t)), \quad t \geq 0, \quad (x, y) \in K. \tag{4.6)$_\lambda$ $(}$$

Definition 4.2. The set $\Sigma(K) = \{\lambda \in \mathbb{R} | (4.6)_\lambda \text{ admits no ED over } K\}$ is called the *Sacker-Sell* or the *dynamic spectrum over* $K$.

Definition 4.3. For $(x, y) \in K$, we define the *Lyapunov exponent* $\lambda(x, y)$ as

$$\lambda(x, y) = \overline{\lim}_{t \to \infty} \frac{\ln \|\Phi(x, y, t)\|}{t}.$$

The number

$$\lambda_K = \sup_{(x,y) \in K} \lambda(x, y)$$

is called the *upper Lyapunov exponent on* $K$. If $\lambda_K \leq 0$, then $K$ is said to be *linearly stable.*

Proposition 4.1. $\sup \Sigma(K) = \lambda_K$.

*Proof.* We note that there are $\bar{C} > 0$, $\omega \in \mathbb{R}$ such that

$$\|\Phi(x, y, t)\| \leq \bar{C} e^{\omega t} \tag{4.7}$$

for any $(x, y) \in K$ and $t \geq 0$. To see this, we claim that there are $\bar{C}, \eta > 0$ such that $\|\Phi(x, y, t)\| \leq \bar{C}$ for any $(x, y) \in K$ and $t \in [0, \eta]$. If not, then there exist $(x_n, y_n) \in K$ and $t_n \to 0+$ such that $\|\Phi(x_n, y_n, t_n)\| \geq n$. It follows from the standard uniform boundedness principle that for some $v \in X$, $\|\Phi(x_n, y_n, t_n)v\|$ is unbounded, a contradiction to (4.2). Now, it is easy to see that (4.7) holds with $\omega = \eta^{-1} \ln \bar{C}$.

Let $\lambda_0 = \sup \Sigma(K)$. First, we consider the case when $\lambda_0, \lambda_K > -\infty$.

By (4.7), $\lambda_0 < \infty$. Hence, for $\epsilon > 0$ and $\lambda_* = \lambda_0 + \epsilon$, there is a $C > 0$ such that

$$\|e^{-\lambda_* t} \Phi(x, y, t)\| \leq C,$$

that is,

$$\|\Phi(x, y, t)\| \leq C e^{\lambda_* t}$$

for $t \geq 0$, $(x, y) \in K$. It follows easily that

$$\lambda_K \leq \lambda_* = \lambda_0 + \epsilon.$$

By taking $\epsilon \to 0$, one has $\lambda_K \leq \lambda_0$. Conversely, since $\lambda_K < \infty$, for any $\epsilon > 0$ and $(x, y) \in K$,

$$e^{-(\lambda_K + \epsilon)t} \|\Phi(x, y, t)\| \to 0 \quad as \quad t \to \infty.$$

This implies that $\lambda_K + \epsilon \in \mathbb{R} \setminus \Sigma(K)$ and $\lambda_0 \leq \lambda_K + \epsilon$. Since $\epsilon$ is arbitrary, $\lambda_0 \leq \lambda_K$. Thus, $\lambda_0 = \lambda_K$.

Next, suppose that $\lambda_0 = -\infty$ or $\lambda_K = -\infty$. By a similar argument as above, one has that $\lambda_K = -\infty$ or $\lambda_0 = -\infty$. □

Corollary 4.2. *If $K$ is linearly stable, then for any $\epsilon > 0$, there is a $C_\epsilon > 0$ such that*

$$\|\Phi(x, y, t)\| \leq C_\epsilon e^{\epsilon t}$$

*for all $t \geq 0$ and $(x, y) \in K$.*

*Proof.* Since $\sup \Sigma(K) \leq 0$, $(4.6)_\epsilon$ has an ED over $K$ with trivial projections (i.e., $P(x, y) \equiv 0$). Let $C_\epsilon > 0$, $\delta_\epsilon > 0$ be the associated ED constants according to $(4.6)_\epsilon$. Then

$$\|e^{-\epsilon t} \Phi(x, y, t)\| \leq C_\epsilon e^{-\delta_\epsilon t} \leq C_\epsilon,$$

that is,

$$\|\Phi(x, y, t)\| \leq C_\epsilon e^{\epsilon t}$$

for all $t \geq 0$ and $(x, y) \in K$. □

**4.2. Strongly Monotone Skew-product Semiflow.**

DEFINITION 4.4.

1) A Banach space $X$ is said to be *strongly ordered* if there is a closed convex cone $X_+ \subset X$ with nonempty interior such that $X_+ \cap (-X_+) = \{0\}$.
2) In the case that $X$ is strongly ordered, we define an *(strong) ordering* on $X$ as follows:

$$\begin{aligned} x_2 \le x_1 \quad &\Longleftrightarrow \quad x_1 - x_2 \in X_+; \\ x_2 < x_1 \quad &\Longleftrightarrow \quad x_1 - x_2 \in X_+ \quad \text{and} \quad x_1 \neq x_2; \\ x_2 \ll x_1 \quad &\Longleftrightarrow \quad x_1 - x_2 \in \operatorname{Int} X_+. \end{aligned} \tag{4.8}$$

DEFINITION 4.5. The skew-product semiflow (4.1) is *strongly monotone* if the phase space $X$ is a strongly ordered Banach space and $\Phi(x, y, t)v \gg 0$ whenever $v > 0$, $(x, y) \in X \times Y$, $t > 0$.

The strong monotonicity is a stronger notion than that of the strongly order preserving for a skew-product semiflow.

THEOREM 4.3. *If the skew-product semiflow* (4.1) *is strongly monotone, then it is strongly order preserving.*

*Proof.* Let $O_+(X, Y) = \{((x_1, y), (x_2, y)) | x_1 - x_2 \in X_+\}$ and consider $h : O_+(X, Y) \to X_+$:

$$h((x_1, y), (x_2, y)) = x_1 - x_2. \tag{4.9}$$

Clearly, $h$ is continuous and onto. It follows that $\operatorname{Int} O_+(X, Y) \neq \emptyset$ with respect to the subset topology. Moreover, if $x_1 - x_2 \in \operatorname{Int} X_+$, then $((x_1, y), (x_2, y)) \in O_+(X, Y)$. Thus, to show that $\Pi$ is strongly order preserving, it is sufficient to prove the following: For any $y \in Y$ and any $x_1, x_2 \in X$ with $x_2 < x_1$, $u(x_2, y, t) \ll u(x_1, y, t)$ $(t > 0)$. For fixed $y \in Y$, $x_1, x_2 \in X$ with $x_2 < x_1$ and $t_0 > 0$, we denote $u(s) = u(x_2 + s(x_1 - x_2), y, t_0)$, $s \in [0, 1]$. Since $u^{'}(s) = \Phi(x_2 + s(x_1 - x_2), y, t_0)(x_1 - x_2) \gg 0$, it is easy to see that there is a neighborhood $N_s$ of $s \in [0, 1]$ such that $u(\tau_2) \gg u(\tau_1)$ for any $\tau_1, \tau_2 \in N_s$ and $\tau_2 > \tau_1$. By taking a finite subcover, we may assume that $[0, 1] \subset \cup_{n=1}^{n_0} N_{s_n}$, where $n_0 > 1$ and $0 = s_1 < s_2 < \cdots < s_{n_0} = 1$. Choose $\tau_n \in N_{s_n} \cap N_{s_n+1}$, $n = 1, 2, \cdots, n_0 - 1$ and define $\tau_0 = 0$, $\tau_{n_0+1} = 1$ such that $\tau_0 < \tau_1 < \cdots < \tau_{n_0+1}$. Then $u(\tau_{n+1}) - u(\tau_n) \gg 0$, $n = 0, 1, 2, \cdots, n_0$. Using convexity of $X_+$, we have $u(x_1, y, t_0) - u(x_2, y, t_0) = u(1) - u(0) = \sum_{n=0}^{n_0} (u(\tau_{n+1}) - u(\tau_n)) \gg 0$. $\square$

**4.3. Continuous separation.**

We now give a continuous separation result which resembles that of Poláčik and Tereščák [22].

Definition 4.6. Let $K \subset X \times Y$ be a compact, positively invariant set of the strongly monotone skew-product semiflow (4.1). $K$ is said to admit a *continuous separation* if there are subspaces $\{X_1(x,y)\}_{(x,y)\in K}$, $\{X_2(x,y)\}_{(x,y)\in K} \subset X$ with the following properties:

1) $X = X_1(x,y) \oplus X_2(x,y)$, $((x,y)\in K)$ and $X_1(x,y), X_2(x,y)$ vary continuously in $(x,y) \in K$.
2) $X_1(x,y) = span\{v(x,y)\}$, where $v(x,y) \in \operatorname{Int} X_+$ and $\|v(x,y)\| = 1$ $((x,y)\in K)$.
3) $X_2(x,y) \cap X_+ = \{0\}$ $((x,y)\in K)$.
4) For any $t > 0$, $(x,y) \in K$,

$$\Phi(x,y,t)X_1(x,y) = X_1(\Pi(x,y,t)), \tag{4.10$_1$}$$

$$\Phi(x,y,t)X_2(x,y) \subset X_2(\Pi(x,y,t)). \tag{4.10$_2$}$$

5) There are $M > 0$, $\delta > 0$ such that for any $(x,y) \in K$, $w \in X_2(x,y)$ with $\|w\| = 1$,

$$\|\Phi(x,y,t)w\| \le Me^{-\delta t}\|\Phi(x,y,t)v(x,y)\| \quad (t > 0). \tag{4.11}$$

Theorem 4.4. *Let $K \subset X \times Y$ be a compact, positively invariant set of the strongly monotone skew-product semiflow* (4.1) *which admits a flow extension. Assume that there is a $T > 0$ such that for any $(x,y) \in K$, $\Phi(x,y,T)$ is compact. Then $K$ admits a continuous separation. Moreover, if $P(x,y) : X \to X_1(x,y)$, $Q(x,y) = I - P(x,y) : X \to X_2(x,y)$ $((x,y) \in K)$ are projections associated to the continuous separation, then the following holds:*

i) *There is a constant $\bar{C} > 0$ such that for all $(x,y) \in K$,*

$$\|P(x,y)\| \le \bar{C}, \quad \|Q(x,y)\| \le \bar{C}.$$

ii) *There is a constant $C > 0$ such that if $z_0 \in X$ and*

$$\|P(x_0,y_0)z_0\| \ge C\|Q(x_0,y_0)z_0\|$$

*for some $(x_0,y_0) \in K$, then $z_0 \in \pm X_+$.*

*Proof.* Consider the vector bundle map $h : X \times K \to X \times K$,

$$h(u,x,y) = (\Phi(x,y,T)u, \Pi(x,y,T)).$$

Clearly, $\Pi(\cdot,T) : K \to K$ is a homeomorphism and $\{\Phi(x,y,T)\}_{(x,y)\in K}$ is a family of relatively compact, strongly positive operators in $L(X,X)$, that is, $u > 0$ implies that $\Phi(x,y,T)u \gg 0$. It follows from [22] that $h$ admits a continuous separation over $K$, that is, there are subspaces $\{X_1(x,y)\}_{(x,y)\in K}$, $\{X_2(x,y)\}_{(x,y)\in K} \subset X$ with the following properties:

a) $X = X_1(x,y) \oplus X_2(x,y)$, $((x,y) \in K)$ and $X_1(x,y), X_2(x,y)$ vary continuously in $(x,y) \in K$.

b) $X_1(x,y) = span\{v(x,y)\}$, where $v(x,y) \in \operatorname{Int} X_+$ and $\|v(x,y)\| = 1$, $((x,y) \in K)$.

c) $X_2(x,y) \cap X_+ = \{0\}$ $((x,y) \in K)$.

d) For any $(x,y) \in K$,

$$\Phi(x,y,T)X_1(x,y) = X_1(\Pi(x,y,T)), \tag{4.12$_1$}$$

$$\Phi(x,y,T)X_2(x,y) \subset X_2(\Pi(x,y,T)). \tag{4.12$_2$}$$

e) There are $M > 0$, $0 < r < 1$ such that for any $(x,y) \in K$ and any $w \in X_2(x,y)$ with $\|w\| = 1$,

$$\|\Phi(x,y,nT)w\| \le M_1 r^n \|\Phi(x,y,nT)v(x,y)\|, \tag{4.13}$$

$n = 1, 2, \cdots$.

For each $(x,y) \in K$, let $P(x,y) : X \to X_1(x,y)$, $Q(x,y) : X \to X_2(x,y)$ be the projections associated to the above. Note that $\|P(x,y)\|$ is bounded for each $(x,y) \in K$. Since $K$ is compact and $P : K \to L(X,X)$ is continuous ($X_1(x,y)$ varies continuously), $\|P(x,y)\|$ is uniformly bounded. This proves i).

We now show that $K$ admits a continuous separation with $X_1(x,y)$, $X_2(x,y)$ $((x,y) \in K)$ as above. Comparing a)-e) above with 1)-5) in Definition 4.6, we only need to check (4.10) and (4.11).

To show (4.11), we let $\delta = -\frac{\ln r}{T}$. By (4.7), there is a $C_1 > 0$ such that

$$\|\Phi(x,y,t)u\| \le C_1 \|u\| \tag{4.14}$$

for any $(x,y) \in K$, $t \in [0,T]$ and $u \in X$.

We claim that there is a constant $C_2 > 0$ such that

$$C_2 \|v\| \le \|\Phi(x,y,t)v\| \tag{4.15}$$

for all $v \in X_1(x,y)$, $(x,y) \in K$, $t \in [0,T]$. If not, then there are sequences $\{(x_n,y_n)\} \subset K$ and $\{t_n\} \subset [0,T]$ such that $\|\Phi(x_n,y_n,t_n)v(x_n,y_n)\| \to 0$ as $n \to \infty$. By the compactness of $K$, we may assume that $(x_n,y_n) \to (x^*,y^*) \in K$ and $t_n \to t^*$. It follows from the continuity of $v(x,y)$ that $v(x_n,y_n) \to v(x^*,y^*)$. By (4.2) and (4.7), we have

$$\|(\Phi(x_n,y_n,t_n) - \Phi(x^*,y^*,t^*))v(x^*,y^*)\| \to 0$$

and

$$\|\Phi(x_n, y_n, t_n)(v(x_n, y_n) - v(x^*, y^*))\| \to 0.$$

This implies that

$$\begin{aligned}\|\Phi(x^*, y^*, t^*)v(x^*, y^*)\| &\le \|(\Phi(x^*, y^*, t^*) - \Phi(x_n, y_n, t_n))v(x^*, y^*)\| \\ &\quad + \|\Phi(x_n, y_n, t_n)(v(x^*, y^*) - v(x_n, y_n))\| \\ &\quad + \|\Phi(x_n, y_n, t_n)v(x_n, y_n)\| \\ &\to 0.\end{aligned}$$

Hence $\Phi(x^*, y^*, t^*)v(x^*, y^*) = 0$. But

$$\Phi(x^*, y^*, t^*)v(x^*, y^*) \begin{cases} = v(x^*, y^*) & \text{for } t^* = 0 \\ \gg 0 & \text{for } t^* > 0, \end{cases}$$

a contradiction.

Now, for any $t > 0$, we write $t = nT + \tau$, where $n \ge 0$ is an integer and $\tau \in [0, T)$. By (4.5), (4.12)-(4.15), if $(x, y) \in K$, $w \in X_2(x, y)$ with $\|w\| = 1$, then

$$\begin{aligned}\|\Phi(x, y, t)w\| &= \|\Phi(\Pi(x, y, nT), \tau)\Phi(x, y, nT)w\| \\ &\le C_1\|\Phi(x, y, nT)w\| \\ &\le C_1 M_1 e^{-\delta nT}\|\Phi(x, y, nT)v(x, y)\| \\ &\le \frac{C_1 M_1}{C_2} e^{-\delta nT}\|\Phi(\Pi(x, y, nT), \tau)\Phi(x, y, nT)v(x, y)\| \\ &= \frac{C_1 M_1}{C_2} e^{\delta\tau} e^{-\delta t}\|\Phi(x, y, t)v(x, y)\| \\ &\le M e^{-\delta t}\|\Phi(x, y, t)v(x, y)\|,\end{aligned}$$

where $M = \frac{C_1 M_1}{C_2} e^{\delta T}$. This proves (4.11).

We now show that $(4.10)_1$ holds for any $t_0 > 0$, $(x_0, y_0) \in K$.

First, we observe that there is a $\delta_0 > 0$ such that

$$\|P(\Pi(x, y, t_0))\Phi(x, y, t_0)v(x, y)\| \ge \delta_0 \tag{4.16}$$

for all $(x, y) \in K$. For otherwise, there is a sequence $\{(x_n, y_n)\} \subset K$ such that

$$\|P(\Pi(x_n, y_n, t_0))\Phi(x_n, y_n, t_0)v(x_n, y_n)\| \to 0.$$

Without loss of generality, assume that $(x_n, y_n)$ converges to some $(x^*, y^*)$. By continuity of $P$ and $v$, we have

$$P(\Pi(x^*, y^*, t_0))\Phi(x^*, y^*, t_0)v(x^*, y^*) = 0,$$

that is, $\Phi(x^*, y^*, t_0)v(x^*, y^*) \in X_2(\Pi(x^*, y^*, t_0))$. Since $\Phi(x^*, y^*, t_0)v(x^*, y^*) \gg 0$, $X_2(\Pi(x^*, y^*, t_0)) \cap X_+ \neq \{0\}$, a contradiction to c).

Next, we note by (4.5) and (4.12) that for any integer $n \geq 0$,

$$\begin{aligned}\Phi(\Pi(x_0, y_0, -nT), nT + t_0)v(\Pi(x_0, y_0, -nT))\\ &= \Phi(x_0, y_0, t_0)\Phi(\Pi(x_0, y_0, -nT), nT)v(\Pi(x_0, y_0, -nT))\\ &= \tilde{C}_1\Phi(x_0, y_0, t_0)v(x_0, y_0),\end{aligned}$$

where $\tilde{C}_1 = \|\Phi(\Pi(x_0, y_0, -nT), nT)v(\Pi(x_0, y_0, -nT))\|$. It follows that

$$\begin{aligned}A_0 &= P(\Pi(x_0, y_0, t_0))\Phi(x_0, y_0, t_0)v(x_0, y_0)\\ &= \tilde{C}_2 P(\Pi(x_0, y_0, t_0 - nT + nT))\Phi(\Pi(x_0, y_0, t_0 - nT), nT)\cdot\\ &\quad \Phi(\Pi(x_0, y_0, -nT), t_0)v(\Pi(x_0, y_0, -nT))\\ &= \tilde{C}_2\Phi(\Pi(x_0, y_0, t_0 - nT), nT)P(\Pi(x_0, y_0, t_0 - nT))\cdot\\ &\quad \Phi(\Pi(x_0, y_0, -nT), t_0)v(\Pi(x_0, y_0, -nT)),\end{aligned}$$

where $\tilde{C}_2 = 1/\tilde{C}_1$. Similarly,

$$\begin{aligned}B_0 &= Q(\Pi(x_0, y_0, t_0))\Phi(x_0, y_0, t_0)v(x_0, y_0)\\ &= \tilde{C}_2\Phi(\Pi(x_0, y_0, t_0 - nT), nT)Q(\Pi(x_0, y_0, t_0 - nT))\cdot\\ &\quad \Phi(\Pi(x_0, y_0, -nT), t_0)v(\Pi(x_0, y_0, -nT)).\end{aligned}$$

By (4.16) and the facts that $Q(x, y)$, $\Phi(x, y, t_0)$ are uniformly bounded over $K$, we see that there is a $C^* > 0$ such that

$$\frac{\|Q(\Pi(x_0, y_0, t_0 - nT))\Phi(\Pi(x_0, y_0, -nT), t_0)v(\Pi(x_0, y_0, -nT))\|}{\|P(\Pi(x_0, y_0, t_0 - nT))\Phi(\Pi(x_0, y_0, -nT), t_0)v(\Pi(x_0, y_0, -nT))\|} \leq C^*. \tag{4.17}$$

If $B_0 \neq 0$, then

$$Q(\Pi(x_0, y_0, t_0 - nT)\Phi(\Pi(x_0, y_0, -nT), t_0)v(\Pi(x_0, y_0, -nT)) \neq 0$$

for all $n \geq 0$.

Combining (4.13), (4.16) and (4.17), one has

$$\frac{\|B_0\|}{\|A_0\|} \leq C^* M_1 r^n \quad (0 < r < 1)$$

for all $n = 1, 2, \cdots$, which implies that $B_0 = 0$, a contradiction. Thus, $B_0 = 0$ at very beginning, that is,

$$\Phi(x_0, y_0, t_0)v(x_0, y_0) \in X_1(\Pi(x_0, y_0, t_0)).$$

Thus,

$$\Phi(x_0, y_0, t_0)X_1(x_0, y_0) = X_1(\Pi(x_0, y_0, t_0)).$$

To show that $(4.10)_2$ holds for any $t_0 > 0$, $(x_0, y_0) \in K$, we take a $u \in X_2(x_0, y_0)$. Express $\Phi(x_0, y_0, t_0)u = u_1 + u_2$, where $u_1 \in X_1(\Pi(x_0, y_0, t_0))$, $u_2 \in X_2(\Pi(x_0, y_0, t_0))$. If $u_1 \neq 0$, then by (4.13),

$$\frac{\|\Phi(\Pi(x_0, y_0, t_0), nT)(u_1 + u_2)\|}{\|\Phi(\Pi(x_0, y_0, t_0), nT)u_1\|} \to 1 \tag{4.18}$$

as $n \to \infty$. Since $\Phi(x_0, y_0, t_0)X_1(x_0, y_0) = X_1(\Pi(x_0, y_0, t_0))$, there is a $\bar{u}_1 \in X_1(x_0, y_0)$ with $\Phi(x_0, y_0, t_0)\bar{u}_1 = u_1$. By (4.13) again,

$$\frac{\|\Phi(\Pi(x_0, y_0, t_0), nT)(u_1 + u_2)\|}{\|\Phi(\Pi(x_0, y_0, t_0), nT)u_1\|} = \frac{\|\Phi(x_0, y_0, nT + t_0)u\|}{\|\Phi(x_0, y_0, nT + t_0)\bar{u}_1\|} \to 0$$

as $t \to \infty$, which contradicts (4.18). Thus, $u_1 = 0$, that is, $\Phi(x_0, y_0, t_0)u \in X_2(\Pi(x_0, y_0, t_0))$.

To prove ii), we use arguments of [21]. Without loss of generality, let us assume that $P(x_0, y_0)z_0 \in X_+ \cap X_1(x_0, y_0)$, that is, $P(x_0, y_0)z_0 = \|P(x_0, y_0)z_0\|v(x_0, y_0)$. Fix $e \gg 0$ with

$$v(x, y) \gg e$$

for all $(x, y) \in K$, and define the order norm

$$\|z\|_e = \inf\{r > 0 \mid -re \leq z \leq re\}.$$

The order norm $\|\cdot\|_e$ is not stronger than the usual norm $\|\cdot\|$ in $X$, that is, there is a $C > 0$ such that

$$\|z\|_e \leq C\|z\|$$

for all $z \in X$.

Now, if

$$C\|Q(x_0, y_0)z_0\| \leq \|P(x_0, y_0)z_0\|,$$

then

$$\begin{aligned} -Q(x_0, y_0)z_0 &\leq \|Q(x_0, y_0)z_0\|_e e \\ &\leq C\|Q(x_0, y_0)z_0\|v(x_0, y_0) \\ &\leq \|P(x_0, y_0)z_0\|v(x_0, y_0) \\ &= P(x_0, y_0)z_0, \end{aligned}$$

that is, $z_0 \in X_+$. □

**4.4. Stable $\omega$-limit Sets.**

We now consider semiflow (4.1) with $(Y, \mathbb{R})$ being distal. Our main results state as follows.

THEOREM 4.5. *Let $K$ be a minimal set of the strongly monotone skew-product semiflow* (4.1). *Assume the following:*

1) *$K$ admits a flow extension;*
2) *There is $T > 0$ such that $\Phi(x, y, T)$ is compact for all $(x, y) \in K$;*
3) *$K$ is linearly stable.*

*Then there is a minimal flow $(\tilde{Y}, \mathbb{R})$ and flow homomorphisms*

$$p_* : (K, \mathbb{R}) \to (\tilde{Y}, \mathbb{R}), \quad \tilde{p} : (\tilde{Y}, \mathbb{R}) \to (Y, \mathbb{R})$$

*such that $\tilde{p}$ is distal and $N$-1 extension for some integer $N \geq 1$, and $p_*$ is an almost* 1-1 *extension. Moreover, if $(Y, \mathbb{R})$ is almost periodic, then $(K, \mathbb{R})$ is almost automorphic.*

LEMMA 4.6. *Let $K$ be as in* Theorem 4.5. *Then there are $\epsilon_0, \bar{M}, \delta_0 > 0$ such that if $(x_1, y), (x_2, y) \in K$ with $\|x_1 - x_2\| \leq \epsilon_0$ and $u(x_1, y, t), u(x_2, y, t)$ are not ordered (that is, $u(x_1, y, t) - u(x_2, y, t) \notin X_+ \cup (-X_+)$) for $t$ in an interval $[0, t_0]$, then*

$$\|u(x_1, y, t) - u(x_2, y, t)\| \leq \bar{M} e^{-\delta_0 t} \|x_1 - x_2\| \tag{4.19}$$

*for all $0 \leq t \leq t_0$.*

*Proof.* We use similar arguments as [21]. Define $g : X \times K \to X$:

$$g(z, x, y) = u(x + z, y, 1) - u(x, y, 1) - \Phi(x, y, 1)z. \tag{4.20}$$

Since $u$ is $C^{1+\alpha}$ in $x$, it is easy to see that there are $\rho_0 > 0$, $C_1 \geq 1$ such that

$$\|u(z + x, y, \tau) - u(x, y, \tau)\| \leq C_1 \|z\|, \tag{4.21}$$

and

$$\|g(z, \Pi(x, y, \tau))\| \leq C_1 \|z\|^{1+\alpha} \tag{4.22}$$

for all $z \in X$, $\|z\| \leq \rho_0$, $(x, y) \in K$, $0 \leq \tau \leq 1$.

Let $M \geq 1$, $\delta > 0$ be the constants, and $P(x, y)$, $Q(x, y) \equiv I - P(x, y)$ be the projections associated to the continuous separation on $K$ (Theorem 4.4) and let $C_2 > 0$ be a constant such that

$$\|P(x, y)\| \leq C_2, \quad \|Q(x, y)\| \leq C_2 \tag{4.23}$$

for all $(x, y) \in K$. By Theorem 4.4 ii), there is a constant $C_* > 0$ such that if $z_* \notin \pm X_+$, then

$$\|P(x, y)z_*\| < C_* \|Q(x, y)z_*\| \tag{4.24}$$

for all $(x,y)\in K$. It follows that if $z_*\notin \pm X_+$, then

$$C_3\|z_*\|\le \|Q(x,y)z_*\|\le C_2\|z_*\|, \tag{4.25}$$

where $C_3=\frac{1}{1+C_*}$, for all $(x,y)\in K$.

Since $K$ is linearly stable, by Corollary 4.2, there is a $C_4\ge 1$ such that

$$\|\Phi(x,y,t)\|\le C_4 e^{\frac{\delta}{2}t} \tag{4.26}$$

$(t\ge 0,\ (x,y)\in K)$.

We now let $(x_1,y),(x_2,y)\in K$ be such that $u(x_1,y,t)$, $u(x_2,y,t)$ are not ordered for $0\le t\le t_0$. Obviously, if $t_0\le 1$, then (4.19) holds with $\epsilon_0=\rho_0$, $\delta_0=\delta/4$, and $\bar M=C_1e^{\delta/4}$. For $t_0>1$, let $N=[t_0]$,

$$z(t)=u(x_2,y,t)-u(x_1,y,t)\quad (t\ge 0),$$

and

$$\begin{aligned} z_n&=z(n),\\ \hat z_n&=Q(\Pi(x_1,y,n))z(n)\end{aligned}$$

for $n=0,1,2,\cdots,N$. It is clear that

$$z_{k+1}=\Phi(\Pi(x_1,y,k),1)z_k+g(z_k,\Pi(x_1,y,k)),$$

and

$$\hat z_{k+1}=\Phi(\Pi(x_1,y,k),1)\hat z_k+Q(\Pi(x_1,y,k+1))g(z_k,\Pi(x_1,y,k)) \tag{4.27}$$

for $k=0,1,2,\cdots,N$.

We note that (4.27) admits the following ‘variational constant’ formula:

$$\begin{aligned}\hat z_n&=\Phi(x_1,y,n)\hat z_0+\\ &\sum_{k=0}^{n-1}\Phi(\Pi(x_1,y,k+1),n-k-1)Q(\Pi(x_1,y,k+1))g(z_k,\Pi(x_1,y,k)),\end{aligned} \tag{4.28}$$

for $n=1,2,\cdots,N$.

Let $\rho>0$ be such that

$$\begin{aligned}&\max\left\{1,\frac{C_3}{2MC_2C_4}\right\}\rho\le\rho_0,\\ &\max\left\{\frac{MC_1C_2C_4e^{\frac{\delta}{4}}}{C_3(1-e^{-\frac{\delta}{4}})},\frac{C_1C_3^\alpha}{2^{1+\alpha}M^{1+\alpha}C_2^\alpha C_4^{1+\alpha}}\right\}\rho^\alpha\le\frac12.\end{aligned} \tag{4.29}$$

Denote $\epsilon_0 = \frac{C_3\rho}{2MC_2C_4}$. We first claim that if $\|z_0\| < \epsilon_0$, then

$$\|z_n\| \le \rho$$

for all $n = 0, 1, 2, \cdots, N$. We prove it by induction. First, by (4.11), (4.22), (4.23), (4.26), (4.28),

$$\begin{aligned}\|\hat{z}_1\| &\le MC_4C_2e^{-\delta/2}\|z_0\| + C_2C_1\|z_0\|^{1+\alpha} \\ &\le MC_4C_2\epsilon_0 + C_2C_1\epsilon_0^{1+\alpha}.\end{aligned}$$

By (4.25), (4.29), then

$$\|z_1\| \le \frac{1}{C_3}\|\hat{z}_1\| \le \rho\left(\frac{1}{2} + \frac{C_1C_3^\alpha\rho^\alpha}{2^{1+\alpha}M^{1+\alpha}C_2^\alpha C_4^{1+\alpha}}\right) \le \rho.$$

Now, assume

$$\|z_k\| \le \rho \quad for \quad 1 \le k \le n-1 \le N.$$

By (4.11), (4.22), (4.23), (4.26), (4.28) again,

$$\begin{aligned}\|\hat{z}_n\| &\le MC_4C_2e^{-\frac{\delta}{2}n}\|z_0\| + \sum_{k=0}^{n-1} MC_1C_4C_2e^{-\frac{\delta(n-k-1)}{2}}\|z_k\|^{1+\alpha} \\ &\le MC_4C_2\epsilon_0 + \frac{\rho^{1+\alpha}MC_1C_2C_4}{1-e^{-\frac{\delta}{2}}}.\end{aligned}$$

By (4.25), (4.29), if $n \le N$, then

$$\begin{aligned}\|z_n\| \le \frac{1}{C_3}\|\hat{z}_n\| &\le \frac{MC_4C_2}{C_3}\epsilon_0 + \frac{\rho^{1+\alpha}MC_1C_2C_4}{C_3(1-e^{-\frac{\delta}{2}})} \\ &= \rho\left(\frac{1}{2} + \frac{\rho^\alpha MC_1C_2C_4}{C_3(1-e^{-\frac{\delta}{2}})}\right) \le \rho.\end{aligned}$$

Next, denote $\bar{z} = \sup_{0\le n\le N} e^{\frac{n}{4}\delta}\|z_n\|$. It follows from (4.28), (4.29) that

$$\begin{aligned}\bar{z} &\le \frac{1}{C_3}\sup_{0\le n\le N} e^{\frac{n}{4}\delta}\|\hat{z}_n\| \\ &\le \sup_{0\le n\le N}\frac{1}{C_3}\left(MC_4C_2\|z_0\| + e^{\frac{1}{4}\delta}MC_1C_2C_4\bar{z}\rho^\alpha\sum_{k=0}^{n-1}e^{-\frac{\delta(n-k-1)}{4}}\right) \\ &\le \frac{MC_4C_2}{C_3}\|z_0\| + \frac{e^{\frac{1}{4}\delta}MC_1C_2C_4\rho^\alpha}{C_3(1-e^{-\frac{\delta}{4}})}\bar{z} \\ &\le \frac{MC_4C_2}{C_3}\|z_0\| + \frac{1}{2}\bar{z},\end{aligned}$$

that is,

$$\bar{z} \leq \frac{2MC_4C_2}{C_3}\|z_0\|.$$

By (4.21) and (4.29),

$$\begin{aligned}\sup_{0\leq t\leq t_0} e^{\frac{\delta}{4}t}\|z(t)\| &= \sup_{0\leq t\leq t_0} e^{\frac{\delta}{4}t}\|u(u(x_2,y,[t]),y\cdot[t],t-[t]) \\ &\qquad - u(u(x_1,y,[t]),y\cdot[t],t-[t])\| \\ &\leq C_1 e^{\frac{\delta}{4}} \sup_{0\leq t\leq t_0} e^{\frac{\delta}{4}[t]}\|u(x_2,y,[t])-u(x_1,y,[t])\| \\ &\leq C_1 e^{\frac{\delta}{4}}\bar{z} \\ &\leq \frac{2MC_1C_2C_4}{C_3} e^{\frac{\delta}{4}}|z_0|.\end{aligned}$$

It follows that if $|x_1 - x_2| \leq \epsilon_0$ and $u(x_2,y,t)$, $u(x_1,y,t)$ are not ordered for all $0 \leq t \leq t_0$, then

$$|u(x_2,y,t) - u(x_1,y,t)| \leq \bar{M}e^{-\delta_0 t}|x_1 - x_2|, \quad 0 \leq t \leq t_0,$$

where $\delta_0 = \frac{\delta}{4}$ and $\bar{M} = \frac{2MC_1C_2C_4}{C_3}e^{\frac{\delta}{4}}$. □

In the next two lemmas, we let $K$ be as in Theorem 4.5. Since a strongly monotone skew-product semiflow is necessarily strongly order preserving (Theorem 4.3), by Theorem 3.2, there is a residual set $Y_0 \subset Y$ such that for any $y \in Y_0$ no two points on $K \cap p^{-1}(y)$ are ordered, where $p : X \times Y \to Y$ denotes the natural projection. We note that the ordering on $X$ induces an ordering '$\leq$' on $X \times Y$ as follows: $(x_1, y_1) \leq (x_2, y_2)$ if and only if $y_1 = y_2$ and $x_2 - x_1 \in X_+$. We denote $O(K)$ as the *order relation* on $K$, that is,

$$O(K) = \{((x_1,y),(x_2,y))|(x_1,y),(x_2,y) \in K, x_2 - x_1 \in \pm X_+\}.$$

LEMMA 4.7. *The proximal relation $P(K)$ on $K$ is an equivalence relation.*

*Proof.* We only need to check the transitivity. Let $((x_1,y),(x_2,y))$, $((x_2,y),(x_3,y)) \in P(K)$.

*Case 1.* There are $t_1 \geq 0$, $t_2 \geq 0$ such that $(\Pi(x_1,y,t_1),\Pi(x_2,y,t_1)) \in O(K)$ and $(\Pi(x_2,y,t_2),\Pi(x_3,y,t_2)) \in O(K)$.

Let $t_0 > \max\{t_1,t_2\}$ and denote $(x_i^*, y^*) = \Pi(x_i,y,t_0)$. By strong monotonicity, both $((x_1^*,y^*),(x_2^*,y^*))$ and $((x_2^*,y^*),(x_3^*,y^*))$ lie in $O(K)$. According to the proof of Corollary 3.3 1), if $y_0 \in Y_0$ and if $\{t_n\}$ is a sequence such that $y^* \cdot t_n \to y_0$, then by taking a subsequence, we may assume that

$$d(\Pi(x_1^*,y^*,t_n),\Pi(x_2^*,y^*,t_n)) \to 0,$$

and

$$d(\Pi(x_2^*, y^*, t_n), \Pi(x_3^*, y^*, t_n)) \to 0,$$

where $d$ denotes the metric on $K$. It follows that

$$d(\Pi(x_1^*, y^*, t_n), \Pi(x_3^*, y^*, t_n)) \to 0.$$

Hence $d(\Pi(x_1, t, t_n + t_0), \Pi(x_3, y, t_n + t_0)) \to 0$, that is, $((x_1, y), (x_3, y)) \in P(K)$.

*Case 2.* There is a $t_0 \geq 0$ such that $(\Pi(x_1, y, t_0), \Pi(x_2, y, t_0)) \in O(K)$, but $(\Pi(x_2, y, t), \Pi(x_3, y, t)) \notin O(K)$ for all $t \geq 0$.

Again, take $y_0 \in Y$ and let $t_n \to \infty$ be such that $y \cdot t_n \to y_0$. By the proof of Corollary 3.3 1), we have

$$\|u(x_1, y, t_n) - u(x_2, y, t_n)\| \to 0. \tag{4.30}$$

Since $((x_2, y), (x_3, y)) \in P(K)$, we may assume that $\|x_3 - x_2\|$ is sufficiently small. Therefore, by Lemma 4.6,

$$\|u(x_2, y, t) - u(x_3, y, t)\| \to 0 \tag{4.31}$$

as $t \to \infty$.

Combining (4.30), (4.31), one has

$$\|u(x_1, y, t_n) - u(x_3, y, t_n)\| \to 0,$$

that is, $((u_1, y), (u_3, y)) \in P(K)$.

*Case 3.* There is a $t_0 \geq 0$ such that $((\Pi(x_2, y, t_0), \Pi(x_3, y, t_0)) \in O(K)$, but $((\Pi(x_1, y, t), \Pi(x_2, y, t)) \notin O(K)$ for all $t \geq 0$.

This is similar to Case 2.

*Case 4.* For all $t \geq 0$, $(\Pi(x_1, y, t), \Pi(x_2, y, t)) \notin O(K)$, $(\Pi(x_2, y, t), \Pi(x_3, y, t)) \notin O(K)$.

By the strong monotonicity, $(\Pi(x_1, y, t), \Pi(x_2, y, t)) \notin O(K)$ for all $t \in \mathbb{R}$. Since $((x_1, y), (x_2, y)) \in P(K)$, there is a $\tau \in \mathbb{R}$ such that $\|u(x_1, y, \tau) - u(x_2, y, \tau)\|$ is sufficiently small. It follows from Lemma 4.6 that

$$\|u(x_1, y, t) - u(x_2, y, t)\| \to 0$$

as $t \to \infty$.

Similarly,

$$\|u(x_2, y, t) - u(x_3, y, t)\| \to 0$$

as $t \to \infty$.

Above all,

$$\|u(x_1, y, t) - u(x_3, y, t)\| \to 0$$

as $t \to \infty$ and $((x_1, y), (x_3, y)) \in P(K)$. $\square$

Lemma 4.8. *$P(K)=O(K)$.*

*Proof.* By Corollary 3.3 1), $O(K)\subset P(K)$. Now let $((x_1,y),(x_2,y))\in P(K)\setminus O(K)$. We first claim that $(x_1,y),(x_2,y)$ are negatively distal. If not, then there is a sequence $t_n\to-\infty$ such that

$$\|u(x_1,y,t_n)-u(x_2,y,t_n)\|\to 0.$$

Let $\epsilon_0$, $\bar{M}$ and $\delta_0$ be as in Lemma 4.6 and let $N$ be such that $\|u(x_1,y,t_n)-u(x_2,y,t_n)\|<\epsilon_0$ for $n\geq N$. Note that $u(\Pi(x_1,y,t_n),t)$, $u(\Pi(x_2,y,t_n),t)$ are not ordered for $0\leq t\leq -t_n$. By Lemma 4.6, one has

$$\begin{aligned}\|x_1-x_2\| &= \|u(\Pi(x_1,y,t_n),-t_n)-u(\Pi(x_2,y,t_n),-t_n)\|\\ &\leq \bar{M}e^{\delta_0 t_n}\|u(x_1,y,t_n)-u(x_2,y,t_n)\|\\ &\leq \epsilon_0\bar{M}e^{\delta_0 t_n}\end{aligned}$$

for $n\geq N$. We let $n\to\infty$ to conclude that $x_1=x_2$, which is a contradiction. It follows that $(x_1,y)$, $(x_2,y)$ are proximal and negatively distal. But this is impossible by Corollary 2.8 of Part I and Lemma 4.7 above. □

*Remark 4.1.* Under conditions of the above lemma, we see that $O(K)$ is an equivalence and (two-sided) invariant relation since $P(K)$ is (in general, $O(K)$ is only positively invariant). Also $P(K)$ is a closed relation since $O(K)$ is. It follows that $K/P(K)$ is compact $T_2$ and metrizable ([16]).

*Proof of Theorem 4.5.* Let $\tilde{Y}=K/P(K)=K/O(K)$. Then, $(K,\mathbb{R})$ induces a flow $(\tilde{Y},\mathbb{R})$ by the invariance of $P(K)$. Clearly, $(\tilde{Y},\mathbb{R})$ is distal. Denote $\tilde{p}:\tilde{Y}\to Y$ as the projection induced by $p$ and denote $p^*:K\to\tilde{Y}=K/P(K)$ as the natural projection to equivalence classes, that is, $p^*(x,y)=[(x,y)]$, $((x,y)\in K)$. Then $p=\tilde{p}\circ p^*$ and by the closeness of $P(K)$, $\tilde{p}$, $p^*$ are continuous. Let $Y_0$ be the residual set given by Theorem 3.2 and fix a $y_0\in Y_0$. Since no two points on $K\cap p^{-1}(y_0)$ are ordered, $K\cap p^{-1}(y_0)=K\cap\tilde{p}^{-1}(y_0)$. Now, if $cardK\cap\tilde{p}^{-1}(y_0)=cardK\cap p^{-1}(y_0)=\infty$, then there is an accumulation point $(x_*,y_0)\in K\cap p^{-1}(y_0)$. Therefore, there is a $(x_0,y_0)\in K\cap p^{-1}(y_0)$ such that $(x_0,y_0)\neq(x_*,y_0)$ and $\|x_0-x_*\|$ is sufficiently small. Since $(x_0,y_0)$, $(x_*,y_0)$ are not ordered, by the invariance of the order relation on $K$, $u(x_0,y_0,t)$, $u(x_*,y_0,t)$ are not ordered for all $t\geq 0$. By Lemma 4.6, one has $\|u(x_0,y_0,t)-u(x_*,y_0,t)\|\to 0$ as $t\to\infty$. It follows that $(x_0,y_0)$, $(x_*,y_0)$ are proximal, a contradiction to Lemma 4.8. Thus, there is an integer $N\geq 1$ such that $cardK\cap\tilde{p}^{-1}(y_0)=N$. By Theorem 2.12 of Part I, $\tilde{Y}$ is an $N$-fold covering of $Y$ and $\tilde{p}:(\tilde{Y},\mathbb{R})\to(Y,\mathbb{R})$ is an $N$-1 extension.

Next, for any $y\in Y_0$ and any $[x,y]\in K\cap\tilde{p}^{-1}(y)$, since $K\cap p^{-1}(y)=K\cap\tilde{p}^{-1}(y)$, one has ${p^*}^{-1}([x,y])=(x,y)$, that is, $cardp^{*-1}([x,y])=1$. Since $\tilde{Y}_0=\{[x,y]\in\tilde{p}^{-1}(y)|y\in Y_0\}$ is residual in $\tilde{Y}$, $p^*:(K,\mathbb{R})\to(\tilde{Y},\mathbb{R})$ is an almost 1-1 extension.

Now, if $(Y, \mathbb{R})$ is almost periodic, then by Theorem 2.12 2) of Part I, $(\tilde{Y}, \mathbb{R})$ is also almost periodic, and by Theorem 2.14 of Part I (see also [38]), $(K, \mathbb{R})$ is almost automorphic. □

COROLLARY 4.9. *Let $\omega(x_0, y_0)$ be an $\omega$-limit set of the strongly monotone skew-product semiflow* (4.1). *If $\omega(x_0, y_0)$ is both linearly and uniformly stable, then $(\omega(x_0, y_0), \mathbb{R}^+)$ admits a minimal flow extension and $(\omega(x_0, y_0), \mathbb{R}) \to (Y, \mathbb{R})$ is a distal and $N$-1 extension. Moreover, $(\omega(x_0, y_0), \mathbb{R})$ is almost periodic if $(Y, \mathbb{R})$ is almost periodic.*

*Proof.* It follows from Theorem 2.8 that $(\omega(x_0, y_0), \mathbb{R}^+)$ has a flow extension $(\omega(x_0, y_0), \mathbb{R})$ which is distal and minimal. By Theorem 4.5, $(\omega(x_0, y_0), \mathbb{R}) \to (Y, \mathbb{R})$ is an $N$-1 extension.

In the case that $(Y, \mathbb{R})$ is almost periodic, the almost periodicity of $(\omega(x_0, y_0), \mathbb{R})$ again follows from Part I, Theorem 2.12. □

*Remark 4.2.* Let $\omega(x_0, y_0)$ be as in the above corollary. If $(Y, \mathbb{R})$ is almost periodic, then $\Pi(x_0, y_0, t)$ is asymptotically almost periodic. To see this, we first observe by Theorem 4.5 and Corollary 4.9 that there is $\epsilon_0 > 0$ such that

$$d((x_1, y), (x_2, y)) \geq \epsilon_0$$

for any $(x_1, y), (x_2, y) \in \omega(x_0, y_0)$.

For each $\epsilon$ with $0 < \epsilon < \epsilon_0$, we denote $\delta(\epsilon)$ as the corresponding modulus of uniform stability of $\Pi(x_0, y_0, t)$. It is easy to see that there are $T > 0$ and $(x, y_0) \in \omega(x_0, y_0)$ such that

$$d(\Pi(x_0, y_0, T), \Pi(x, y_0, T)) < \delta,$$

and therefore

$$d(\Pi(x_0, y_0, T + t), \Pi(x, y_0, T + t)) < \epsilon < \epsilon_0$$

for all $t > 0$. This implies that

$$d(\Pi(x_0, y_0, T + t), \Pi(x, y_0, T + t)) \to 0$$

as $t \to \infty$, that is, $\Pi(x_0, y_0, t)$ is asymptotically almost periodic.

Let $K \subset X \times Y$ be a compact invariant set of the strongly monotone skew-product semiflow $\Pi$. The linear stability of $K$ can be verified in terms of the continuous separation on $K$. Let $X_1(x, y)$, $X_2(x, y)$ $((x, y) \in K)$ be the subspaces associated to the continuous separation of $(K, \mathbb{R})$. Recall that $X_1(x, y) = \text{span}\{v(x, y)\}$, where $v(x, y) \in \text{Int}\, X_+$, $\|v(x, y)\| = 1$ for $(x, y) \in K$. Denote

$$c(x, y, t) = \|\Phi(x, y, t)v(x, y)\|, \tag{4.32}$$

$(x, y) \in K$, $t \in \mathbb{R}$. By the invariance of $X_1(x, y)$, (4.32) generates a linear skew-product flow $\tilde{\Pi} : \mathbb{R}^1 \times K \times \mathbb{R} \to \mathbb{R}^1 \times K$,

$$\tilde{\Pi}(z, (x, y), t) = (c(x, y, t)z, (x, y) \cdot t). \tag{4.33}$$

Proposition 4.10. *Let $K$ be as above. Then the upper Lyapunov exponent of* (4.4) *coincides with the upper Lyapunov exponent of* (4.33)*.*

*Proof.* Let $\lambda_K$, $\tilde{\lambda}_K$ be the upper Lyapunov exponents of (4.4), (4.33) respectively. Then $\tilde{\lambda}_K = \sup_{(x,y)\in K} \overline{\lim}_{t\to\infty} \frac{\ln c(x,y,t)}{t}$. By (4.32),

$$\tilde{\lambda}_K \leq \lambda_K. \tag{4.34}$$

Let $\lambda_0$ be the exponent associated to the continuous separation on $K$. For any $(x_*, y_*) \in K$, we let $v_* = v_1 + v_2 \in X$, where $\|v_*\| = 1$ and $v_i \in X_i(x_*, y_*)$ $(i = 1, 2)$. Then

$$\|\Phi(x_*, y_*, t)v_2\| \leq k\|v_2\|e^{-\lambda_0 t}c(x_*, y_*, t)$$

for some $k > 0$ and all $t \geq 0$. This implies that

$$\|\Phi(x_*, y_*, t)\| \leq \tilde{C}(ke^{-\lambda_0 t} + 1)c(x_*, y_*, t)$$

for some $\tilde{C} > 0$. It follows that

$$\overline{\lim_{t\to\infty}} \frac{\ln \|\Phi(x_*, y_*, t)\|}{t} \leq \overline{\lim_{t\to\infty}} \frac{\ln c(x_*, y_*, t)}{t},$$

that is,

$$\lambda_K \leq \tilde{\lambda}_K. \tag{4.35}$$

By (4.34) and (4.35), $\lambda_K = \tilde{\lambda}_K$. □

*Remark 4.3.*

1) We shall see in Part III that, for strongly monotone skew-product semiflow, the almost $N$-1 extension in Theorem 4.5 may not be replaced by an $N$-1 extension in general. Moreover, subharmonic phenomena (i.e., $N > 1$ in Theorem 4.5) may be observed.

2) A linearly stable minimal set of a strongly monotone skew-product semiflow $\Pi$ need not be stable. Conversely, uniform stability need not imply linear stability either. However, if the flow on a minimal set or an $\omega$-limit set $K$ of $\Pi$ is uniquely ergodic (e.g., $(K, \mathbb{R})$ is an almost periodic minimal flow), then uniform stability does imply the linear stability of $K$. To see this, we note by Proposition 4.10 that the unique ergodicity of $(K, \mathbb{R})$ simply implies that the Sacker-Sell spectrum of (4.33) reduces to a point $\{\lambda_M\}$ which coincides with the upper Lyapunov exponent of the linearized skew-product semiflow (4.4) over $K$. It is then not difficult to see that $\{\lambda_K\}$ is also a (degenerate) spectrum interval of (4.4). Now, if $\lambda_K > 0$, then an unstable manifold of $\Pi$ over $K$ exists (see [2]). Therefore, $K$ can not be stable.

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**Part III. Applications to Differential Equations**

Wenxian Shen and Yingfei Yi

# PART III
# APPLICATIONS TO DIFFERENTIAL EQUATIONS

## 1. Introduction

In the current part, we apply the theory developed in the previous parts to study dynamics for a large class of almost periodic ordinary, partial and functional differential equations. Main issues to be considered are the existence and harmonic properties of almost automorphic and almost periodic solutions as well as the long time behavior of bounded solutions. Our primary goal in studying these issues is to address fundamental roles played by the notion of almost automorphy in the qualitative study of almost periodic differential equations. The following two significances of almost automorphic dynamics shall be emphasized through our study: 1) An almost periodic time variation often produces almost automorphic dynamics rather than almost periodic one. 2) The study of almost automorphic dynamics provides a deep understanding to the existence of almost periodic one.

Our study concerns with both linear and nonlinear equations. Due to essential dynamical differences, we also divide the nonlinear study into 'one dimensional' and 'higher dimensional' cases.

In the linear study, we only consider linear (non-homogeneous) systems of almost periodic ODE's (see also [60] for abstract linear equations). Similar to the $N$- almost periodic case ([30]), we use a Favard type of result to investigate the existence of almost automorphic solutions. For instance, we are able to show the following.

THEOREM A. *If a traceless, 2-dimensional, linear and non-homogeneous system of almost periodic ODE's admits a bounded solution, then its associated skew-product flow has an almost automorphic minimal set $E$. Moreover, for any almost automorphic solution $x$ lying in $E$, $\mathcal{M}(x)$ is contained in the frequency module of the coefficients.*

Among the 'one dimensional' nonlinear systems, we are particularly interested in scalar almost periodic ODE's and parabolic PDE's in 1-space dimension. Autonomous and periodic cases of parabolic PDE's in 1-space dimension have been

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Dedicated to Professor R. Ellis on the Occasion of His 70th Birthday

given considerable amount of attentions in recent years (see [1], [2], [5]-[10], [22], [24], [34], [37]-[39], [50]-[51] and references therein). The almost periodic cases were studied in our recent works ([46]-[49]) in which almost automorphic dynamics were shown to be essential. In this work, for the sake of completeness, we give some new and unified treatments to our previous results summarizing below.

Theorem B. *Let* $\Pi$ *be a skew-product semiflow (flow) generated by a scalar almost periodic ODE or a parabolic PDE in* 1*-space dimension. Then the following holds.*

1) *Any* $\omega$*-limit set of* $\Pi$ *contains at most two minimal sets;*
2) *Any minimal set of* $\Pi$ *is harmonically almost automorphic (that is, if* $u$ *is an almost automorphic solution, then the frequency module* $\mathcal{M}(u)$ *of* $u$ *is always contained in the frequency module of the vector field);*
3) *A minimal set of* $\Pi$ *is almost periodic if it is either uniformly stable or hyperbolic.*

Regarding to 'higher dimensional' nonlinear systems, we shall mainly study those of certain monotone natures which particularly include almost periodic parabolic equations in higher space dimensions, almost periodic cooperative system of ordinary and delay differential equations. In this context, dynamical behavior of a solution should be closely related to its stability since an unstable motion in these systems may well be chaotic even in autonomous and periodic situations (see [36]). Concerning with the existence of almost automorphic and almost periodic dynamics, we have the following results.

Theorem C. *Let* $\Pi$ *denotes the skew-product semiflow generated by monotone systems mentioned above.*

1) *Any linearly stable minimal set* $E$ *of* $\Pi$ *is almost automorphic and is either harmonic or subharmonic, that is, there is a positive integer* $N$ *such that if* $u$ *is an almost automorphic solution lying in* $E$*, then* $N\mathcal{M}(u) \subset \mathcal{M}(f)$*, where* $f$ *denotes the vector field.*
2) *If* $E$ *in* 1) *is also uniformly stable, then it becomes almost periodic.*

We note that subharmonic phenomena (that is, $N > 1$) can occur even in periodic monotone systems (see [12], [36], [52]).

Following works of Casten & Holland ([4]), Matano ([34]), and Hess ([22]) in autonomous and periodic parabolic equations, we also investigate effects of stabilities to behavior of solutions in almost periodic parabolic equations. The result is as follows.

Theorem D. *Consider a spatially homogeneous parabolic equation with the non-flux Neumann boundary condition. Then any linearly stable almost automorphic (almost periodic) solution* $u(x,t)$ *is spatially homogeneous.*

Concerning with the long time behavior of a bounded solution in almost periodic differential equations, we shall study asymptotic almost periodicity and the existence of an almost periodic global attractor under certain dissipative conditions. We shall also discuss certain irregular asymptotic phenomena resulting from almost automorphic dynamics and make comparisons with periodic systems.

The current part is organized as follows. In Section 2, we show constructions of skew-product flows or semiflows associated to non-autonomous ordinary, parabolic and delay differential equations following the original framework of Miller ([35]) and Sell ([44], [45]). Section 3 deals with almost automorphic and almost periodic dynamics of scalar almost periodic ODE's and parabolic PDE's in 1-space dimension. Many interesting examples exhibiting ergodic or non-ergodic, non-almost periodic almost automorphic dynamics are provided. Other issues such as ergodicity of minimal sets and asymptotic almost periodicity of bounded solutions are also discussed. Systems of (linear and cooperative nonlinear) almost periodic ordinary differential equations are discussed in Section 4. An example of almost periodic cooperative system of ODE's which admits a linearly stable, subharmonic, non-almost periodic, almost automorphic solution is also given. Section 5 is devoted to the study of almost automorphic and almost periodic dynamics for almost periodic parabolic equations in higher space dimensions. Other related problems such as spatial homogeneity of solutions and the existence of an almost periodic global attractor etc will also be considered. In Section 6, we investigate almost automorphic and almost periodic dynamics for certain almost periodic delay differential equations along with some discussions on the unique backward extension of solutions.

## 2. Skew-product Flows or Semiflows Generated by Differential Equations

### 2.1. Ordinary Differential Equations.

Consider an ODE system

$$x' = f(x,t), \quad x \in \mathbb{R}^n, \tag{2.1}$$

where $f$ is a $C^2$ admissible (see Part I, Section 3) and (*uniformly*) *minimal function*, that is, the time translated flow $(H(f), \mathbb{R})$ on the compact hull $H(f)$ is minimal (e.g., $f$ is uniformly almost automorphic or almost periodic, see Part I, Definition 3.3). By Part I, Theorem 3.1, each $g \in H(f)$ is also $C^2$ admissible, and $f$ has a unique extension to a continuous function $F : \mathbb{R}^n \times H(f) \to \mathbb{R}^n$ with $F(x, g \cdot t) \equiv g(x,t)$. Moreover, $F$ is $C^2$ in $x$ and Lipschitz in $g \in H(f)$.

For each $g \in H(f)$, $x_0 \in \mathbb{R}^n$, we define $x(x_0, g, t)$ as the solution of

$$x' = g(x,t) \equiv F(x, g \cdot t) \tag*{$(2.2)_g$}$$

with initial value $x_0$. By the standard theory of local existence, uniqueness and continuity of solutions of ODE's ([18]), equation (2.1) gives rise to a (local) skew-product flow $\Pi : \mathbb{R}^n \times H(f) \times \mathbb{R} \to \mathbb{R}^n \times H(f)$:

$$\Pi(x_0, g, t) = (x(x_0, g, t), g \cdot t), \tag{2.3}$$

where $x(x_0, g, t)$ is $C^2$ in $x_0$.

We note that the family $(2.2)_g$ ($g \in H(f)$) consists of only translated and limiting equations of (2.1). This means that the (local) skew-product flow (2.3) is the right class which reflects the 'dynamics' of (2.1), in particular when considering the long time behavior of bounded solutions. Now, if $x(x_0, g, t)$ is a bounded solution of $(2.2)_g$ for $t$ in its interval of existence, by standard theory of ODE's, then $x(x_0, g, t)$ exists for all $t \geq 0$, and its forward orbit $\{x(x_0, g, t) \mid t \geq 0\} \subset \mathbb{R}^n$ is relatively compact. In terms of the (local) skew-product flow (2.3), this implies that the $\omega$-limit set $\omega(x_0, g)$ is well defined, compact and invariant, that is, $\Pi$ restricted to $\omega(x_0, g)$ defines a global flow. An $\alpha$-limit set and a minimal set of $\Pi$ can be defined in a similar fashion. The above discussion also justifies the role played by $H(f)$. To define a dynamical system for a nonautonomous equation, one alternative way would be to add one dimension and make it autonomous since the system

$$\begin{cases} x' = f(x, t) \\ t' = 1 \end{cases} \tag{2.4}$$

clearly defines a (local) flow $\tilde{\Pi}$ on $\mathbb{R}^n \times \mathbb{R}$. Nevertheless, in the dynamical system (2.4), all $\omega$, $\alpha$-limit sets will be empty due to the lack of compactness of solutions.

### 2.2. Parabolic Equations.

Consider a scalar parabolic equation

$$\begin{cases} u_t = \Delta u + f(u, \bigtriangledown u, x, t), & t > 0, \quad x \in \Omega \\ u|_{\partial\Omega} = 0 \quad \text{or} \quad \frac{\partial u}{\partial n}|_{\partial\Omega} = 0, & t > 0, \end{cases} \tag{2.5}$$

where $\Omega$ is a bounded, connected and smooth domain in $\mathbb{R}^n$, $f : (\mathbb{R}^1 \times \mathbb{R}^n \times \mathbb{R}^n) \times \mathbb{R} \to \mathbb{R}^1$ is a $C^2$ admissible and (uniformly) minimal function. By the standard theory of parabolic equations ([16]), for each $U_0 \in C^1(\bar{\Omega})$ satisfying the boundary condition of (2.5) and for each $g \in H(f)$, the equation

$$\begin{cases} u_t = \Delta u + g(u, \bigtriangledown u, x, t) \equiv \Delta u + F(u, \bigtriangledown u, x, g \cdot t), & t > 0, \quad x \in \Omega \\ u|_{\partial\Omega} = 0 \quad \text{or} \quad \frac{\partial u}{\partial n}|_{\partial\Omega} = 0, & t > 0 \end{cases} \tag{2.6$_g$}$$

locally admits a unique classical solution $u(U_0, g, x, t)$ with initial value $U_0$, where $F(u, p, x, g)$ is the extension of $f$ on $\mathbb{R}^1 \times \mathbb{R}^n \times \mathbb{R}^n \times H(f)$ according to Part I, Theorem 3.1.

We now define a (local) skew-product semiflow over $(H(f), \mathbb{R})$ similarly to the case of ODE's. Let $X$ be a fractional power space ([21]) associated to the operator $u \to -\Delta u$, $\mathcal{D} \to L^p(\Omega)$ that satisfies $X \hookrightarrow C^1(\bar{\Omega})$, where $\mathcal{D} = \{u \in H^{2,p}(\Omega) : u|_{\partial\Omega} = 0 \quad or \quad \frac{\partial u}{\partial n}|_{\partial\Omega} = 0\}$, $p > n$. One can show that if $U_0 \in X$, $g \in H(f)$, then $u(U_0, g, \cdot, t) \in X$ is $C^2$ in $U_0$ and is continuous in $g, t$ within its (time) interval of existence. In other word, there is a well defined (local) skew-product semiflow $\Pi : X \times H(f) \times \mathbb{R}^+ \to X \times H(f)$:

$$\Pi(U_0, g, t) = (u(U_0, g, \cdot, t), g \cdot t), \quad t > 0 \tag{2.7}$$

associated to (2.5), where $u(U_0, g, t)$ is $C^2$ in $U_0$.

Using *a priori* estimates of parabolic equations ([16], [21]), if $u(U_0, g, \cdot, t)$ is a bounded solution of $(2.6)_g$ for $t$ in its interval of existence, then $u(U_0, g, \cdot, t)$ exists for all $t > 0$, and for any $\delta > 0$, $\{u(U_0, g, \cdot, t) \mid t \geq \delta\}$ is relatively compact, hence its $\omega$-limit set $\omega(U_0, g)$ is well defined and compact. Moreover, by [20], [21], $\Pi$ restricted to $\omega(U_0, g)$ is a (global) semiflow which admits a flow extension $(\omega(U_0, g), \mathbb{R})$. A minimal set $E$ of (2.7) can be defined in the same fashion, that is, $E = \omega(x_0, g)$ for some $(x_0, g) \in E$ and $(E, \mathbb{R})$ is minimal in the usual sense.

We remark that, using (2.7), dynamics of (2.5) is relatively independent of the choice of a phase space $X$ as long as the class of solutions under considerations possess enough regularity. In fact, using *a priori* estimates of parabolic equations, for any $U_0 \in X$, if $u(U_0, g, \cdot, t)$ is $X$-bounded, then it is $H^{2,p}$-bounded and moreover $\omega(U_0, g)|_{X \times H(f)}$ coincides with $\omega(U_0, g)|_{H^{2,p}(\Omega) \times H(f)}$ ([21]).

We finally note that by the comparison principle of parabolic equations ([16]), a natural condition which guarantees the existence of a $H^{2,p}(\Omega)$-bounded (hence $X$-bounded) solution for (2.5) is the following: There is a $M > 0$ such that

$$uf(u, p, x, t) \leq 0, \quad |u| \geq M, \quad p \in \mathbb{R}^n, \quad x \in \Omega, \quad t \in \mathbb{R}. \tag{2.8}$$

### 2.3. Delay-differential Equations.

Consider a system of delay differential equations

$$x'(t) = f(x(t), x(t-1), t), \tag{2.9}$$

where $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a $C^2$ admissible and (uniformly) minimal function. Let $X = C([-1, 0], \mathbb{R}^n)$. Then by the standard theory of delay differential equation ([19]), for each $\phi \in X$ and each $g \in H(f)$,

$$x'(t) = g(x(t), x(t-1), t) \tag*{$(2.10)_g$}$$

locally admits a unique solution $x(\phi, g, t)$ with initial value $\phi$, that is, $x(\phi, g, t) = \phi(t)$ for $t \in [-1, 0]$. Moreover, if $x_t(\phi, g) \in X$ is such that $x_t(\phi, g)(\theta) = x(\phi, g, t+\theta)$ for $\theta \in [-1, 0]$ and $t > 0$, then $x_t(\phi, g)$ is $C^2$ in $\phi \in X$ and Lipschitz in

$g \in H(f)$ (see [19]). Therefore, there is a well defined (local) skew-product semiflow $\Pi : X \times H(f) \times \mathbb{R}^+ \to X \times H(f)$,

$$\Pi(\phi, g, t) = (x_t(\phi, g), g \cdot t) \tag{2.11}$$

associated to (2.9), where $x_t(\phi, g)$ is $C^2$ in $\phi$.

We shall prove in Section 6 that if $x(\phi, g, t) \in \mathbb{R}$ is a bounded solution of $(2.10)_g$ for $t$ in its existence interval, then $x_t(\phi, g)$ exists for all $t > 0$ and $\{x_t(\phi, g) \mid t \geq 1 + \delta\}$ is relatively compact in $X$ for any $\delta > 0$, hence $\omega(\phi, g)$ is well defined and compact. Moreover, under some appropriate condition (see Part II, Section 2 and Section 6 in the current part), $\Pi$ restricted to $\omega(\phi, g)$ extends to a global flow.

## 3. Scalar Ordinary Differential Equations and Parabolic Equations in 1-Space Dimension

In this section, we consider scalar ordinary differential equations and parabolic equations in 1-space dimension, that is, $n = 1$ in both (2.1) and (2.5). Without loss of generality, we assume that the domain $\Omega$ in (2.5) is simply the interval $[0, 1]$. Thus, equation (2.5) can be rewritten into the form

$$\begin{cases} u_t = u_{xx} + f(u, u_x, x, t), & t > 0, \quad 0 < x < 1 \\ \beta u(i, t) + (1 - \beta) u_x(i, t) = 0, & t > 0, \quad i = 0, 1, \end{cases} \tag{3.1}$$

where $\beta = 0$ or $1$, which indicates either the Neumann or the Dirichlet boundary condition. We note that, when $\beta = 0$ and $f \equiv f(u, t)$ in (3.1), solutions of the scalar ODE

$$u' = f(u, t) \tag{3.2}$$

are nothing but spatially homogeneous solutions of (3.1). Therefore, results for (3.1) certainly hold for (3.2). In what follows, we denote $\Pi : X \times H(f) \times \mathbb{R}^+ \to X \times H(f)$:

$$\Pi(U_0, g, t) = (u(U_0, g, t), g \cdot t) \tag{3.3}$$

as the (local) skew-product semiflow or flow generated by (3.1) or (3.2) according to constructions in the previous section, that is, $X = \mathbb{R}^1$ in the case of (3.2) and $X$ is a fractional power space associated to $-\frac{\partial^2}{\partial x^2}$ and the boundary condition, in the case of (3.1).

In (3.3), $u(U_0, g, t)$ either denotes the solution $u(U_0, g, x, t)$ of

$$\begin{cases} u_t = u_{xx} + g(u, u_x, x, t), & t > 0, \quad 0 < x < 1 \\ \beta u(i, t) + (1 - \beta) u_x(i, t) = 0, & t > 0, \quad i = 0, 1 \end{cases} \tag{$3.4_g$}$$

with initial value $U_0 \in X$, or denotes the solution of

$$u' = g(u, t) \tag{$3.5_g$}$$

with initial value $U_0 \in \mathbb{R}^1$. Without loss of generality, we assume that $u(U_0, g_0, t)$ exists for all $t > 0$. Throughout the section, $p : X \times H(f) \to H(f)$ is denoted as the natural projection.

### 3.1. Ordering Defined by Zero Numbers.

Zero number properties, initiated in [33], play a crucial role in studying dynamics of a 1-dimensional parabolic equation (see applications in [2], [6], [8]-[10], [17], [34], [46]-[49], etc.). It is known that similar properties are unable to hold in higher space dimensions ([17]), which makes the 1-dimensional case rather special.

Definition 3.1. For a given $C^1$ function $v : [0,1] \to \mathbb{R}^1$, the zero number of $v$ is defined as

$$Z(v(\cdot)) = \#\{x \in (0,1) \mid v(x) = 0\}.$$

The following result of zero number properties can be found originally in [1], [33] and has recently been improved in [7].

Lemma 3.1. *Consider the scalar linear parabolic equation:*

$$\begin{cases} v_t = a(x,t)v_{xx} + b(x,t)v_x + c(x,t)v, & t > 0, \quad x \in (0,1), \\ \beta v(i,t) + (1-\beta)v_x(i,t) = 0, & t > 0, \quad i = 0,1, \end{cases} \tag{3.6}$$

$(\beta = 0,1)$*, where* $a$*,* $a_t$*,* $a_x$*,* $b$ *and* $c$ *are bounded continuous functions, and there is a constant* $\delta > 0$ *such that* $a \geq \delta$*. Let* $v(x,t)$ *be a nontrivial classical solution of* (3.6)*. Then the following holds.*

1) $Z(v(\cdot,t))$ *is finite for* $t > 0$ *and is nonincreasing as* $t$ *increases;*
2) $Z(v(\cdot,t))$ *decreases and only decreases at* $t = t_0$ *such that* $v(t_0,\cdot)$ *admits a multiple zero in* $[0,1]$*;*
3) $Z(v(\cdot,t))$ *can only drop finitely many times, and there exists a* $t^* > 0$ *such that* $v(t,\cdot)$ *has only simple zeros in* $[0,1]$ *as* $t \geq t^*$ *(hence* $Z(v(t,\cdot)) =$ *constant as* $t > t^*$*).*

Let

$$\Delta(X, H(f)) = \{((U_1,g),(U_2,g)) \mid U_1, U_2 \in X, g \in H(f)\} \tag{3.7}$$

with the subset topology.

Corollary 3.2. *For fixed* $\beta = 0,1$*, define* $h : \Delta(X, H(f)) \to \mathbb{R}^1$*:*

$$h((U_1,g),(U_2,g)) = (1-\beta)(U_1(0) - U_2(0)) + \beta(U_{1x}(0) - U_{2x}(0)). \tag{3.8}$$

*Then the following holds.*

1) *If* $(U_1,g) \neq (U_2,g)$*, then there is a* $T > 0$ *such that* $h(\Pi(U_1,g,t), \Pi(U_2,g,t))$ *is nonzero and has constant sign for all* $t \geq T$*.*
2) $(U_1,g) = (U_2,g)$ *if and only if there is a* $T > 0$ *such that* $h(\Pi(U_1,g,t), \Pi(U_2,g,t)) \equiv 0$ *for all* $t \geq T$*.*

*Proof.* Denote $u(x,t) = u(U_1, g, t)(x) - u(U_2, g, t)(x)$. Then $u(x,t)$ is a classical solution of

$$
(3.9) \qquad \begin{cases} u_t = u_{xx} + a(x,t)u_x + b(x,t)u, & t > 0, \quad 0 < x < 1 \\ \beta u(i,t) + (1-\beta)u_x(i,t) = 0, & t > 0, \quad i = 0, 1, \end{cases}
$$

where

$$
a(x,t) = \int_0^1 g_p(u(U_2, g, t)(x), u_x(U_2, g, t)(x) + s u_x(x,t), x, t) ds,
$$

and

$$
b(x,t) = \int_0^1 g_u(u(U_2, g, t)(x) + s u(x,t), u_x(U_2, g, t)(x), x, t) ds.
$$

By Lemma 3.1, if $U_1 \neq U_2$, then there is a $T > 0$ such that all zeros of $u(\cdot, t)$ in $[0,1]$ are simple and $Z(u(\cdot,t))$ does not decrease for all $t \geq T$. Applying the boundary conditions in (3.9), one sees that $h(\Pi(U_1, g, t), \Pi(U_2, g, t))$ does not vanish for all $t \geq T$. For otherwise, $x = 0$ would be a multiple zero of $u(x,t)$ for some $t \geq T$. This proves 1).

Now, if $U_1 = U_2$, by the uniqueness of solutions of $(3.4)_g$, then $h(\Pi(U_1, g, t), \Pi(U_2, g, t)) \equiv 0$ for all $t > 0$. Suppose that there is a $T > 0$ such that $h(\Pi(U_1, g, t), \Pi(U_2, g, t)) \equiv 0$ for all $t \geq T$. Then $U_1 = U_2$ by 1). □

Definition 3.2. For each $g \in H(f)$, we define an *ordering* on $p^{-1}(g)$ as follows: $(U_1, g), (U_2, g) \in p^{-1}(g)$ and $(U_1, g) \geq (>) (U_2, g)$ if there is a $T > 0$ such that

$$
h(\Pi(U_1, g, t), \Pi(U_2, g, t)) \geq (>) 0
$$

for all $t \geq T$, where $h$ is defined by (3.8).

An immediate consequence of Corollary 3.2 is the following.

Theorem 3.3. *'$\geq$' is a total ordering on each $p^{-1}(g)$ $(g \in H(f))$ and $\Pi$ is order preserving, that is, $(U_1, g) > (U_2, g)$ implies $\Pi(U_1, g, t) > \Pi(U_2, g, t)$ for all $t > 0$.*

### 3.2. Dynamics of Π.

Let $\Pi$ be the skew-product semiflow (3.3).

Theorem 3.4. (Minimal sets) *Let $E \subset X \times H(f)$ be a minimal set of $\Pi$. Then the following holds.*

1) (*Almost automorphy*)
   a) *$(E, \mathbb{R})$ is an almost 1-1 extension of $(H(f), \mathbb{R})$;*
   b) *$(E, \mathbb{R})$ is almost automorphic if and only if $(H(f), \mathbb{R})$ is almost automorphic;*
   c) *Let $f$ in (3.1) or (3.2) be uniformly almost automorphic (hence $(H(f), \mathbb{R})$ is almost automorphic) and let $(U, g) \in E$ be an almost automorphic point (there are residually many), then $u(U, g, t)$ is a (uniform) almost automorphic solution of $(3.4)_g$ or $(3.5)_g$, and moreover $\mathcal{M}(u) \subset \mathcal{M}(f)$.*

2) (*Ergodicity*)

a) $(E,\mathbb{R})$ *is uniquely ergodic if and only if* $(H(f),\mathbb{R})$ *is uniquely ergodic* (*e.g., $f$ is uniformly almost periodic*) *and* $\mu(Y_0)=1$*, where $\mu$ is the ergodic measure on* $(H(f),\mathbb{R})$ *and* $Y_0=\{y\in Y \mid card p^{-1}(y)\cap E=1\}$*;*

b) *If* $(E,\mathbb{R})$ *is uniquely ergodic, then it is isomorphic* (*topologically conjugate*) *to a subflow of* $(\mathbb{R}^1\times H(f),\mathbb{R})$.

*Proof.* We first define a strong ordering on each fiber $E\cap p^{-1}(g)$ $(g\in H(f))$ as follows: $(U_1,g)\gg(U_2,g)$ if there are neighborhoods $\mathcal{N}_1$, $\mathcal{N}_2$ of $(U_1,g)$, $(U_2,g)$ in $E\cap p^{-1}(g)$ respectively such that $(U_1^*,g)>(U_2^*,g)$ for any $(U_i^*,g)\in\mathcal{N}_i$ $(i=1,2)$. Let $Y_0$ be the residual set which satisfies Part I, Lemma 2.16. For $g_0\in Y_0$ with $card E\cap p^{-1}(g_0)>1$, we denote

$$Z(g_0)=\min_{\substack{(U_1,g_0),(U_2,g_0)\in E\cap p^{-1}(g_0)\\ (U_1,g_0)\neq(U_2,g_0)}}\ \inf_{t>0} Z(u(U_1,g_0,t)-u(U_2,g_0,t)). \tag{3.10}$$

By the proof of Corollary 3.2, there are $(U_1^*,g_0),(U_2^*,g_0)\in E\cap p^{-1}(g_0)$ and $T>0$ such that $Z(g_0)\equiv Z(u(U_1^*,g_0,t)-u(U_2^*,g_0,t))$, and $u(U_1^*,g_0,t)-u(U_2^*,g_0,t)$ admits only simple zeros as $t\geq T$. We claim that such $(U_1^*,g_0)$, $(U_2^*,g_0)$ forms a strongly order preserving pair in the sense of Definition 3.2, Part II. To see this, assume without loss of generality that $(U_1^*,g_0)>(U_2^*,g_0)$. Note that $X\hookrightarrow C^1(0,1)$ and $u(U_1^*,g_0,T)-u(U_2^*,g_0,T)$ admits only simple zeros. One can choose neighborhoods $\tilde{\mathcal{N}_1}$, $\tilde{\mathcal{N}_2}$ of $(U_1^*,g_0)$, $(U_2^*,g_0)$ in $E$ respectively such that $Z(u(U_1,g,T)-u(U_2,g,T))\equiv Z(g_0)$ and $h(\Pi(U_1,g,T)-\Pi(U_2,g,T))>0$ for all $(U_i,g)\in\tilde{\mathcal{N}_i}$ $(i=1,2)$. Now, for any $(U_i,g_0)\in\tilde{\mathcal{N}_i}\cap p^{-1}(g_0)$, $(i=1,2)$, by the definition of $Z(g_0)$, one has $Z(u(U_1,g_0,t)-u(U_2,g_0,t))\equiv Z(g_0)$ for all $t\geq T$. Hence $u(U_1,g_0,t)-u(U_2,g_0,t)$ admits only simple zeros as $t\geq T$. Thus, $h(\Pi(U_1,g_0,t),\Pi(U_2,g_0,t))>0$ for all $t\geq T$, that is, $(U_1,g_0)>(U_2,g_0)$. This shows that $(U_1^*,g_0)\gg(U_2^*,g_0)$. Next, let $(U_i,g_0)\in E\cap p^{-1}(g_0)$ $(i=1,2)$ be such that $(U_i,g_0)\cdot t_0\in\tilde{\mathcal{N}_i}$ $(i=1,2)$ for some $t_0<0$. By the same arguments as above, one has that $Z(u(U_1,g_0,t)-u(U_2,g_0,t))\equiv Z(g_0)$ and $u(U_1,g_0,t)-u(U_2,g_0,t)$ admits only simple zeros as $t>-t_0+T$. Again, it follows that $(U_1,g_0)$, $(U_2,g_0)$ forms a strongly ordered pair. Since $(U_1,g_0)\cdot t_0>(U_2,g_0)\cdot t_0$, by Theorem 3.3,

$$(U_1,g_0)=((U_1,g_0)\cdot t_0)\cdot(-t_0)>(U_2,g_0)=((U_2,g_0)\cdot t_0)\cdot(-t_0),$$

that is, $(U_1,g_0)\gg(U_2,g_0)$. Above all, $(U_1^*,g_0)$, $(U_2^*,g_0)$ is a strongly order preserving pair.

Now, 1) a) follows immediately from Theorem 3.1, Part II.

If $(E,\mathbb{R})$ is almost automorphic, $(Y,\mathbb{R})$ is clearly almost automorphic, where $Y=H(f)$. We now assume that $(Y,\mathbb{R})$ is almost automorphic. Then the set $\hat{Y}$ of almost automorphic points of $Y$ is residual (Part I, Remark 2.6 2)). It follows that $\hat{Y}\cap Y_0$ is residual and each singleton $\{(U,g)\}=p^{-1}(g)\cap E$ $(g\in\hat{Y}\cap Y_0)$ is an almost automorphic point of $X$. 1) b) is then proved.

To show 1) c), we first note that if $(U,g) \in E$ is an almost automorphic point, then $g = p((U,g))$ is an almost automorphic point in $H(f)$ and $p^{-1}(g) \cap E = \{(U,g)\}$ (see Part I, Corollary 2.15). It follows that for any sequence $\alpha \subset \mathbb{R}$, whenever $T_\alpha g = g$, then $T_\alpha u(U,g,\cdot) = u(U,g,\cdot)$. By Corollary 3.7 and Theorem 3.8 of Part I, $\mathcal{M}(u(U,g,\cdot)) \subset \mathcal{M}(g) = \mathcal{M}(f)$.

The proof of 2) a), b) was given in [49] by using invariant foliations and zero number properties. We only show a simple argument from [27] for 2) a) in the case of scalar ODE's. In this case, $X = \mathbb{R}^1$. Define $a_i : E \to \mathbb{R}^1$ $(i = 1,2)$ as

$$a_1(g) = \max\{U \in \mathbb{R}^1 \mid (U,g) \in E \cap p^{-1}(g)\},$$
$$a_2(g) = \min\{U \in \mathbb{R}^1 \mid (U,g) \in E \cap p^{-1}(g)\}.$$

Suppose that $(H(f),\mathbb{R})$ is uniquely ergodic with $\mu(Y_0) = 1$ and denote

$$E_0 = \{(U,g) \in E \mid p^{-1}p((U,g)) = \{(U,g)\}\}.$$

Then $pE_0 = Y_0$. Let $\nu$ be an invariant probability measure on $(E,\mathbb{R})$. Recall that $p$ induces an onto map $p^* : M(E) \to M(H(f))$ of spaces of (Borel) probability measures and $p^*$ preserves invariant measures. We have $p^*(\nu) = \mu$ and $\nu(A) = \mu(P(A))$ for all Borel sets $A \subset E_0$, that is, $\nu$ is unique.

Conversely, suppose that $(E,\mathbb{R})$ is uniquely ergodic. Clearly, $(H(f),\mathbb{R})$ is also uniquely ergodic. Again, we denote $\mu$ as the ergodic measure on $(H(f),\mathbb{R})$. If $\mu(Y_0) = 0$, then the functionals $l_i : C(E) \to \mathbb{R}$,

$$l_i(f) = \int_Y f(a_i(g),g)d\mu$$

$(i = 1,2)$ would define distinct invariant measures on $(E,\mathbb{R})$, a contradiction. Since $Y_0$ is invariant, $\mu(Y_0) = 1$. □

Theorem 3.5. ($\omega$-limit sets) *Consider* $\Pi$. *The following holds.*

1) *Each $\omega$-limit set contains at most two minimal sets.*
2) *Let $\omega(U_0,g_0)$ be an $\omega$-limit set. Then $p : (\omega(U_0,g_0),\mathbb{R}) \to (H(f),\mathbb{R})$ is a* 1-1 *extension (hence $\omega(U_0,g_0)$ is minimal) provided that one of the following conditions holds:*
   a) $\Pi(U_0,g_0,t)$ *is uniformly stable;*
   b) $\omega(U_0,g_0)$ *is* hyperbolic*, that is, the linear skew-product (semi-) flow generated by*

$$\begin{cases} u_t = u_{xx} + a((U,g)\cdot t,x)u_x + b((U,g)\cdot t,x)u, & t > 0, \quad 0 < x < 1 \\ \beta u(i,t) + (1-\beta)u_x(i,t) = 0, & t > 0, \quad i = 0,1, \end{cases}$$

   *where* $a((U,g),x) \equiv F_p(U,U_x,x,g)$, $b((U,g),x) \equiv F_u(U,U_x,x,g)$, $\beta = 0,1$, *or by*

$$u' = F_u((U,g)\cdot t)$$

   $((U,g) \in \omega(U_0,g_0))$ *admits an exponential dichotomy.*
   c) $f_u \le 0$.

*Proof.* 1) See [46] for details.

2) a) is an easy consequence of Theorem 3.4 and Part II, Theorem 2.8. We note that in Part II, Theorem 2.8, $(Y,\mathbb{R}) := (H(f),\mathbb{R})$ was assumed to be distal only to ensure flow extension of an $\omega$-limit set. In the current case, flow extension on an $\omega$-limit set is automatic (see Section 2.2).

2) b) is proved in [47] by using center manifold theory ([11]) and the Floquet theory developed in [10].

To prove 2 c), we note by the maximum principle ([16], [40]) that $L((U_1, g), (U_2, g)) \equiv \max_{x\in[0,1]} |U_1(x) - U_2(x)|$ is a Lyapunov function, that is, $\Pi$ is contracting (see Part II, Definition 2.10). By Part II, Lemma 2.10, $(\omega(U_0, g_0),\mathbb{R})$ is distal, hence a 1-1 extension by Theorem 3.4 1). □

Corollary 3.6. (Asymptotic almost periodicity) *Assume that $f$ is uniformly almost periodic. A solution $u(U, f, t)$ of* (3.1) *or* (3.2) *is* asymptotically almost periodic (*that is, there is an almost periodic solution $u(U^*, f, t)$ of* (3.1) *or* (3.2) *such that $\|u(U, f, t) - u(U^*, f, t)\| \to 0$ as $t \to \infty$*) *if and only if $\omega(U, f)$ is almost periodic and minimal.*

*Proof.* If $u(U, f, t)$ is asymptotically almost periodic, then it is clear that $\omega(U, f) = \omega(U^*, f)$. Since $(U^*, f)$ is an almost periodic point, $cl\{(U^*, f)\cdot t\} = \omega(U^*, f)$ is minimal and almost periodic. By Theorem 3.4, $\omega(U, f)$ is a 1-cover of $H(f)$.

Conversely, let $\omega(U, f)$ be a 1-cover of $H(f)$ and denote $(U^*, f) = \omega(U, f) \cap p^{-1}(f)$. $u(U^*, f, t)$ is clearly almost periodic. Since $\omega(U^*, f) \subset \omega(U, f)$ and they are both 1-covers of $H(f)$, $\omega(U^*, f) = \omega(U, f)$. It follows that for any sequence $t_n \to \infty$, $\|u(U, f, t_n) - u(U^*, f, t_n)\| \to 0$, that is, $\|u(U, f, t) - u(U^*, f, t)\| \to 0$ as $t \to \infty$. □

### 3.3. Comments and Remarks.

1) By Theorem 3.4 1) a) or b), if $f$ is uniformly almost periodic, then any minimal set is of course almost automorphic. Theorem 3.4 1) b) simply says that dynamics of $\Pi$ is always closed within the category of almost automorphy although it need not be closed within that of almost periodicity. This indicates an essentialness of the notion of almost automorphy. We shall show in the next section that all results in Theorems 3.4, 3.5 are sharp, that is, when $f$ is uniformly almost periodic, an $\omega$-limit set of $\Pi$ need not be minimal and may contains two minimal sets, a minimal set need not be almost periodic and need not be uniquely ergodic. Therefore, these results indicate major differences between a periodic time dependence and the almost periodic one. In the case that $f$ is periodic in $t$, as shown in [2], [8], [9], each $\omega$-limit set (minimal set) of $\Pi$ is always periodic minimal with the same period as $f$ (hence uniquely ergodic).

2) Since an $\omega$-limit set contains at least one minimal set, Theorem 3.4 in the case that $f$ is uniformly almost periodic, particularly implies the existence of almost automorphic solutions over the hull in the following sense: If the original equation (3.1) or (3.2) admits a forward bounded solution $u(U, f, t)$, then

for residually many $g \in H(f)$, equation $(3.4)_g$ or $(3.5)_g$ admits an almost automorphic solution which 'lies' in $\omega(U, f)$. However, this does not mean that the original equation can have an almost automorphic solution, simply because, $f$ need not lie in the residual set of $H(f)$ which gives almost automorphic motions. This observation again justifies the important role played by the notion of skew-product semiflow. It suggests that, to study dynamics of a (non-periodic) nonautonomous equation, one has to study its 'compactification' rather than the equation alone. We remark that if $f$ is uniformly almost periodic and if one of the conditions in Theorem 3.5 2) holds, then there certainly exists an almost periodic solution in the original equation (3.1) or (3.2).

3) Let $f$ be uniformly almost periodic and let $E$ be a non-almost periodic almost automorphic minimal set of $\Pi$. A natural question is the following: what cause the appearance of such a minimal set?

i) (Stability and hyperbolicity): By Theorem 3.5 2), such a minimal set is neither uniformly stable nor hyperbolic, that is, it may only possesses a weaker stability or a weaker hyperbolicity. Therefore, it can be very sensitive to small perturbations.

ii) (Spatial variations): We say $(E, \mathbb{R})$ is a *pure PDE flow* if it is not isomorphic to any subflow of $(\mathbb{R}^1 \times H(f), \mathbb{R})$. Thus, in a pure PDE flow, the space variable $x \in (0, 1)$ should somehow effect its dynamics. By Theorem 3.4 2), pure PDE flows (which are largely expected) are certainly non-almost periodic almost automorphic since they are not uniquely ergodic.

In fact, the ergodicity issue should reflect somewhat complicated dynamical features of $\Pi$ which result from non-periodic time variations because, by Theorem 3.4 2) a), in a non-uniquely ergodic minimal set $E$ of $\Pi$, the set of almost automorphic points carries zero measure with respect to any invariant measure on $(E, \mathbb{R})$. As we remarked in the introduction of Part I, among certain symbolic dynamical systems, it is shown (see [32]) that non-ergodic almost automorphic minimal sets having positive topological entropy is generic. This point is certainly worthy for a further study in differential equations. Another evidence is an example in the next section in which a non-uniquely ergodic minimal set $E$ of $\Pi$ presents certain complicated topological features.

4) In the case that $f$ is uniformly almost periodic, the onset of almost automorphic dynamics represents a kind of 'non-uniform' asymptotic behavior of bounded solutions. Even though an $\omega$-limit set $\omega(U_0, f)$ is minimal and $\omega(U_0, f) \cap p^{-1}(f) = \{(U_*, f)\}$ is a single almost automorphic point, $\omega(U_0, f, t)$ need not be asymptotically almost automorphic since there is no guarantee that $\|u(U_0, f, t) - u(U_*, f, t)\| \to 0$ unless $\omega(U_0, f)$ is almost periodic minimal. Instead, one can generally conclude that $u(U_0, f, t)$ is 'almost' asymptotically almost automorphic since $\omega(U_0, g)$, as an almost automorphic minimal set, contains residually many almost automorphic points.

5) Let $f$ be uniformly almost automorphic (almost periodic). By Theorem 3.4, we see that if $u(U, g, t)$ is any (uniform) almost automorphic or almost periodic

solution of $(3.4)_g$ or $(3.5)_g$, then $\mathcal{M}(u(U,g,\cdot)) \subset \mathcal{M}(f)$ since $E \equiv cl\{(U,g)\cdot t\}$ is minimal. In other word, no subharmonic (uniform) almost automorphic or almost periodic solution may occur. This '1-dimensional' property need not hold in 'higher dimensions' (see the next two sections). There are some properties which do not depend on either the phase dimension of an ODE or the space dimension of a parabolic PDE. For examples, in certain circumstances, one can have only one minimal set in an $\omega$-limit set, one can also give an explicit condition so that a global attractor exists. We leave details to later sections.

**3.4. Examples.**

We give a few examples of almost periodic scalar ODE's and PDE's which exhibit almost automorphic phenomena. Those examples of ODE's may be also viewed as examples of parabolic PDE's with a Neumann boundary condition.

1) *Existence of a non-almost periodic almost automorphic minimal set.*

EXAMPLE 3.1. A linear almost periodic scalar ODE

$$x' = A(t)x + B(t) \tag{3.11}$$

is constructed by Johnson in [26] satisfying following properties:

a) $A(t)$, $B(t)$ are uniform limits of $2^n$-periodic functions $A_n(t)$, $B_n(t)$ respectively;
b) $\int_0^t A(s)ds \to \infty$ as $t \to \infty$;
c) If $x_0(t)$ is the solution of (3.11) with $x_0(0) = 0$, then $|x_0(t)| \le 1$, and as $n \ge 4$,

$$x_0(2^n) = \begin{cases} \frac{1}{5}, & n \text{ odd} \\ 0, & n \text{ even.} \end{cases} \tag{3.12}$$

Let $\Pi_t : \mathbb{R}^1 \times Y \to \mathbb{R}^1 \times Y$ be the skew-product flow generated by (3.11), where $Y = H(A,B)$. Since $\lim_{t\to\infty} \int_0^t A(s)ds = \infty$, all nontrivial solutions of

$$x' = A(t)x \tag{3.13}$$

are unbounded. It follows that $x_0(t)$ is the only bounded solution of (3.11), that is, $E = \omega(0, y_0)$ is the only minimal set of $\Pi_t$, and $card p^{-1}(y_0) \cap E = 1$, where $y_0 = (A,B)$, $p : R^1 \times Y \to Y$ is the natural projection. Since $(0, y_0)$ is an almost automorphic point (Part I, Corollary 2.15), $x_0(t)$ is an almost automorphic solution of (3.11) with $\mathcal{M}(x_0) \subset \mathcal{M}(A,B)$ (Theorem 3.4 1) c)).

We claim that $x_0(t)$ is not almost periodic. For otherwise, $E$ would be a 1-cover of $Y$. Since $\{2^n | n = 1, 2, \cdots\} \subset \mathcal{M}(A,B)$, both $\lim_{n\to\infty} A(2^n)$ and $\lim_{n\to\infty} B(2^n)$ exist uniformly in $t \in \mathbb{R}$ by the classical theory of almost periodic functions ([15], [30]). It follows that $\lim_{n\to\infty} x_0(2^n)$ also exists, which contradicts (3.12). Thus, $E$ is a non-almost periodic almost automorphic minimal set.

It is shown in Johnson [27] that this minimal set $E$ is uniquely ergodic.

We now give an example of quasi-periodic time dependence.

Example 3.2. Consider a differential equation on the torus

$$x' = f(x,t) \tag{3.14}$$

where $f(x,t+1) = f(x+1,t) = f(x,t)$. Let $x(t,\eta)$ be the solution of (3.14) with $x(0,\eta) = \eta$. Consider the Poincaré map $\psi$: $\eta \mapsto x(1,\eta)$. It is well known that when the rotation number $\rho$ of $\psi$ is irrational, the limit set $S'$ of $\{\psi^n(\eta)(\text{mod } 1) | n = 1, 2, \cdots\}$ is either $[0,1]$ or a Cantor set (see [18]).

We let $f$ in (3.14) be such that $S'$ is a cantor set and consider

$$x' = f(x+\rho t, t) - \rho. \tag{3.15}$$

Clearly, (3.15) is quasi-periodic in $t$ with frequences $2\pi$ and $2\pi\rho$. As shown in Fink [15], equation (3.15) admits a bounded solution but no almost periodic solution. Therefore, by Theorem 3.4, the skew-product flow generated by (3.15) admits a non-almost periodic almost automorphic minimal set.

2) *An $\omega$-limit set which contains two minimal sets.*

Example 3.3. Consider the scalar ODE

$$u' = -(A(t)\cos u + B(t)\sin u)\sin u, \tag{3.16}$$

where $f(t) \equiv (A(t), B(t))$ is as in Example 3.1.

For $(U,g) \equiv (U, a_g, b_g) \in \mathbb{R}^1 \times H(f)$, denote by $u(U,g,t)$ the solution of

$$u' = -(a_g(t)\cos u + b_g(t)\sin u)\sin u \tag{3.17}$$

with $u(U,g,0) = U$. Then

$$\Pi_t(U,g) = (u(U,g,t), g\cdot t) \tag{3.18}$$

is the skew product flow on $\mathbb{R}^1 \times H(f)$ generated by (3.16).

Clearly, $E_1 = \{0\} \times H(f)$ is an almost periodic minimal set of $\Pi_t$. We now consider transformation $x(t) = \cot u(t)$ to (3.16). A simple calculation shows that $x(t)$ satisfies (3.11). By Example 3.1, $E = cl\{(x_0(t), f\cdot t) \mid t \in \mathbb{R}^1\}$ is a non-almost periodic, almost automorphic minimal set of the skew-product flow generated by (3.11), that is, $E_2 = cl\{\Pi_t(\pi/2, f) \mid t \in \mathbb{R}^1\} \subset (0,\pi) \times H(f)$ is a non-almost periodic, almost automorphic minimal set of (3.18). Define $u(g) = \min\{U \mid (U,g) \in E_2\}$. We shall show that there are $g_0 \in H(f)$, $U_0 \in (0, u(g_0))$ such that

$$E_1 \cup E_2 \subset \omega(U_0, g_0).$$

To do so, for each $(U,g) = (U, a_g, b_g) \in E_2$, we consider the transformation

$$\cot u = \frac{\cot \tilde{u}}{\sin u(U,g,t)} + \cot u(U,g,t) \tag{3.19}$$

to (3.17). The equation for $\tilde{u}$ reads

$$\tilde{u}' = \beta((U,g)\cdot t)\sin\tilde{u}\cos\tilde{u}, \tag{3.20}$$

where $\beta((U,g)\cdot t) = -a_g(t)\sin^2 u(U,g,t) + b_g(t)\sin u(U,g,t)\cos u(U,g,t)$.

Let $\widetilde{\Pi}_t$ denote the skew-product flow on $\mathbb{R}^1 \times E_2$ generated by (3.20). Then minimal sets $\widetilde{E}_1 = \{0\}\times E_2$, $\widetilde{E}_2 = \{\pi/2\}\times E_2$ of $\widetilde{\Pi}_t$ correspond to $E_1$ and $E_2$ respectively in terms of the transformation (3.19). By arguments in [25], one has

$$\lim_{|t|\to\infty}\frac{1}{t}\int_0^t \beta((U,g)\cdot s)ds = 0. \tag{3.21}$$

Since $E$ is the only minimal set of the skew-product flow generated by (3.11), $E_1$ and $E_2$ are only minimal sets of $\Pi_t$ in $[0,\pi)\times H(f)$. It follows that $\tilde{E}_1$ and $\tilde{E}_2$ are only minimal sets of $\tilde{\Pi}_t$ in $[0,\frac{\pi}{2}]\times E_2$. Note that $V \equiv \cot\tilde{u}$ satisfies

$$V_t = -\beta((U,g)\cdot t)V, \tag{3.22}$$

that is, $\cot\tilde{u} = \cot\tilde{U}e^{-\int_0^t \beta((U,g)\cdot s)ds}$ ($\tilde{u}(0) = \tilde{U}$). We now take $(\tilde{U},U,g) \in (0,\pi/2)\times E_2$. It is easy to see that there is a sequence $t_n \to \infty$ such that if $\tilde{u}(t) \equiv \tilde{u}(t,\widetilde{U},U,g)$ is the solution of (3.20) with $\tilde{u}(0) = \widetilde{U}$, then $\tilde{u}(t_n)$ converges to either 0 or $\pi/2$, that is, $\cot\tilde{u}(t_n)$ converges to either $+\infty$ or 0. Thus, $\int_0^{t_n}\beta((U,g)\cdot s)ds$ converges to either $+\infty$ or $-\infty$. In any case, $\int_0^t \beta((U,g)\cdot s)ds$ is unbounded. Using this fact and (3.21), one has by [25] that the set

$$E_0 = \left\{(U,g)\in E_2 \mid \limsup_{t\to\infty}\int_0^t \beta((U,g)\cdot s)ds = \infty,\right. \\ \left.\liminf_{t\to\infty}\int_0^t \beta((U,g)\cdot s)ds = -\infty\right\} \tag{3.23}$$

is a residual subset of $E_2$. Fix a $(\widetilde{U},U,g_0)\in(0,\pi/2)\times E_0$. It follows from (3.23) that $\widetilde{E}_1\cup\widetilde{E}_2 \subset \omega(\widetilde{U},U,g_0)$. Let $U_0 = \cot^{-1}\left(\frac{\cot\widetilde{U}}{\sin U} + \cot U\right)$. Then $E_1\cup E_2 \subset \omega(U_0,g_0)$.

3) *An $\omega$-limit set which is not minimal and contains only one minimal set.*

EXAMPLE 3.4. Consider

$$\begin{cases} u_t = u_{xx} + (f(t)-\lambda_1)u, & 0<x<1, \quad t>0, \\ u(t,0) = u(t,1) = 0, & t>0, \end{cases} \tag{3.24}$$

where $f(t)$ is an almost periodic function with the Fourier series $f(t) \sim -\sum_{k=1}^{\infty} 2^{-k}\pi\sin(2^{-k}\pi t)$, $\lambda_1$ is the first eigenvalue of $\frac{\partial^2}{\partial x^2} : H_0^2(0,1)\to L^2(0,1)$.

We consider the skew product semiflow $\Pi_t$ on $X \times H(f)$:

$$\Pi_t(U, g) = (u(U, g, \cdot, t), g \cdot t) \tag{3.25}$$

generated by (3.24) in the usual way, where $X$ is a fractional power space. Let $U_1$ be the first eigenfunction of $\frac{\partial^2}{\partial x^2} : H_0^2(0,1) \to L^2(0,1)$. It is easy to see that $u(t, \cdot, U_1, f) = e^{\int_0^t f(s)ds} U_1$ is a solution of (3.24). By discussions in [43], $\phi(t) = e^{\int_0^t f(s)ds}$ satisfies the following properties:

1) $\phi(t)$ is bounded for $t \geq 0$;
2) There is a sequence $t_n \to \infty$ such that $\phi(t_n) \to 0$ and $\phi(2^n) \geq e^{-2\pi-2}$ for $n = 1, 2, \cdots$;
3) For any sequence $t_n \to \infty$ such that $\lim_{n\to\infty} \phi(t + t_n) = \phi^*(t)$ exists, $\phi^*(t)$ is not almost periodic if it is non-zero.

It is clear by the above properties that $\omega(U_1, f)$ is not minimal, and, $\{0\} \times H(f)$ is the only minimal set in $\omega(U_1, f))$. We note that $\omega(U_1, g)$ is neither uniformly stable nor hyperbolic by Theorem 3.5 2).

4) *Existence of non-uniquely ergodic minimal set.*

We have seen that the minimal set in Example 3.1 is uniquely ergodic, almost automorphic, but not almost periodic.

An idea of constructing examples of non-uniquely ergodic almost automorphic minimal set is suggested by Johnson ([25], [28]) as follows. Consider linear systems of ODE's

$$z' = A(y \cdot t)z, \tag{3.26}$$

where $z \in \mathbb{R}^2$, $y \in Y$, $(Y, \mathbb{R})$ is an almost periodic minimal flow. Let $\theta = arg z$. Then $\theta$ satisfies a scalar equation

$$\theta' = f(\theta, y \cdot t), \tag{3.27}$$

where $f$ is periodic in $\theta$ with period $\pi$. Thus, (3.27) induces a skew-product flow $\tilde{\Pi}_t$ on $P^1 \times Y$, where $P^1$ is the real projective 1-space. By Sacker-Sell spectrum theory ([42]), the spectrum of (3.26) is either a point, two points, or a nondegenerate closed interval. Suppose that the Sacker-Sell spectrum $\Sigma$ of (3.26) is a nondegenerate closed interval. It is shown in [29], [42] that $\tilde{\Pi}_t$ contains a unique minimal set $\tilde{E}$, and in [25] that there are exactly two ergodic measures on $\tilde{E}$. Let $\Pi_t$ be the skew-product flow on $\mathbb{R}^1 \times Y$ generated by

$$x' = f(x, y \cdot t), \quad x \in \mathbb{R}^1. \tag{3.28}$$

Then $\Pi_t$ has a minimal set with exactly two ergodic measures (therefore, it can not be almost periodic).

EXAMPLE 3.5. A typical such system is constructed by Vinograd ([56]) as follows.

Consider

$$x' = A(y \cdot t)x \equiv \begin{pmatrix} 0 & 1 + a(y \cdot t) \\ 1 - a(y \cdot t) & 0 \end{pmatrix} x, \tag{3.29}$$

where $y \in T^2$, $y \cdot t = (y_1 + t, y_2 + \alpha t)$, $\alpha$ is irrational.

Let $\theta = \arg x$. Then $\theta$ satisfies

$$\theta' = -a(y \cdot t) + \cos 2\theta. \tag{3.30}$$

The equation (3.29) has the following properties.

1) $a(y)$ is the limit of a nondecreasing sequence $\{a_n(y)\}$ and $a_n(y) \geq 0$.
2) For $y_0 = (0, 0) \in T^2$ and for each $n$, the equation

$$\theta' = -a_n(y_0 \cdot t) + \cos 2\theta \tag*{$(3.31)_n$}$$

has two solutions $\{\theta_1^n(t)\}$, $\{\theta_2^n(t)\}$ such that

$$-\frac{\pi}{4} < \theta_1^n(t) < \theta_1^{n+1}(t) < \theta_2^{n+1}(t) < \theta_2^n(t) < \frac{\pi}{4} \quad (n \geq 1), \tag{3.32}$$

$$0 < \inf_t d(\theta_1^n(t), \theta_2^n(t)) \equiv \gamma_n \to 0. \tag{3.33}$$

3) The equation

$$x' = \begin{pmatrix} 0 & 1 + a_n(y \cdot t) \\ 1 - a_n(y \cdot t) & 0 \end{pmatrix} x \tag*{$(3.34)_n$}$$

has two Lyapunov exponents $\beta_n$, $-\beta_n$ with $\beta_n > \frac{1}{2}$.

We now summarize some additional properties of (3.29) studied in Johnson [28].

1) The Sacker-Sell spectrum of $(3.34)_n$ is $\Sigma_n = \{-\beta_n, \beta_n\}$ but the Sacker-Sell spectrum $\Sigma$ of (3.29) is a nondegenerate interval containing $[-\frac{1}{2}, \frac{1}{2}]$.
2) $E_1^n = cl\{(\theta_1^n(t), y_0 \cdot t)\}$, $E_2^n = cl\{(\theta_2^n(t), y_0 \cdot t)\}$ are disjoint almost periodic minimal sets of the skew-product flow $\Pi_t^n$ on $P^1 \times T^2$ which is generated by $(3.31)_n$, that is, $E_1^n$, $E_2^n$ are 1-covers of $T^2$.
3) Let $E_1^n = \{(g_n(y), y) \mid y \in T^2\}$, $E_2^n = \{(h_n(y), y) \mid y \in T^2\}$. Then

$$-\frac{\pi}{4} < g_n(y) \leq g_{n+1}(y) < h_{n+1}(y) \leq h_n(y) < \frac{\pi}{4}.$$

4) Let $g(y) = \lim_{n\to\infty} g_n(y)$, $h(y) = \lim_{n\to\infty} h_n(y)$. Then $Y_0 = \{y \in T^2 \mid g(y) = h(y)\}$ is a residual subset of $T^2$. Denote $\tilde{E} = \{(\theta, y) \in P^1 \times T^2 \mid g(y) \le \theta \le h(y)\}$ and let $E = cl\{(g(y_0 \cdot t), y_0 \cdot t) \mid t \in \mathbb{R}\}$ for a fixed $y_0 \in Y_0$. Then $E \subset \tilde{E}$ is the unique almost automorphic minimal set of $\Pi_t$, and moreover $E$ supports exactly two ergodic measures.
5) $\tilde{E}$ is an isolated invariant set. $\tilde{E}$ has the following complicated nature: a) $\tilde{E}$ is connected; b) $\tilde{E}$ is locally connected at all points where $g(y) = h(y)$; c) $\tilde{E}$ is not locally connected at all points.

This example shows that, in the case of scalar almost periodic ODE's (thus in scalar parabolic PDE's in 1-space dimension with a Neumann boundary condition), if a minimal set $E$ in the associated skew-product flow is almost automorphic but not uniquely ergodic, then some complicated (topological or dynamical) natures on $E$ or in vicinity of $E$ may be expected.

## 4. System of Ordinary Differential Equations

### 4.1. Linear System.

Consider

$$x' = A(t)x + B(t), \quad x \in \mathbb{R}^n, \tag{4.1}$$

where $A : \mathbb{R} \to L(\mathbb{R}^n, \mathbb{R}^n)$, $B : \mathbb{R} \to \mathbb{R}^n$ are almost periodic. Let $Y = H(A, B)$. By Section 2, there are continuous functions $a : Y \to L(\mathbb{R}^n, \mathbb{R})$, $b : Y \to \mathbb{R}^n$ which extend $A$ and $B$ respectively, that is, $a(y_0 \cdot t) \equiv A(t)$, $b(y_0 \cdot t) \equiv B(t)$, where $y_0 \equiv (A, B)$. Now, the family of equations

$$x' = a(y \cdot t)x + b(y \cdot t) \tag*{$(4.2)_y$}$$

($y \in Y$) generates a skew-product flow $\Pi : \mathbb{R}^n \times Y \times \mathbb{R} \to \mathbb{R}^n \times Y$ in the usual way:

$$\Pi(x_0, y, t) = (x(x_0, y, t), y \cdot t), \tag{4.3}$$

where $x(x_0, y, t)$ denotes the solution of $(4.2)_y$ with initial value $x_0$.

With our terminology, the classical Favard theorem states as follows:

Theorem 4.1. (Favard [13], [14]) *If for all $y \in Y$, any nontrivial bounded solution $x(t)$ of*

$$x' = a(y \cdot t)x \tag*{$(4.4)_y$}$$

*satisfies* $\inf_{t\in\mathbb{R}} | x(t)| > 0$ *and if* (4.1) *admits a bounded solution, then* (4.1) *has an almost periodic solution $x_0(t)$ with $\mathcal{M}(x_0) \subset \mathcal{M}(A, B)$.*

Similar to the study for $N$-almost periodic solutions ([30]), we now look for weaker conditions which gurantee the existence of an almost automorphic solution in (4.1).

THEOREM 4.2. *If any nontrivial bounded solution $x(t)$ of*

$$x' = A(t)x \tag{4.5}$$

*satisfies* $\inf_{t\in\mathbb{R}} |x(t)| > 0$ *and if* (4.1) *admits a bounded solution, then* (4.1) *has an almost automorphic solution* $x_0(t)$ *with* $\mathcal{M}(x_0) \subset \mathcal{M}(A, B)$.

*Proof.* We use arguments of [54].

Denote $\|\cdot\|$ as the norm in $L^\infty(\mathbb{R}, \mathbb{R}^n)$. Our condition simply implies that (4.1) has a unique bounded solution $x_0(t)$ with minimum $\|\cdot\|$ norm (see [14], [15]). For any sequence $\alpha \subset \mathbb{R}$, it is easy to see that

$$\|T_\alpha x_0\| \le \|x_0\|. \tag{4.6}$$

Let $x_* = T_{-\alpha}T_\alpha x_0$. Applying (4.6) with $-\alpha$, we have

$$\|x_*\| \le \|T_\alpha x\| \le \|x_0\|. \tag{4.7}$$

Since $E = cl\{\Pi(x_0, y_0, t) \mid t \in \mathbb{R}\}$ is invariant to (4.3) and $(x_*(t), y_* \cdot t) = T_{-\alpha}T_\alpha(x_0(t), y_0 \cdot t) = T_{-\alpha}T_\alpha\Pi(x_0(0), y_0, t) \in E$, $x_*(t)$ is a bounded solution of (4.1). Note that $x_0$ is the unique minimum norm solution of (4.1), by (4.7), $x_* = T_{-\alpha}T_\alpha x_0 = x_0$, that is, $x_0(t)$ is an almost automorphic solution of (4.1). It follows that $(E, \mathbb{R})$ is an almost automorphic minimal flow. In fact, it is easy to see that $p : (E, \mathbb{R}) \to (Y, \mathbb{R})$ is an almost automorphic extension and $\{(x_0(0), y_0)\} = p^{-1}(y_0)$, where $p : \mathbb{R}^n \times Y \to Y$ denotes the natural projection. Clearly, for any sequence $\alpha \subset \mathbb{R}$, $T_\alpha y_0 = y_0$ implies $T_\alpha x_0 = x_0$. By Part I, Theorem 3.8, $\mathcal{M}(x_0) \subset \mathcal{M}(y_0) = \mathcal{M}(A, B)$. □

*Remark 4.1.*

1) We first note that conditions of Theorem 4.2 only make references to equation (4.1) itself.

2) By exactly the same arguments, Theorem 4.2 also holds when $A(t)$, $B(t)$ are continuous almost automorphic.

3) The Favard's theorem can be viewed as a corollary of Theorem 4.2. Using arguments of [55], for any $y \in Y$, let $\alpha = \{t_n\}$ be a sequence such that $y_0 \cdot t_n \to y$. One has that $x_* = T_\alpha x_0$ is the unique minimum norm solution of $(4.2)_y$. Thus, by Theorem 4.2, $x_*$ is an almost automorphic solution of $(4.2)_y$. Since each point $(x_*, y) \in E$ is almost automorphic, $(E, \mathbb{R})$ must be almost periodic (Part I, Remark 2.6 3)).

4) Let $A(t)$, $B(t)$ be almost periodic functions as in Example 3.1. Recall that

$$x' = A(t)x + B(t) \tag{4.8}$$

has no almost periodic solution. Let $Y = H(A, B)$ and $a : Y \to \mathbb{R}$, $b : Y \to \mathbb{R}$ be extensions of $A$, $B$ respectively. Then, for each $y \in Y$, no equation

$$x' = a(y \cdot t)x + b(y \cdot t) \tag*{$(4.8)_y$}$$

admits an almost periodic solution. By [26], there is a $\tilde{y} \in Y$ so that $\int_0^t a(\tilde{y} \cdot s)ds \geq -M > -\infty$ for all $t$. Let $\tilde{A}(t) \equiv a(\tilde{y} \cdot t)$, $\tilde{B}(t) \equiv b(\tilde{y} \cdot t)$. Then the equation

$$x' = \tilde{A}(t)x + \tilde{B}(t)$$

is such that conditions of Theorem 4.2 are satisfied but there is no almost periodic solution.

COROLLARY 4.3. *Consider the the skew-product flow* $\Pi$ *in* (4.3) *and assume that* (4.1) *has a bounded solution.*

1) *If for some* $y_0 \in Y$, *all nontrivial solutions* $x(t)$ *of* $(4.4)_{y_0}$ *are bounded and satisfy* $\inf_{t\in\mathbb{R}} \mid x(t)| > 0$, *then* $\Pi$ *admits an almost periodic minimal set* $E$ *and* $p : (E, \mathbb{R}) \to (Y, \mathbb{R})$ *is a* 1-1 *extension. Moreover, for any* $(x_0, y) \in E$, $x(x_0, y, t)$ *is an almost periodic solution of* $(4.2)_y$ *with* $\mathcal{M}(x) \subset \mathcal{M}(A, B)$.
2) *If for some* $y_0 \in Y$, *equation* $(4.4)_{y_0}$ *admits no nontrivial bounded solution, then* $\Pi$ *has a unique minimal set* $E$ *and* $p : (E, \mathbb{R}) \to (Y, \mathbb{R})$ *is an almost* 1-1 (*hence almost automorphic*) *extension. Moreover, if* $x(t)$ *is an almost automorphic solution of* $(4.2)_y$ *for some* $y \in Y$, *then* $\mathcal{M}(x) \subset \mathcal{M}(A, B)$.
3) *If for all* $y \in Y$, $(4.4)_y$ *has a unique bounded solution up to linear dependence, then* $\Pi$ *either admits a unique minimal set* $E$ *and* $p : (E, \mathbb{R}) \to (Y, \mathbb{R})$ *is an almost* 1-1 (*hence almost automorphic*) *extension, or admits infinitely many minimal sets which are all* 1-*covers of* $Y$. *Moreover, if* $x(t)$ *is an almost automorphic* (*almost periodic*) *solution of* $(4.2)_y$ *for some* $y \in Y$, *then* $\mathcal{M}(x) \subset \mathcal{M}(A, B)$.
4) *If for all* $y \in Y$, $(4.4)_y$ *admits no nontrivial bounded solution, then* $\Pi$ *admits a unique minimal set* $E$ *which is a* 1-*cover of* $Y$. *Moreover, for any* $y \in Y$, *bounded solution* $x(x_0, y, t)$ *of* $(4.2)_y$ *is unique, almost periodic, and* $\mathcal{M}(x) \subset \mathcal{M}(A, B)$.

*Proof.* 1) We first note our conditions imply that for all $y \in Y$, any nontrivial solution $x(t)$ of $(4.4)_y$ is bounded and satisfies $\inf_{t\in\mathbb{R}} \mid x(t)| > 0$. Therefore, 1) follows from Theorem 4.1.

2) In this case, $\Pi$ can not have two minimal sets, for otherwise, equation $(4.4)_y$ would have a nontrivial bounded solution for all $y \in Y$. Let $E$ be the unique minimal set of $\Pi$. The same reason as above indicates that $p^{-1}(y_0) \cap E$ must be a singleton, that is, $p : (E, \mathbb{R}) \to (Y, \mathbb{R})$ is an almost automorphic (hence almost 1-1) extension. The module containment result follows from Part I, Remark 2.4 1) and Theorem 3.8.

3) For any $y \in Y$, let $x(\xi(y), t)$ denote the bounded solution of $(4.4)_y$ with $x(\xi(y), 0) = \xi(y)$ and $\|\xi(y)\| = 1$.

First, assume that for all $y \in Y$, $\inf_{t\in\mathbb{R}} \|x(\xi(y), t)\| = 0$. Let $E$ be any minimal set of $\Pi$ and $Y_0 \subset Y$ be the residual set associated to $E$ according to Part I, Theorem 2.16. By compactness of $E$, there is a finite interval $[-c_0, c_0] \in \mathbb{R}$ such that whenever $(x_1, y), (x_2, y) \in E$,

$$x(x_1, y, t) - x(x_2, y, t) = cx(\xi(y), t) \tag{4.10}$$

for some $c \in [-c_0, c_0]$. It follows that $p : (E, \mathbb{R}) \to (Y, \mathbb{R})$ is a proximal extension, in particular, the proximal relation $P(E)$ on $E$ is an equivalence relation. Therefore, the Ellis semigroup $E(E)$ admits a unique minimal (left) ideal $I$ (Part I, Theorem 2.7). For a fixed $y_0 \in Y_0$, let us assume that $p^{-1}(y_0) \cap E$ contains two points, say $(x_0, y_0)$ and $(x_1, y_0)$. By Part I, Theorem 2.2, there is an idempotent point $u \in I$ such that $u(x_*, y_0) = (x_0, y_0)$ for all $(x_*, y_0) \in p^{-1}(y_0) \cap E$, that is, if $t_\alpha \to \infty$ is a net in $\mathbb{R}$ with $\Pi(\cdot, t_\alpha) \to u$ in $E(E)$, then

$$\lim_\alpha x(x_*, y_0, t_\alpha) = x_0 \tag{4.11}$$

for all $(x_*, y_0) \in p^{-1}(y_0) \cap E$. By (4.10), (4.11), it is clear that

$$\lim_\alpha x(\xi(y_0), t_\alpha) = 0.$$

By (4.10) and the above, we see that the convergence in (4.11) is in fact uniform on $p^{-1}(y_0) \cap E$. On the other hand, by arguments of Part I, Theorem 2.16, there exists a net $(x_\alpha, y_0) \subset p^{-1}(y_0) \cap E$ such that

$$\lim_\alpha x(x_\alpha, y_0, t_\alpha) = x_1,$$

a contradiction. Thus $p^{-1}(y_0) \cap E$ is a singleton, that is, $p : (E, \mathbb{R}) \to (Y, \mathbb{R})$ is an almost 1-1 extension. Similar arguments also show that $E$ is the only minimal set of $\Pi$. The module containment result again follows from Part I, Remark 2.4 1) and Theorem 3.8.

Next, suppose that $\inf_{t \in \mathbb{R}} \|x(\xi(y_0), t)\| > 0$ for some $y_0 \in Y$. Then

$$\inf_{t \in \mathbb{R}} \|x(\xi(y), t)\| > 0 \tag{4.12}$$

for all $y \in Y$. Let $\tilde{\Pi}$ denote the linear skew-product flow generated by $(4.4)_y$ and let $\tilde{E}_0 \subset cl\{\tilde{\Pi}(\xi(y_0), y_0, t) | t \in \mathbb{R}\}$ be a minimal set. Then $\tilde{E}_0$ is distal, and by the arguments of Theorem 3.4, a 1-cover of $Y$. Denote

$$\tilde{E}_0 = \{(\tilde{\xi}(y), y) | y \in Y\},$$

where $\tilde{\xi} : Y \to \mathbb{R}^n$ is a continuous function. Then all minimal sets of $\tilde{\Pi}$ have the form

$$\tilde{E} = \{(c\tilde{\xi}(y), y) | y \in Y\} \tag{4.13}$$

for some constant $c$. Therefore, all minimal sets $\tilde{E}$ of $\tilde{\Pi}$ are 1-covers of $Y$.

Now, by (4.12) and Theorem 4.1, there is a minimal set $E_0$ of $\Pi$ which is a 1-cover of $Y$. For each $c \in \mathbb{R}$, it is clear that

$$E = \{(x_0 + c\tilde{\xi}(y), y) | (x_0, y) \in E_0\} \tag{4.14}$$

is a compact invariant set of $\Pi$ which is also a 1-cover of $Y$, that is, $E$ is an almost periodic minimal set. In fact, any minimal set of $\Pi$ is given by (4.14) for some $c \in \mathbb{R}$. To see this, for any $y \in Y$, we let $x(x_*, y, t)$ be any bounded solution of $(4.2)_y$. Then there is a $c \in \mathbb{R}$ such that

$$x(x_*, y, t) = x(x_0, y, t) + cx(\tilde{\xi}(y), t),$$

where $x(\tilde{\xi}(y), t)$ is the solution of $(4.4)_y$ with $x(\tilde{\xi}(y), 0) = \tilde{\xi}(y)$. It follows that $x(t) \equiv x(x_*, y, t)$ is an almost periodic solution and by (4.13) and Theorem 4.1, $\mathcal{M}(x) \subset \mathcal{M}(A, B)$. By Part I, Theorem 3.8, $E = cl\{\Pi(x_*, y, t) | t \in \mathbb{R}\}$ is a 1-cover of $Y$. Clearly, $E$ must have the form of (4.14).

4) It is a easy consequence of 2). □

*Remark 4.2.*

1) Almost automorphy in Corollary 4.3 2) can not be replaced by almost periodicity in general. Let $A$, $B$ be as in Example 3.1. It is easy to see that the equation

$$\begin{cases} x_1' = A(t)x_1 + B(t) \\ x_2' = -A(t)x_2, \end{cases}$$

satisfies the conditions of Corollary 4.3 2), but does not admit any almost periodic solution.

2) Assume the conditions of Corollary 4.3 1). By the classical Liouvill's theorem ([18]), the conditions are equivalent to that $\int_0^t trA(s)ds$ is bounded and all solutions of $(4.4)_y$ for some $y \in Y$ are bounded. By [25], there is a continuous function $C : Y \to \mathbb{R}$ such that $C(y \cdot t) - C(y) = \int_0^t tra(y \cdot s)ds$. Hence, the almost periodic transformation

$$x = e^{\frac{C(y \cdot t) - C(y)}{n}} z \tag{4.15}$$

transforms $(4.2)_y$ equivalently to

$$z' = \tilde{a}(y \cdot t)z + \tilde{b}(y \cdot t) \tag*{$(4.16)_y$}$$

where $\tilde{a}(y \cdot t) = a(y \cdot t) - \frac{1}{n} tra(y \cdot t)I$, $\tilde{b}(y \cdot t) = e^{-\frac{C(y \cdot t) - C(y)}{n}} b(y \cdot t)$.

We now let $n = 2$ and assume that (4.5) admits a nontrivial almost periodic solution. By the almost periodic Floquet theory ([25]), there is a *strong Perron transformation*, $P : Y \to GL(n)$ (that is, $P$ is continuous, and, for each $y \in Y$, the map $\mathbb{R} \to GL(n)$: $t \to P(y \cdot t)$ is continuously differentiable, moreover, $y \to \frac{d}{dt}P(y \cdot t)|_{t=0}$ is continuous) such that

$$z = P(y \cdot t)\hat{z} \tag{4.17}$$

transforms $(4.16)_y$ to

$$\hat{z}' = \hat{a}(y \cdot t)\hat{z} + P^{-1}(y \cdot t)\tilde{b}(y \cdot t),$$

where $\hat{a}$ is a constant matrix $\begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix}$. Therefore, combinning (4.15), (4.17), we see that for any $y \in Y$, all solutions of $(4.2)_y$ are almost periodic and satisfy the following

$$x(t) = e^{\frac{C(y_0 \cdot t) - C(y_0)}{2}} P(y_0 \cdot t) \begin{pmatrix} \cos \omega_0 t & -\sin \omega_0 t \\ \sin \omega_0 t & \cos \omega_0 t \end{pmatrix} c + x_0(t),$$

where $c \in \mathbb{R}^2$ is arbitrary and $x_0(t)$ is an almost periodic solution of $(4.2)_y$ given by Corollary 4.3 1). Clearly, $\mathcal{M}(x) \subset \mathcal{M}(A, B)$ if and only if $\omega_0 \in \mathcal{M}(A, B)$ or $c = 0$.

3) The condition in 4) of the above corollary is equivalent to the fact that the linear skew-product flow $\tilde{\Pi}$ generated by $(4.4)_y$ admits an exponential dichotomy (see [41]). Therefore, Corollary 4.3 4) also follows from the standard theory of exponential dichotomy, that is, for each $y \in Y$, $(4.2)_y$ has a unique bounded solution which is in fact almost periodic ([15]).

4) Let $\Sigma$ denotes the Sacker-Sell spectrum associated to $\tilde{\Pi}$. In the case of Corollary 4.3 1), $\Sigma = \{0\}$ since there is no unbounded solution to any equation $(4.4)_y$, and in the case of Corollary 4.3 4), $0 \notin \Sigma$ because of the existence of an exponential dichotomy. In the other two cases of the above corollary where almost automorphic dynamics are claimed, $\Sigma$ may contain a spectral interval $[-\beta, \beta]$ for some $\beta > 0$.

Corollary 4.4. *Consider the the skew-product flow* $\Pi$ *in* (4.3) *with* $n = 2$ *and assume that* a) (4.1) *has a bounded solution;* b) $\int_0^t trA(s)ds$ *is bounded. Then* $\Pi$ *admits an almost automorphic minimal set* $E$. *Moreover, if* $(x_0, y) \in E$ *is an almost automorphic point, then* $x(x_0, y, t)$ *is an almost automorphic solution of* $(4.2)_y$ *with* $\mathcal{M}(x) \subset \mathcal{M}(A, B)$.

*Proof.* If for some $y_0 \in Y$, all solutions of $(4.4)_{y_0}$ are bounded, then the classical Liouvill's theorem implies that the norm of all nontrivial solutions are bounded away from 0. Therefore, by Corollary 4.3, we only need to consider the case when $(4.4)_y$ admits a nontrivial bounded solution and an unbounded solution for all $y \in Y$. But this reduces to Corollary 4.3 3). □

*Remark 4.3.* Consider an almost periodic linear oscillator

$$x'' + a(t)x' + b(t)x = f(t) \tag{4.18}$$

and assume that $\int_0^t a(s)ds$ is bounded. Let $Y = H(a, b, f)$ and $\tilde{a}, \tilde{b}, \tilde{f} : Y \to \mathbb{R}$ be extensions of $a, b, f$ respectively. By the above corollary, if (4.18) has a bounded solution with bounded derivative, then there is a residual set $Y_0 \in Y$ such that for each $y \in Y_0$, the equation

$$x'' + \tilde{a}(y \cdot t)x' + \tilde{b}(y \cdot t)x = \tilde{f}(y \cdot t) \tag*{$(4.19)_y$}$$

admits an almost automorphic solution $x(y, t)$ with $\mathcal{M}(x) \subset \mathcal{M}(a, b, f)$. In terms of the equation (4.18), this implies that there is a bounded solution $x(t)$ such that for 'almost all sequence' $t_n \to \infty$, $x(t+t_n)$ limits to an almost automorphic solution $x_*(t)$ of $(4.19)_y$ for some $y \in Y_0$, pointwise in $C^1$ sense. This fact should be significant in study multi-frequency linear oscillations especially when an interval (Sacker-Sell) spectrum associated to the linear (phase) system of the homogeneous part of (4.18) occurs.

### 4.2. Cooperative Systems.

We consider a system of ODE's

$$x' = f(x, t), \quad x \in \mathbb{R}^n \quad (n \geq 2), \tag{4.20}$$

where $f$ is $C^2$ admissible and uniformly almost periodic in $t$. Again, let $\Pi : \mathbb{R}^n \times H(f) \times \mathbb{R} \to \mathbb{R}^n \times H(f)$ denote the skew-product flow generated by (4.20), that is,

$$\Pi(x_0, g, t) = (x(x_0, g, t), g \cdot t), \tag{4.21}$$

where $x(x_0, g, t)$ is the solution of

$$x' = g(x \cdot t) \equiv F(x, g \cdot t) \tag*{$(4.22)_g$}$$

with initial value $x_0$. Recall that $F : \mathbb{R}^n \times H(f) \to \mathbb{R}^n$ is the extension of $f$ with $F(x, f \cdot t) \equiv f(x, t)$.

Definition 4.1. Denote $x = (x_1, x_2, \cdots, x_n)^\top$, $f = (f_1, f_2, \cdots, f_n)^\top$.

1) (4.20) is said to be *cooperative* if for any $i \neq j$,

$$\frac{\partial f_i}{\partial x_j}(x, t) \geq 0 \quad (x \in \mathbb{R}^n, t \in \mathbb{R}).$$

It is said to be *strongly cooperative* if there is a $\delta > 0$ such that for any $i \neq j$,

$$\frac{\partial f_i}{\partial x_j}(x, t) \geq \delta \quad (x \in \mathbb{R}^n, t \in \mathbb{R}).$$

2) (4.20) is said to be *strongly irreducible* if there is a $\delta_0 > 0$ such that if two nonempty subsets $S$, $S'$ of $\{1, 2, \cdots, n\}$ form a partition of $\{1, 2, \cdots, n\}$, then for any $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, there exist $i \in S$, $k \in S'$ such that

$$\left| \frac{\partial f_i}{\partial x_k}(x, t) \right| \geq \delta_0.$$

*Remark 4.4.*

1) Strongly irreducibility implies that for any $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, the matrix $(\frac{\partial f_i}{\partial x_j}(x, t))$ is an irreducible matrix.

2) It is easy to see that if (4.20) is cooperative (strongly cooperative, strongly irreducible), then so are $(4.22)_g$ $(g \in H(f))$ with the same constants $\delta$, $\delta_0$.

3) A strongly cooperative system is clearly cooperative and strongly irreducible. A simple example of strongly cooperative system is the following

$$\begin{cases} x_1' = a_1(x_1, t) + x_2 + x_1^2 x_2^3 \\ x_2' = a_2(x_2, t) + x_1 + x_2^2 x_1^3, \end{cases}$$

where $a_1$, $a_2$ are both uniformly almost periodic in $t$.

The following system is easily seen to be cooperative and strongly irreducible but not strongly cooperative:

$$\begin{cases} x_1' = x_4 + x_1 \\ x_2' = x_1 + x_2 \\ x_3' = x_2 + x_3 \\ x_4' = x_3 + x_4. \end{cases}$$

In the following, we denote $X_+ = \{x = (x_1, x_2, \cdots, x_n)^\top \mid x_i \geq 0, i = 1, 2, \cdots, n\}$ as the positive cone. Note that $\operatorname{Int} X_+ = \{x = (x_1, x_2, \cdots, x_n)^\top \in \mathbb{R}^n \mid x_i > 0, i = 1, 2, \cdots, n\}$. It is clear that $\mathbb{R}^n$ is strongly ordered by $X_+$ as follows: For any $x = (x_1, x_2, \cdots, x_n)^\top$, $y = (y_1, y_2, \cdots, y_n)^\top \in \mathbb{R}^n$,

$$\begin{aligned} x \leq y \quad &\Longleftrightarrow \quad y - x \in X_+; \\ x < y \quad &\Longleftrightarrow \quad x \leq y, \quad x \neq y; \\ x \ll y \quad &\Longleftrightarrow \quad y - x \in \operatorname{Int} X_+. \end{aligned}$$

LEMMA 4.5. *If* (4.20) *is cooperative and strongly irreducible, then the skew-product flow* (4.21)*, when viewed as a skew-product semiflow* $\Pi = (\mathbb{R}^n \times H(f), \mathbb{R}^+)$*, is strongly monotone.*

*Proof.* For given $x_0 \in \mathbb{R}^n$, $g \in H(f)$, let $\Phi(x_0, g, t) = \frac{\partial x}{\partial x_0}(x_0, g, t)$. Then $\Phi(x_0, g, t)$ is the fundamental matrix of

$$z' = A(t)z$$

with $\Phi(x_0, g, 0) = I$, where $A(t) = \frac{\partial g}{\partial x}(x(x_0, g, t), t)$. Since (4.20) is cooperative, $\Phi(x_0, g, t) > 0$ in the sense that $\Phi_{ij}(x_0, g, t) \geq 0$ and $\sum_{k=1}^n \Phi_{ik}(x_0, y, t)$, $\sum_{k=1}^n \Phi_{kj}(x_0, y, t) \neq 0$ for any $t > 0$, $i, j \in \{1, 2, \cdots, n\}$.

We claim that $\Phi_{ij}(x_0, g, t) > 0$ for any $t > 0$, $i, j \in \{1, 2, \cdots, n\}$. For otherwise, there are $t_0 > 0$, $i_0, j_0 \in \{1, 2, \cdots, n\}$ such that $\Phi_{i_0 j_0}(x_0, g, t_0) = 0$. Define $S = \{i \mid \Phi_{ij_0}(x_0, g, t_0) = 0\}$, $S' = \{1, 2, \cdots, n\} \setminus S$. Then $i_0 \in S$ and $S' \neq \emptyset$. Let $\lambda > 0$ be such that $(A(t_0) + \lambda I)_{ij} \geq 0$ for any $i, j = 1, 2, \cdots, n$ and $t \in [0, t_0]$. Note that $N(t) = e^{\lambda t}\Phi(x_0, g, t)$ satisfies $N_{ij_0}(t_0) = 0$, $N_{kj_0}(t_0) > 0$ $(i \in S,\, k \in S')$,

$$N'(t) = (A(t) + \lambda I)N(t),$$

and, $N_{ij}(t) \geq 0$ for all $i, j = 1, 2, \cdots, n$, $t > 0$. It follows that

$$N'_{ij}(t) = \sum_{k=1}^{n} (A_{ik}(t) + \delta_{ik}\lambda) N_{kj}(t) \geq 0$$

for all $i, j = 1, 2, \cdots, n$, $t \in [0, t_0]$. Therefore, each $N_{ij}(t)$ is monotonely increasing in $t \in [0, t_0]$.

By strong irreducibility of $(4.20)_g$, there are $i \in S$, $k \in S'$ such that $A_{ik}(t_0) + \delta_{ik}\lambda > 0$. Since $k \notin S$, $N_{kj_0}(t_0) > 0$. Hence, $N'_{ij_0}(t_0) > 0$. Now, $N_{ij_0}(t)$ is monotonely increasing and $N_{ij_0}(t) \geq 0$ for $0 \leq t \leq t_0$, one must have $N_{ij_0}(t_0) > 0$ (for otherwise, $N'_{ij_0}(t_0) = 0$), a contradiction.

Therefore, $\Phi_{ij}(x_0, g, t) > 0$ for any $t > 0$, $i, j \in \{1, 2, \cdots, n\}$. It follows that for any $v \in \mathbb{R}^n$ with $v > 0$, $\Phi(x_0, g, t)v \gg 0$ for all $t > 0$, that is, $\Pi$ is strongly monotone. □

THEOREM 4.6. *Consider* (4.21) *and assume that* (4.20) *is cooperative and strongly irreducible. Then the following holds.*

1) *Any linearly stable minimal set $E$ of $\Pi$ is almost automorphic, and there is an integer $N \geq 1$ such that if $(x_0, g) \in E$ is an almost automorphic point, then $x(x_0, g, t)$ is an almost automorphic solution of $(4.22)_g$ with $N\mathcal{M}(x(x_0, g, \cdot)) \subset \mathcal{M}(f)$.*
2) *If $x(x_*, g_*, t)$ is a forward bounded solution of $(4.22)_{g_*}$ for some $g_* \in H(f)$ and $(\omega(x_*, g_*), \mathbb{R})$ is both uniformly and linearly stable, then $(\omega(x_*, g_*), \mathbb{R})$ is an almost periodic minimal flow. Moreover, there is an integer $N \geq 1$ such that if $(x_0, g) \in \omega(x_*, g_*)$, then $x(x_0, g, t)$ is an almost periodic solution of $(4.22)_g$ with $N\mathcal{M}(x(x_0, g, \cdot)) \subset \mathcal{M}(f)$.*
3) *If $\omega(x_*, g_*)$ is such that $(x_*, g_*) \geq (\leq)(x, g_*)$ $((x, g_*) \in \omega(x_*, g_*))$, then $\omega(x_*, g_*)$ contains a unique minimal set $E$ and $(E, \mathbb{R}) \to (H(f), \mathbb{R})$ is an almost 1-1 extension. Moreover, if $(x_0, g_0) \in E$ is an almost automorphic point, then the almost automorphic solution $x(x_0, g_0, t)$ of $(4.22)_{g_0}$ satisfies $\mathcal{M}(x(x_0, g_0, \cdot)) \subset \mathcal{M}(f)$.*

*Proof.* 1) Since $\Pi = (\mathbb{R}^n \times H(f), \mathbb{R}^+)$ is strongly monotone, by Part II, Theorem 4.5, if $E$ is a linearly stable minimal set of (4.21), then there is an almost periodic minimal flow $(Y_*, \mathbb{R})$ and flow homomorphisms $p_0$, $p_*$ such that

$$(E, \mathbb{R}) \xrightarrow{p_0} (Y_*, \mathbb{R}) \xrightarrow{p_*} (H(f), \mathbb{R}), \tag{4.23}$$

where $p_*$ is an $N$-1 extension for some positive integer $N$ and $p_0$ is an almost 1-1 extension. Thus, $(E, \mathbb{R})$ is almost automorphic by Part I, Theorem 2.14. Now if $(x_0, g) \in E$ is an almost automorphic point, then by Part I, Remark 2.4 1) and Theorem 3.8, $N\mathcal{M}(x(x_0, g, \cdot)) \subset \mathcal{M}(f)$.

2) follows from Part II, Corollary 4.9. The module containment result follows from 1).

3) follows from Part II, Proposition 3.4 and Part I, Theorem 3.6. □

*Remark 4.6.*

1) Consider (4.21) and assume that (4.20) is strongly irreducible and cooperative. Let $K \subset \mathbb{R}^n \times H(f)$ be a minimal set of (4.21) and denote $X_1(x_0, g) = span\{v(x_0, g)\}$, $X_2(x_0, g)$ $((x_0, g) \in K)$ as subspaces associated to the continuous separation on $K$ (see Part II, Section 4). By Proposition 4.10 of Part II, it is easy to see that the upper Lyapunov exponent $\lambda_K$ over $K$ can be calculated as follows:

$$\lambda_K = \sup_{(x_0,g)\in K} \overline{\lim_{t\to\infty}} \frac{1}{t} \int_0^t \mu(x_0, g, s)ds,$$

where $\mu(x_0, g, s) = \langle g_x((x_0, g) \cdot t, t)v((x_0, g) \cdot t), v((x_0, g) \cdot t)\rangle$, here $\langle \cdot, \cdot \rangle$ denotes the standard inner product of $\mathbb{R}^n$. Thus, if $\lambda_K = 0$, then $K$ is linearly stable, and, if $\lambda_K < 0$, then $K$ is uniformly and asymptotically stable.

2) Let$(x_*, y_*)$ be as in Theorem 4.6 2). Then by Remark 4.2 of Part II, $x(x_*, y_*, t)$ is asymptotically almost periodic.

### 4.3. An example.

Based on Example 3.1 and an example in [52], we now construct an example of cooperative and strongly irreducible system which exhibits the subharmonic phenomena indicated in Theorem 4.6.

First, fix an integer $k_0 > 1$ and let $T_{k_0}(u)$ be the $k_0$-th Chebyshev's polynomial. Then $T_{k_0}(u)$ is uniquely determined by $T_{k_0}(u) = \cos(k_0 arc \cos u)$ for $u \in [0, 1]$ (see [52]). Let $f : \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1$ be defined by $f(\xi, u) = [\cos(k_0\xi) - T_{k_0}(u)]T'_{k_0}(u)$. Then $f$ is a polynomial in $u$ of degree $2k_0 - 1$ with the leading coefficient $-k_0 2^{2(k_0-1)}$ ([52]). Moreover, $f$ is $2\pi/k_0$-periodic in $\xi$ and $\mathcal{M}(f) = \{mk_0 \mid m \in \mathbb{Z}\}$. Consider for each $\lambda > 0$,

$$\begin{cases} u_1' = \lambda u_4 + f_1^\lambda(t, u_1) \\ u_2' = \lambda u_1 + f_2^\lambda(t, u_2) \\ u_3' = \lambda u_2 + f_3^\lambda(t, u_3) \\ u_4' = \lambda u_3 + f_4^\lambda(t, u_4), \end{cases} \tag{4.24$_\lambda$}$$

where $f_1^\lambda(t,u) = f(\lambda t, u)$, $f_2^\lambda(t,u) = f(\lambda t - \pi/2, u)$, $f_3^\lambda(t,u) = f(\lambda t + \pi, u)$, $f_4^\lambda(t,u) = f(\lambda t + \pi/2, u)$. It is easy to see that $\mathcal{M}(f_i^\lambda) = \{mk_0\lambda \mid m \in \mathbb{Z}\}$, $i = 1,2,3,4$. We note that $u^*(t) \equiv (\cos\lambda t, \sin\lambda t, -\cos\lambda t, -\sin\lambda t)^\top$ is a $2\pi/\lambda$-periodic solution of $(4.24)_\lambda$ (see [52] for the case $\lambda = 1$) with $\mathcal{M}(u^*) = \{m\lambda \mid m \in \mathbb{Z}\}$.

Next, let $A(t)$, $B(t)$ be the almost periodic functions in Example 3.1. Recall that

$$x' = A(t)x + B(t) \tag{4.25}$$

admits no almost periodic but an almost automorphic solution $x_0(t)$.

Now, let $\lambda$ be such that $\mathcal{M}(f_1^\lambda) \cap \mathcal{M}(A,B) = \{0\}$ and consider

$$\begin{cases} u_1' = \lambda u_4 + u_5 + u_6 + f_1^\lambda(t,u_1) \\ u_2' = \lambda u_1 + u_5 + u_6 + f_2^\lambda(t,u_2) \\ u_3' = \lambda u_2 + u_5 + u_6 + f_3^\lambda(t,u_3) \\ u_4' = \lambda u_3 + u_5 + u_6 + f_4^\lambda(t,u_4) \\ u_5' = u_1 + u_2 + u_3 + u_4 + A(t)u_5 + B(t) \\ u_6' = u_1 + u_2 + u_3 + u_4 + A(t)u_6 - B(t). \end{cases} \tag{4.26}$$

It is easy to verify that (4.26) is a cooperative and strongly irreducible system. Therefore, it induces a strongly monotone skew-product flow on $\mathbb{R}^6 \times H(F)$, where $F(t,u) = (f_1^\lambda(t,u_1), f_2^\lambda(t,u_2), f_3^\lambda(t,u_3), f_4^\lambda(t,u_4), A(t)u_5 + B(t), A(t)u_6 - B(t))^\top$ for $u = (u_1, u_2, \cdots, u_6)^\top$. Clearly, $\mathcal{M}(F) = \mathcal{M}(f_1^\lambda, A, B)$.

Let $\tilde{u}(t) = (u^*(t), x_0(t), -x_0(t))^\top = (\cos\lambda t, \sin\lambda t, -\cos\lambda t, -\sin\lambda t, x_0(t), -x_0(t))^\top$. Then $\tilde{u}$ is a non-almost periodic, almost automorphic solution of (4.26). It is easy to see that $\mathcal{M}(\tilde{u}) \not\subset \mathcal{M}(F)$ and $k_0\mathcal{M}(\tilde{u}) \subset \mathcal{M}(F)$ (that is, $(\omega(\tilde{u}, F), \mathbb{R})$ is an almost $k_0$-1 extension of $(H(F), \mathbb{R})$).

## 5. Parabolic Equations in Higher Space Dimension

Let $\Omega$, $f$, $X$ be as in Section 2.2. In addition, we assume that $f$ is uniformly almost periodic. Recall that the parabolic equation

$$\begin{cases} u_t = \Delta u + f(u, \nabla u, x, t), & t > 0, \quad x \in \Omega \\ u|_{\partial\Omega} = 0 \quad \text{or} \quad \frac{\partial u}{\partial n}|_{\partial\Omega} = 0, & t > 0 \end{cases} \tag{5.1}$$

generates a (local) skew-product semiflow $\Pi : X \times H(f) \times \mathbb{R}^+ \to X \times H(f)$:

$$\Pi(U, g, t) = (u(U, g, \cdot, t), g \cdot t), \tag{5.2}$$

where $u(U, g, x, t)$ is the solution of

$$\begin{cases} u_t = \Delta u + g(u, \nabla u, x, t), & t > 0, \quad x \in \Omega \\ u|_{\partial\Omega} = 0 \quad \text{or} \quad \frac{\partial u}{\partial n}|_{\partial\Omega} = 0, & t > 0 \end{cases} \tag{$5.3)_g$}$$

with $u(U, g, x, 0) \equiv U(x)$.

**5.1. Stable Minimal Sets.**

Let $X_+ = \{u \in X \mid u(x) \geq 0, x \in \bar{\Omega}\}$. We first observe that $\operatorname{Int} X_+ \neq \emptyset$. This is because, in the case of Dirichlet boundary condition, the set $\{u \in X \mid u(x) > 0$ for $x \in \Omega$, $\frac{\partial u}{\partial n}(x) < 0$ for $x \in \partial\Omega\} \subset \operatorname{Int} X_+$ is nonempty, while in the case of Neumann boundary condition, the set $\{u \in X \mid u(x) > 0$ for $x \in \bar{\Omega}\} \subset \operatorname{Int} X_+$ is nonempty. It follows that $X_+$ defines a strong ordering on $X$ as follows:

$$\begin{aligned} u_1 \leq u_2 \quad &\Longleftrightarrow \quad u_1(x) \leq u_2(x) \quad \text{for all} \quad x \in \Omega; \\ u_1 < u_2 \quad &\Longleftrightarrow \quad u_1 \leq u_2 \quad \text{but} \quad u_1 \neq u_2; \\ u_1 \ll u_2 \quad &\Longleftrightarrow \quad u_2 - u_1 \in \operatorname{Int} X_+. \end{aligned}$$

Lemma 5.1. *The skew-product semiflow* $\Pi$ *in* (5.2) *is strongly monotone.*

*Proof.* For given $u_0 \in X$, $g \in H(f)$, consider

$$\begin{cases} v_t = \Delta v + a(x,t) \bigtriangledown v + b(x,t)v, & t > 0, \quad x \in \Omega \\ v|_{\partial\Omega} = 0 \quad \text{or} \quad \frac{\partial v}{\partial n}|_{\partial\Omega} = 0, & t > 0, \end{cases} \tag{5.4}$$

where

$$a(x,t) = \frac{\partial g}{\partial p}(u(u_0,g,x,t), \bigtriangledown u(u_0,g,x,t), x, t),$$

$$b(x,t) = \frac{\partial g}{\partial u}(u(u_0,g,x,t), \bigtriangledown u(u_0,g,x,t), x, t).$$

Let $\Phi(u_0,g,t)$ be the evolutional operator of (5.4). Then for any $v \in X$, $\Phi(u_0,g,t)v$ is the solution of (5.4) with initial value $v$. By the maximum principle and the Hopf boundary principle for parabolic equations ([16], [40]), if $v > 0$, then $\Phi(u_0,g,t)v \gg 0$ for $t > 0$. Therefore, $\Phi(u_0,g,t)v \in \operatorname{Int} X_+$, that is, $\Pi$ is strongly monotone. $\square$

Theorem 5.2.

1) *Any linearly stable minimal set* $E$ *of* $\Pi$ *is almost automorphic, and there is an integer* $N \geq 1$ *such that if* $(U,g) \in E$ *is an almost automorphic point, then* $u(U,g,x,t)$ *is a uniform almost automorphic solution of* $(5.3)_g$*, and* $N\mathcal{M}(u) \subset \mathcal{M}(f)$.
2) *Let* $(U_0,g_0) \in X \times H(f)$ *be such that* $\{u(U_0,g_0,\cdot,t) \mid t \geq \delta\}$ *is bounded for some* $\delta \geq 0$. *If* $(\omega(U_0,g_0),\mathbb{R})$ *is both uniformly and linearly stable, then it is minimal and almost periodic. Moreover, there is a positive integer* $N$ *such that for any* $(U,g) \in \omega(U_0,g_0)$, $u(U,g,x,t)$ *is a uniform almost periodic solution of* $(5.3)_g$ *with* $N\mathcal{M}(u) \subset \mathcal{M}(f)$.
3) *If* $\omega(u_*,g_*)$ *is such that* $(u_*,g_*) \geq (\leq)(u,g_*)$ $((u,g_*) \in \omega(u_*,g_*))$*, then* $\omega(u_*,g_*)$ *contains a unique minimal set* $E$ *and* $(E,\mathbb{R}) \to (H(f),\mathbb{R})$ *is an almost* 1-1 *extension. Moreover, if* $(u_0,g_0) \in E$ *is an almost automorphic point, then the uniform almost automorphic solution* $u(u_0,g_0,x,t)$ *of* $(5.3)_{g_0}$ *satisfies* $\mathcal{M}(u) \subset \mathcal{M}(f)$.

*Proof.* It follows from Lemma 5.1 and arguments of Theorem 4.6. □

*Remark 5.1.*

1) The almost automorphy in Theorem 5.2 1) can not be replaced by almost periodicity even in 1-space dimension.

Consider the following example

$$(5.4)_y \qquad \begin{cases} u_t = u_{xx} + a(y\cdot t)u + b(y\cdot t), & t>0, \quad 0<x<1 \\ u_x(0,t) = u_x(1,t) = 0, & t>0, \end{cases}$$

where $a(y)$, $b(y)$ are extensions of $A$, $B$ in Example 3.1 on $Y = H(A,B)$ respectively. We know already that the skew-product flow $\tilde{\Pi}$ generated by

$$(5.5)_y \qquad u' = a(y\cdot t)u + b(y\cdot t)$$

admits only one minimal set $E$ which is non-almost periodic almost automorphic. It follows that the mean value $M(a)$ of $a$ is 0, for otherwise $(5.5)_y$ would admit a (unique) almost periodic solution. Clearly, $E$ is also a minimal set of the skew-product semiflow $\Pi$ generated by $(5.4)_y$. The linearized equation of $(5.4)_y$ along $E$ reads

$$(5.6)_{(U,y)} \qquad \begin{cases} u_t = u_{xx} + a(y\cdot t)u, & t>0, \quad 0<x<1 \\ u_x(0,t) = u_x(1,t) = 0, & t>0 \end{cases}$$

($(U,y)\in E$). Let $X_1(U,y)$, $X_2(U,y)$ be subspaces associated to the continuous separation of $(5.5)_{(U,y)}$ on $E$. Since $e^{\int_0^t a(y\cdot s)ds} \in \operatorname{Int} X_+$ is a solution of $(5.6)_{(U,y)}$ for all $(U,y)\in E$, $X_1(U,y) = span\{1\}$. It follows from Proposition 4.10 of Part II that the upper Lyapunov exponential $\lambda_E$ over $E$ is $M(a) = 0$, that is, $E$ is linearly stable.

2) Subharmonic phenomena ($N > 1$ in Theorem 5.2) are often observed in higher space dimensions. There is an example in [12] (see also [53]) in which a periodic time dependent parabolic equation on an annulus domain admits a stable (hence linearly stable, see [12]) periodic solution with a multiple period. We note that in the periodic case, the multiplicity $N$ can be estimated within a global attractor (see [23] for details).

3) In the periodic case, since an almost $N$-1 extension of a periodic minimal set is necessarily an $N$-1 extension, Theorem 5.2 particularly implies that any linearly stable minimal set of $\Pi$ is periodic. This has already been shown in [38]. In fact, a similar result to Theorem 5.2 in the periodic case implies generic convergence, that is, for 'almost all' initial value, the corresponding solution is asymptotically periodic (see [37], [38] for details). Such a generic convergence result no longer holds in the almost periodic case. To see this, we consider

$$(5.7) \qquad \begin{cases} u_t = u_{xx} + f(u,t), & t>0, \quad 0<x<1 \\ u_x(0,t) = u_x(1,t) = 0, & t>0, \end{cases}$$

where $f(u,t) = -(A(t)\cos u + B(t)\sin u)\sin u$ is as in Example 3.3. Recall that the skew-product flow $\tilde{\Pi}$ generated by

$$u' = f(u,t) \tag{5.8}$$

has the following property: there are two minimal sets $E_1$, $E_2$, with one almost automorphic and one almost periodic, such that for any $(U_0, g) \in \mathbb{R}^1 \times H(f)$ 'in between' $E_1$ and $E_2$, $E_1 \cup E_2 \subset \omega(U_0, g)$. Using the comparison principle for parabolic equations ([16], [40]), it can be shown that there is a neighborhood $\mathcal{N}$ of $(U_0, g)$ in $X \times H(f)$ such that for any $(U, g) \in \mathcal{N}$, $E_1 \cup E_2 \subset \omega(U, g)$ (which is not even minimal). For this example, generic convergence fails within either the category of almost automorphy or almost periodicity since if the skew-product semiflow $\Pi$ generated by (5.7) has a trajectory $\Pi(U, g, t)$ which is asymptotically almost periodic or almost automorphic, then $\omega(U, g)$ has to be an almost periodic or almost automorphic minimal set.

4) Let $K$ be a minimal set of (5.2) and denote $X_1(U, g) = \text{span}\{v(U, g)\}$, $X_2(U, g)$ $((U, g) \in K)$ as the subspaces associated to the continuous separation on $K$. By Proposition 4.10 of Part II, it is easy to see that the upper Lyapunov exponent $\lambda_K$ of $K$ can be calculated as follows:

$$\lambda_K = \sup_{(U,g)\in K} \overline{\lim_{t\to\infty}} \frac{1}{t} \int_0^t \mu(U, g, s)ds,$$

where

$$\begin{aligned}
\mu(u, g, s) = \frac{1}{\int_\Omega [v((U, g)\cdot s)(x)]^2 dx} \int_\Omega \{ & -[\nabla v((U, g)\cdot s)(x)]^2 \\
& + \langle a(x, s) v((U, g)\cdot s)(x), \nabla v((U, g)\cdot s)(x)\rangle \\
& + b(x, s)[v((U, g)\cdot s)(x)]^2\} dx,
\end{aligned}$$

$$\begin{aligned}
a(x, s) &= g_p(u(U, g, x, s), \nabla u(U, g, x, s), x, s), \\
b(x, s) &= g_u(u(U, g, x, s), \nabla u(U, g, x, s), x, s).
\end{aligned}$$

5) Let $(U_0, g_0)$ be as in Theorem 5.2 2). Then by Remark 4.2 of Part II, $\Pi(U_0, g_0, t)$ is asymptotically almost periodic.

### 5.2. Spatially Homogeneous Solutions.

As suggested by [4], [34] and [22], in spatially homogeneous autonomous and periodic parabolic equations with Neumann boundary condition, stable equilibria or periodic solutions are not suppose to have spatial variations. We now investigate almost periodic equations of the following form:

$$\begin{cases} u_t = \Delta u + f(u, \nabla u, t), & t > 0, \quad x \in \Omega \\ \frac{\partial u}{\partial n}|_{\partial\Omega} = 0, & t > 0, \end{cases} \tag{5.9}$$

where $f$ is $C^3$ and uniformly almost periodic, $\Omega \subset \mathbb{R}^n$ is a bounded, convex and smooth domain.

We let $\Pi : X \times H(f) \times \mathbb{R}^+ \to X \times H(f)$ be the strongly monotone skew-product semiflow generated by (5.9).

DEFINITION 5.1. A bounded solution $u(x,t) \equiv u(U,f,x,t)$ of (5.9) is *linearly stable* if the following holds.

i) $(\omega(U,f),\mathbb{R})$ is linearly stable in the usual sense.

ii) Let $\Psi(t,s) \equiv \Phi((U,f)\cdot s, t-s)$ be the evolutional operator of the following linearized equation along $u(x,t)$:

$$
(5.10)\qquad \begin{cases} v_t = \Delta v + f_p(u(x,t), \bigtriangledown u(x,t), t) \bigtriangledown v + \\ \qquad\qquad \bigtriangledown f_u(u(x,t), u(x,t), t) v, & t > 0, \quad x \in \Omega, \\ \frac{\partial v}{\partial n}|_{\partial\Omega} = 0, & t > 0. \end{cases}
$$

Then for any $v_0 \in X$, $\sup_{t \geq 0} \|\Psi(t,0)v_0\| < \infty$.

THEOREM 5.3. *If* $u(x,t) \in C^3(\bar{\Omega} \times \mathbb{R}^1)$ *is a linearly stable almost automorphic* (*almost periodic* ) *solution of* (5.9), *then it is spatially homogeneous, that is,* $u(t) \equiv u(x,t)$ *is a solution of*

$$
(5.11)\qquad u' = f(u, 0, t),
$$

*and* $\mathcal{M}(u) \subset \mathcal{M}(f)$.

*Proof.* Denote $L' = \frac{\partial}{\partial t} - \Delta - \sum_{j=1}^N \frac{\partial f}{\partial p_j}(t,u,\bigtriangledown u)\frac{\partial}{\partial x_j}$, $m_0(t,x) = \frac{\partial f}{\partial u}(t,u,\bigtriangledown u)$. For any $1 \leq i \leq N$, we note that $u_{x_i}$ satisfies

$$
(5.12)\qquad L' u_{x_i} = m_0(t,x) u_{x_i}.
$$

Suppose that $u(x,t)$ is not spatially homogeneous. Let $v = (\sum_{i=1}^N u_{x_i}^2)^{1/2}$, $v_\epsilon = (v^2 + \epsilon^2)^{1/2}$ for $\epsilon > 0$. Then

$$
v_\epsilon L' v_\epsilon + \sum_{i,j=1}^N (u_{x_i x_j})^2 - \sum_{j=1}^N (v_{\epsilon x_j})^2 = m_0(t,x) v^2
$$

holds ([22]). It follows that

$$
L' v_\epsilon \leq m_0 \frac{v^2}{v_\epsilon}.
$$

Since $\frac{\partial u}{\partial n}(t,x) = 0$ $(x \in \Omega)$ and $\Omega$ is convex, one has $\frac{\partial v_\epsilon^2}{\partial n} = \frac{\partial v^2}{\partial n} \leq 0$ ([22]). Thus, $v_\epsilon$ satisfies

$$
(5.13)\qquad \begin{cases} L' v_\epsilon \leq m_0 \frac{v^2}{v_\epsilon}, & (t,x) \in \mathbb{R}^1 \times \Omega \\ \frac{\partial v_\epsilon}{\partial n} \leq 0, & (t,x) \in \mathbb{R}^1 \times \partial\Omega. \end{cases}
$$

Now, choose $k > \|m_0\|_{C(\mathbb{R}^1\times\bar{\Omega})}$ sufficiently large such that both

$$(5.14)\qquad \begin{cases} (L'+k)w_\epsilon = m_0\frac{v^2}{v_\epsilon} + kv_\epsilon, & (t,x)\in\mathbb{R}^1\times\Omega,\\ \frac{\partial w_\epsilon}{\partial n} = 0, & (t,x)\in\mathbb{R}^1\times\partial\Omega \end{cases}$$

and

$$(5.15)\qquad \begin{cases} (L'+k)w = (m_0+k)v, & (t,x)\in\mathbb{R}^1\times\Omega,\\ \frac{\partial w}{\partial n} = 0, & (t,x)\in\mathbb{R}^1\times\partial\Omega \end{cases}$$

admit unique globally and asymptotically stable almost automorphic (almost periodic) solutions, say $w_\epsilon(t,x)$, $w(t,x)$ respectively. This can always be done, since when $k \gg 1$, both linear parts corresponding to (5.14) and (5.15) admit an exponential dichotomy, hence (5.14) and (5.15) admit unique bounded solutions ([15]) which are in fact almost automorphic (almost periodic) if $u(x,t)$ is. Now by Theorem 5.4 and Remark 5.2 below, these bounded solutions are globally and asymptotically stable. Since $m_0\frac{v^2}{v_\epsilon} + kv_\epsilon \to (m_0+k)v$ as $\epsilon\to 0$, one has

$$w_\epsilon(t,x) \to w(t,x)$$

in $X$ as $\epsilon \to 0$ ([21]). By the maximum principle ([16], [40]), $w_\epsilon(t,x) \geq v_\epsilon(t,x) > 0$. This implies that $w(t,x)\geq v(t,x)\geq 0$, and

$$(L'+k)w = (m_0+k)v \leq (m_0+k)w.$$

Moreover, $w(t,\cdot) \gg 0$ for all $t\in\mathbb{R}$. By the assumption that $u(x,t)$ is not spatially homogeneous, it is not difficult to see that $w(t,x)\geq\delta$ ($t\in\mathbb{R}$, $x\in\bar{\Omega}$) for some $\delta > 0$.

In what follows, we denote $w(t,\cdot)$ and $v(t,\cdot)$ by $w(t)$ and $v(t)$ respectively. Let $h(t) = (m_0(t,\cdot)+k)(v(t)-w(t))$. Then $h\leq 0$, and

$$(5.16)\qquad (L'-m_0)w = h.$$

We first prove that $h(0) = 0$.

Let $U(t,s)$ $(t\geq s)$ be the evolutional operator of

$$(5.17)\qquad \begin{cases} w_t = \Delta w + \sum_{j=1}^{N}\frac{\partial f}{\partial p_j}(t,u,\bigtriangledown u)\frac{\partial w}{\partial x_j} + \frac{\partial f}{\partial u}(t,u,\bigtriangledown u)w, & t>0, \quad x\in\Omega,\\ \frac{\partial w}{\partial n} = 0, & t>0, \quad x\in\partial\Omega. \end{cases}$$

Then $w(t) = U(t,0)w(0) + \int_0^t U(t,\tau)h(\tau)d\tau$ for all $t>0$, and $U(t,0)$ is strongly positive, that is, $U(t,0)w_0 \gg 0$ for any $w_0>0$ and $t>0$. By the definition of linear stability of $u(t,x)$, one has that for any $w_0$, $U(t,0)w_0$ is bounded in $t>0$.

Suppose that $h(0) \neq 0$. Then $h(0) < 0$ and $U(t,0)h(0) \ll 0$ for all $t > 0$. It follows that

$$\int_0^t U(t,\tau)h(\tau)d\tau \ll 0 \quad \text{for all} \quad t > 0.$$

Thus, $w(t) \ll U(t,0)w(0)$ $(t > 0)$. Let $X_1(U,g)$, $X_2(U,g)$ $((U,g) \in \omega(u(0),f))$ be linear subspaces associated to the continuous separation on $\omega(u(0),f)$ and write $w(0) = av^1 + v^2$, where $v^1 \in X_1(u,f)$ with $\|v^1\| = 1$, $v^2 \in X_2(u,f)$. Then $U(t,0)v^1$ is bounded for all $t \geq 0$ and $U(t,0)v^2 \to 0$ as $t \to \infty$. Note that for $t > 0$, $w(t) \gg 0$ and $w(t)$ is also bounded away from zero. It follows easily that $U(t,0)v^1$ is bounded away from zero and is almost automorphic (almost periodic). Let $t_n \to \infty$ be such that $w(t_n) \to w(0)$, $U(t_n,0)v^1 \to v^1$. Then

$$w(0) = av^1 + v^2 \leq av^1.$$

Therefore $v^2 = 0$, and

$$w(t) = aU(t,0)v^1 + \int_0^t U(t,\tau)h(\tau)d\tau.$$

Now let $\delta_0 > 0$ and $s_n \to \infty$ be such that $w(s_n+\delta_0) \to w(0)$, $U(s_n+\delta_0,0)v^1 \to v^1$, and

$$\frac{\partial f}{\partial p_j}(s_n+\tau, u(s_n+\tau,x), \bigtriangledown u(s_n+\tau,x)) \to \frac{\partial f}{\partial p_j}(\tau, u(\tau,x), \bigtriangledown u(\tau,x)),$$

$$\frac{\partial f}{\partial u}(s_n+\tau, u(s_n+\tau,x), \bigtriangledown u(s_n+\tau,x)) \to \frac{\partial f}{\partial u}(\tau, u(\tau,x), \bigtriangledown u(\tau,x)),$$

$$h(s_n+\tau,x) \to h(\tau,x)$$

uniformly for $\tau \in [0,\delta_0]$. Then $U(s_n+\delta_0, s_n+\tau) \to U(\delta_0,\tau)$ as $n \to \infty$ uniformly for $\tau \in [0,\delta_0]$ (see [21]). Therefore,

$$\int_0^{\delta_0} U(s_n+\delta_0, s_n+\tau)h(s_n+\tau)d\tau \to \int_0^{\delta_0} U(\delta_0,\tau)h(\tau)d\tau$$

as $n \to \infty$. This implies that

$$\begin{aligned}
\lim_{n\to\infty} w(s_n+\delta_0) &= w(0) \\
&= \lim_{n\to\infty}\left[U(s_n+\delta_0,0)av^1 + \int_0^{s_n+\delta_0} U(s_n+\delta_0,\tau)h(\tau)d\tau\right] \\
&= av^1 + \lim_{n\to\infty}\int_0^{s_n+\delta_0} U(s_n+\delta_0,\tau)h(\tau)d\tau \\
&\le av^1 + \lim_{n\to\infty}\int_{s_n}^{s_n+\delta_0} U(s_n+\delta_0,\tau)h(\tau)d\tau \\
&= av^1 + \lim_{n\to\infty}\int_0^{\delta_0} U(s_n+\delta_0,s_n+\tau)h(s_n+\tau)d\tau \\
&= av^1 + \int_0^{\delta_0} U(\delta_0,\tau)h(\tau)d\tau \\
&\ll av^1 = w(0),
\end{aligned}$$

a contradiction. Hence $h(0)=0$, that is, $v(0,x)\equiv w(0,x)$.

Since $w(0,\cdot)\gg 0$, one has $v(0,\cdot)\gg 0$. Let $M$ be the set of all local maximum points of $u(0,\cdot)$. If there is a $x^*\in M\cap\Omega$, then $\triangledown u(0,x^*)=0$, that is, $v(0,x^*)=0$, a contradiction. Therefore, $M\subset\partial\Omega$. We now take $x^*\in M$. Then $\frac{\partial u}{\partial n}(0,x^*)=0$, and, for any unit vector $\nu$ pointing outward of $\Omega$, $\triangledown u(0,x^*)\cdot\nu\ge 0$. This is possible only if $\triangledown u(0,x^*)=0$, that is, $v(0,x^*)=0$, a contradiction again. Thus $u(0,x)$ is independent of $x$. By the uniqueness of solutions of (5.9) and the almost automorphy (almost periodicity) of $u(t,x)$, $u(t,x)$ is spatially homogeneous, hence $u(x,t)\equiv u(t)$ is a solution of (5.11). It follows from Theorem 3.4 that $\mathcal{M}(u)\subset\mathcal{M}(f)$. $\square$

**5.3. Global Attractor.**

We end this section by giving an explicit condition which guarantees the existence of an almost periodic global attractor for (5.1) or (5.2).

THEOREM 5.4. *Consider* (5.1) *and assume the following:*

1) *There is a* $\delta>0$ *such that* $f_u(u,p,x,t)\le-\delta$ *for all* $(u,p,x,t)\in\mathbb{R}^1\times\mathbb{R}^k\times\mathbb{R}^k\times\mathbb{R}^1$*;*
2) (5.1) *admits a bounded solution.*

*Then there is a unique almost periodic solution* $u(U_0,f,x,t)$ *of* (5.1) *with* $\mathcal{M}(u)\subset\mathcal{M}(f)$ *such that* $E=cl\{\Pi(U_0,f,t)\mid t\in\mathbb{R}\}$ *is a global attractor of* $\Pi$*, that is, if* $u(U,g,x,t)$ *is any bounded solution of* $(5.3)_g$ $(g\in H(f))$*, then*

$$\|u(U,g,\cdot,t)-u(U^*,g,\cdot,t)\|\to 0 \quad as \quad t\to\infty,$$

*where* $(U^*,g)=p^{-1}(g)\cap E$.

*Proof.* Define $L : \tilde{Z} = \{((U_1, g), (U_2, g)) \mid (U_i, g) \in X \times H(f), i = 1, 2\} \to \mathbb{R}^+$ as

$$L((U_1, g), (U_2, g)) = \|U_1 - U_2\|_{L^\infty(\Omega)}.$$

By embedding $X \hookrightarrow L^\infty(\Omega)$, $L$ is continuous. Clearly, $L((U_1, g), (U_2, g)) = 0$ if and only if $(U_1, g) = (U_2, g)$. Now, by the strong maximum principle for parabolic equations ([16], [40]), if $(U_1, g) \neq (U_2, g)$, then

$$L(\Pi(U_1, g, t), \Pi(U_2, g, t)) < L((U_1, g), (U_2, g))$$

for all $t > 0$, that is, $L$ is strictly contracting (see Part II, Definition 2.10). Therefore, the theorem follows from Part II, Theorem 2.9. □

## 6. Functional Differential Equations

We consider the skew-product semiflow $\Pi : X \times H(f) \times \mathbb{R}^+ \to X \times H(f)$,

$$\Pi(\phi, g, t) = (x_t(\phi, g), g \cdot t) \tag{6.1}$$

which is generated by the following delay differential equation

$$x'(t) = f(x(t), x(t-1), t), \quad x \in \mathbb{R}^n, \tag{6.2}$$

where $X$, $f$ are as in section 2.3, $x_t(\phi, g)(\theta) \equiv x(\phi, g, t + \theta)$ $(\theta \in [-1, 0])$, and $x(\phi, g, t)$ is the solution of

$$x'(t) = g(x(t), x(t-1), t), \quad x \in \mathbb{R}^n \tag*{$(6.3)_g$}$$

with $x(\phi, g, t) = \phi(t)$ for $t \in [-1, 0]$. In addition, we assume that $f$ is uniformly almost periodic.

### 6.1. Cooperative and Irreducible Equations.

Definition 6.1.

1) (6.2) is said to be *cooperative* (*strongly cooperative*) with respect to $x(t)$ if

$$\frac{\partial f_i}{\partial \xi_j}(\xi_1, \cdots, \xi_n, \eta_1, \cdots \eta_n, t) \geq 0 (\geq \delta > 0)$$

for any $i, j \in \{1, 2, \cdots, n\} (i \neq j)$, $(\xi_1, \cdots, \xi_n)$, $(\eta_1, \cdots, \eta_n) \in \mathbb{R}^n$, $t \in \mathbb{R}^1$.

2) (6.2) is said to be *irreducible* (*strongly irreducible*) with respect to $x(t)$ if for any two subsets $S, S' \subset \{1, 2, \cdots, n\}$ which form a partition of $\{1, 2, \cdots, n\}$, and any $(\xi_1, \cdots, \xi_n)$, $(\eta_1, \cdots, \eta_n) \in \mathbb{R}^n$, $t \in \mathbb{R}$, there exist $i \in S$, $k \in S'$ such that

$$\left|\frac{\partial f_i}{\partial \xi_k}(\xi_1, \cdots, \xi_n, \eta_1, \cdots, \eta_n, t)\right| > 0 (\geq \delta > 0).$$

3) (6.2) is said to be *monotone* (*strongly monotone*) with respect to $x(t-1)$ if

$$\frac{\partial f_i}{\partial \eta_j}(\xi, \eta, t) > 0 (\geq \delta > 0)$$

for any $i, j = 1, 2, \cdots, n$, $\xi, \eta \in \mathbb{R}^n$, $t \in \mathbb{R}$.

We note that if $f$ satisfies any strong conditions of 1)-3) above, then so does every $g \in H(f)$ (with the same constant $\delta$). In the following, we say $x \geq 0$ ($x > 0$), ($x \gg 0$) for $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ if $x_i \geq 0$ ($x_i \geq 0$ but $\sum_{i=1}^n x_i^2 \neq 0$), ($x_i > 0$) for $i = 1, 2, \cdots, n$. Let $X_+ = \{\phi \in X \mid \phi(\theta) \geq 0 \text{ for all } \theta \in [-1, 0]\}$. Since $\operatorname{Int} X_+ = \{\phi \in X \mid \phi(\theta) \gg 0 \text{ for all } \theta \in [-1, 0]\}$ is non-empty, $X_+$ defines a strong ordering on $X$ as follows:

$$\begin{aligned} \phi_1 \leq \phi_2 \quad &\Longleftrightarrow \quad \phi_1(\theta) \leq \phi_2(\theta) \quad \text{for all} \quad \theta \in [-1, 0]; \\ \phi_1 < \phi_2 \quad &\Longleftrightarrow \quad \phi_1 \leq \phi_2 \quad \text{and} \quad \phi_1 \neq \phi_2; \\ \phi_1 \ll \phi_2 \quad &\Longleftrightarrow \quad \phi_2 - \phi_1 \in \operatorname{Int} X_+. \end{aligned}$$

LEMMA 6.1. *Denote* $\Phi(\phi, g, t) = \frac{\partial x_t(\phi, g)}{\partial \phi}$, $(\phi, g) \in X \times H(f)$ *and assume that* (6.2) *is cooperative and strongly irreducible with respect to* $x(t)$, *and is strongly monotone with respect to* $x(t-1)$. *Then the skew-product semiflow* $\Pi$ *defined in* (6.1) *is strongly monotone in the following sense:*

1) *For any* $v \in X$ *with* $v > 0$, $\Phi(\phi, g, t)v > 0$ *if* $t > 0$ *and* $\Phi(\phi, g, t)v \gg 0$ *if* $t \geq 2$.
2) *For any* $v \in X$ *with* $v \gg 0$, $\Phi(\phi, g, t)v \gg 0$ *if* $t \geq 0$.

*Proof.* For given $g \in H(f)$, $\phi \in X$, $v \in X$ with $v > 0$, let $y(v, t)$ be the solution of

$$y'(t) = A(t)y(t) + B(t)y(t-1) \tag{6.4}$$

with $y(v, \theta) = v(\theta)$ for $\theta \in [-1, 0]$, where $A(t) = \frac{\partial g}{\partial \xi}(x(\phi, g, t), x(\phi, g, t-1), t)$, $B(t) = \frac{\partial g}{\partial \eta}(x(\phi, g, t), x(\phi, g, t-1), t)$. We note that by the strong monotonicity of (6.2) with respect to $x(t-1)$, $(B(t))_{ij} > 0$ for any $i, j = 1, 2 \cdots, n$ and $t \in \mathbb{R}$. Denote $y_t(v) \in X$ by $y_t(v)(\theta) \equiv y(v, t+\theta)$ ($\theta \in [-1, 0]$). Then $\Phi(\phi, g, t)v = y_t(v)$, and $\Phi(\phi, g, t)v = y_t(v) > 0 (\gg 0)$ if and only if $y(v, s) \geq 0 (\gg 0)$ and $y(v, s) \not\equiv 0$ for any $s \in [-1 + t, t]$. Since $v > 0$ and $y(v, \theta) = v(\theta)$ ($\theta \in [-1, 0]$), one has

$$y(v, s) \geq 0 \quad \text{and} \quad y(v, s) \not\equiv 0 \quad \text{for all} \quad s \in [-1, 0]. \tag{6.5}$$

Let $U(t, s)$ be the evolutional operator generated by

$$z'(t) = A(t)z(t), \quad x \in \mathbb{R}^n. \tag{6.6}$$

By Remark 4.4 2) and Lemma 4.5, $U(t,s)$ is strongly positive in the sense that for any $z_0 \in \mathbb{R}^n$ with $z_0 > 0$, $U(t,s)z_0 \gg 0$ if $t > s$.

By (6.5) and the following variation of constants formula

$$y(v,t) = U(t,0)y(v,0) + \int_0^t U(t,s)B(s)y(v,s-1)ds, \quad t > 0,$$

one has that

$$y(v,s) \geq 0, \quad y(v,s) \not\equiv 0 \quad \text{for all} \quad s \in [0,1] \quad \text{and} \quad y(v,1) \gg 0. \tag{6.7}$$

Also, (6.5) and (6.7) imply that $\Phi(\phi, g, t)v > 0$ if $t \in [0,1]$.

Next, using the variation of constants formula

$$y(v,t) = U(t,1)y(v,1) + \int_1^t U(t,s)B(s)y(v,s-1)ds, \quad t > 1,$$

one sees that

$$y(v,s) \gg 0 \quad \text{for all} \quad s \in [1,2]. \tag{6.8}$$

By (6.7) and (6.8), $\Phi(\phi, g, t)v > 0$ if $t \in [1,2]$, and $\Phi(\phi, g, 2)v \gg 0$.

Applying the above arguments inductively on every successive closed interval with positive integer boundaries, one shows that $y(v,s) > 0$ and $y(v,s) \not\equiv 0$ if $s \in [0,1]$, and $y(v,s) \gg 0$ if $s \geq 1$, which imply that $\Phi(\phi, g, t)v > 0$ for all $t \geq 0$ and $\Phi(\phi, g, t)v \gg 0$ for all $t \geq 2$. 1) is proved.

To prove 2), it is sufficient to show similarly to the above that $y(v,t) \gg 0$ for all $t \geq -1$ and $v \gg 0$. We omit the details. □

LEMMA 6.2. *For given $\phi \in X$, $g \in H(f)$, if $x_t(\phi, g)$ is bounded for $t \geq 0$, then $\{\Pi(\phi, g, t) \mid t \geq 1 + \delta\}$ is relatively compact for any $\delta > 0$.*

*Proof.* Let $x(t) = x(\phi, g, t) = x_t(\phi, g)(0)$. Then

$$x'(t) = g(x(t), x(t-1), t) \quad (t > 0).$$

By the boundedness of $x_t(\phi, g)$, there is a $M > 0$ such that $|g(x(t), x(t-1), t)| \leq M$, that is, $|x'(t)| \leq M$ for all $t > 0$. Applying Ascoli's theorem, one sees that $\{x_t(\phi, g) \mid t \geq 1 + \delta\}$ is relatively compact in $X$ for any $\delta > 0$. Therefore, $\{\Pi(\phi, g, t) = (x_t(\phi, g), g \cdot t) \mid t \geq 1 + \delta\}$ is relatively compact in $X \times H(f)$ for any $\delta > 0$. □

LEMMA 6.3. *Assume that* $\frac{\partial g}{\partial \eta}(\xi, \eta, t) = (\frac{\partial g_i}{\partial \eta_j})$ *is a positive (negative) definite matrix for any* $g \in H(f)$, $\xi, \eta \in \mathbb{R}^n$, $t \in \mathbb{R}$. *If* $x_t(\phi, g)$ *is bounded for* $t \geq 0$, *then* $(\omega(\phi, g), \mathbb{R}^+)$ *admits a flow extension.*

*Proof.* First, by Lemma 6.2 above and Proposition 2.1 of Part II, for any $(\phi^*, g^*) \in \omega(\phi, g)$, $\Pi(\phi^*, g^*, t)$ admits a negative orbit.

Suppose that for some $(\phi^*, g^*) \in \omega(\phi, g)$, there are two negative orbits of $\Pi(\phi^*, g^*, t)$, say $x_t^1$, $x_t^2$ ($x_t^1 \not\equiv x_t^2$, $t < 0$). Let $y(t) \equiv x_t^1(0) - x_t^2(0)$. Then $y(t) = 0$ if $t \geq -1$ and $y(t) \not\equiv 0$, and

$$y'(t) = A(t)y(t) + B(t)y(t-1)$$

for $t \in \mathbb{R}^1$, where

$$A(t) = \int_0^t \frac{\partial g^*}{\partial \xi}(sx_t^1(0) + (1-s)x_t^2(0), x_t^1(-1), t)ds,$$

$$B(t) = \int_0^1 \frac{\partial g^*}{\partial \eta}(x_t^2(0), sx_t^1(-1) + (1-s)x_t^2(-1), t)ds.$$

Since $B(t)$ is nonsingular,

$$y(t-1) = B^{-1}(t)[y'(t) - A(t)y(t)]$$

for any $t \in \mathbb{R}^1$. It follows that $y(t) \equiv 0$ if $t \geq -2$. Inductively, one has $y(t) \equiv 0$, a contradiction. The lemma is then proved by Part II, Theorem 2.3. □

LEMMA 6.4. *Assume the conditions in* Lemma 6.1 *and* Lemma 6.3. *If* $x_t(\phi, g)$ *is bounded for* $t > 0$, *then* $\omega(\phi, g)$ *admits a continuous separation.*

*Proof.* By Lemma 6.2, $\omega(\phi, g)$ is compact, and by Lemma 6.3, $\Pi$ admits a flow extension on $\omega(\phi, g)$. The lemma then follows from Lemma 6.1 and similar arguments of Part II, Theorem 4.4. □

THEOREM 6.5. *Consider* (6.2) *and assume that*

a) (6.2) *is cooperative, strongly irreducible with respect to* $x(t)$ *and strongly monotone with respect to* $x(t-1)$;
b) $\frac{\partial g}{\partial \eta}(\xi, \eta, t)$ *is positive definite for any* $g \in H(f)$, $\xi, \eta \in \mathbb{R}^n$, *and* $t \in \mathbb{R}$.

*Then the following holds.*

1) *Any linearly stable minimal set* $E$ *of* $\Pi$ *is almost automorphic, and there is an integer* $N \geq 1$ *such that if* $(\phi, g) \in E$ *is an almost automorphic point, then* $x_g(t) \equiv x_t(\phi, g)(0)$ *is an almost automorphic solution of* $(6.3)_g$ *with* $N\mathcal{M}(x_g) \subset \mathcal{M}(f)$.
2) *Let* $(\phi_0, g_0) \in X \times H(f)$ *be such that* $\{x_t(\phi_0, g_0) \mid t \geq 1 + \delta\}$ *is bounded for some* $\delta \geq 0$. *If* $(\omega(\phi_0, g_0), \mathbb{R})$ *is both uniformly and linearly stable, then it is minimal and almost periodic. Moreover, for any*

*$(\phi, g) \in \omega(\phi_0, g_0)$, $x_g(t) \equiv x_t(\phi, g)(0)$ is an almost periodic solution of $(6.3)_g$ with $N\mathcal{M}(x_g) \subset \mathcal{M}(f)$.*

3) *If $\omega(\phi_*, g_*)$ is such that $(\phi_*, g_*) \geq (\leq)(\phi, g_*)$ $((\phi, g_*) \in \omega(\phi_*, g_*))$, then $\omega(\phi_*, g_*)$ contains a unique minimal set $E$ and $(E, \mathbb{R}) \to (H(f), \mathbb{R})$ is an almost 1-1 extension. Moreover, there is a positive integer $N$ such that if $(\phi_0, g_0) \in E$ is an almost automorphic point, then the almost automorphic solution $x_{g_0}(t) \equiv x_t(\phi_0, g_0)(0)$ of $(6.3)_{g_0}$ satisfies $\mathcal{M}(x_{g_0}) \subset \mathcal{M}(f)$.*

*Proof.* It follows from Lemma 6.4 and arguments of Theorem 4.6. □

*Remark 6.1..*

1) In the case 2) of the above theorem, the condition b) is not necessary (see Part II, theorem 2.8).

2) Let $(\phi_0, g_0)$ be as in Theorem 6.5 2). Then by Remark 4.2 of Part II, $\Pi(\phi_0, g_0, t)$ is asymptotically almost periodic.

### 6.2. Global Attractor.

We now give an explicit condition for a scalar delay differential equation to admit an almost periodic global attractor.

Theorem 6.6. *Consider (6.2) with $n = 1$ and assume the following:*

1) *There is a $\delta_0 > 0$ such that $f_\xi(\xi, \eta, t) < -\delta_0$, $f_\eta(\xi, \eta, t) < -\delta_0$ for any $\xi, \eta, t \in \mathbb{R}^1$ ;*
2) *(6.2) admits a bounded solution.*

*Then there is a unique almost periodic solution $x_*(t)$ of (6.2) with $\mathcal{M}(x_*) \subset \mathcal{M}(f)$ such that any bounded solution of (6.2) is asymptotic to $x_*$ as $t \to \infty$.*

*Proof.* Suppose that $x(\phi^*, f, t)$ is a bounded solution of (6.2). By Lemma 6.2 and Lemma 6.3, the $\omega$-limit set $\omega(\phi^*, f)$ is well defined and admits a flow extension.

We now show that $\omega(\phi^*, f)$ is globally, uniformly and asymptotically stable. Take any $(\phi_0, g) \in \omega(\phi^*, f)$, $\phi \in X$ and let $y(t) = x(\phi, g, t) - x(\phi_0, g, t)$. Then $y(t)$ is a solution of

$$x'(t) = a(t)x(t) + b(t)x(t-1), \tag{6.9}$$

where

$$a(t) = \int_0^1 g_\xi(sx(\phi, g, t) + (1-s)x(\phi_0, g, t), x(\phi, g, t-1), t)ds,$$

$$b(t) = \int_0^1 g_\eta(x(\phi_0, g, t), sx(\phi, g, t-1) + (1-s)x(\phi_0, g, t-1), t)ds.$$

By condition 1),

$$a(t) \leq -\delta_0, \quad b(t) \leq -\delta_0. \tag{6.10}$$

Denote $\Psi(\psi, t)$ as the solution of (6.9) with $\Psi(\psi, t) = \psi(t)$ for $t \in [-1, 0]$, where $\psi \in X$. By the zero number properties of delay differential equations (see [3], [31] for details), if $\psi(\theta) > 0 (< 0)$ for all $\theta \in [-1, 0]$, then $\Psi(\psi, t) > 0 (< 0)$ for all $t > 0$. Moreover, by (6.10), $\Psi(\psi, t)$ is strictly decreasing (increasing) as $t$ increases. It follows that $|\Psi(\psi, t)| \searrow 0$ as $t \to \infty$.

For any $\delta > 0$, we let $M_\delta = \|\phi_0 - \phi\|_X + \delta$. By the same zero number properties,

$$|x(\phi_0, g, t) - x(\phi, g, t)| \leq \Psi(M_\delta, t) \searrow 0 \tag{6.11}$$

as $t \to \infty$. This implies that $\omega(\phi^*, f)$ is globally, uniformly and asymptotically stable, and, $\omega(\phi^*, f) \to H(f)$ is a 1-1 extension. The rest of the proof follows from Part I, Theorem 3.8. □

## 7. References


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Wenxian Shen, Department of Mathematics, Auburn University, Auburn University, AL 36849

*E-mail address*: `ws@math.auburn.edu`

Yingfei Yi, School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160

*E-mail address*: `yi@math.gatech.edu`