Ergodicity of Minimal Sets in Scalar Parabolic Equations

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* Partially supported by NSF grant DMS-9402945
** Partially supported by NSF grant DMS 9207069

1. Introduction

We shall study ergodic properties for a minimal set of the (local) skew product semiflow generated by the following family of scalar parabolic equations:

$$u_t = u_{xx} + f(y \cdot t, x, u, u_x), \quad t > 0, \quad 0 < x < 1, \tag{1.1$_y$}$$

with Dirichlet

$$u(t,0) = 0, \quad u(t,1) = 0, \quad t > 0, \tag{1.2$_D$}$$

or Neumann

$$u_x(t,0) = 0, \quad u_x(t,1) = 0, \quad t > 0, \tag{1.2$_N$}$$

boundary conditions, where $y \in Y$, $(Y, \mathbb{R})$ is a minimal flow with compact metric phase space Y, the function $f(y, x, u, p)$ is Lipschitz in y, and for any $y \in Y$, $f(y \cdot t, x, u, p)$ is C^2 in t, x, u, and p.

To be more precise, let X be a fractional power space ([14]) associated with the operator $u \mapsto -u_{xx} : \mathcal{D} \to L^2(0,1)$ that satisfies the imbedding $X \hookrightarrow C^1[0,1]$, where $\mathcal{D} = \{u | u \in H^2(0,1),\ u \text{ satisfies } (1.2)_D \text{ or } (1.2)_N\}$. Then equations $(1.1)_y$-$(1.2)_D$ or $(1.1)_y$-$(1.2)_N$ generate a (local) skew-product semiflow ([31], [32], [33]) Π_t on $X \times Y$:

$$\Pi_t(U, y) = (u(t, \cdot, U, y), y \cdot t), \tag{1.3}$$

where $u(t, x, U, y)$ is the solution of $(1.1)_y$-$(1.2)_D$ or $(1.1)_y$-$(1.2)_N$ with $u(0, x, U, y) = U(x)$, $y \cdot t$ denotes the flow on Y.

Now, let $(U_0, y_0) \in X \times Y$ be such that $\{\Pi_t(U_0, y_0) | t > 0\}$ is bounded in its existence interval. Then $\Pi_t(U_0, y_0)$ is globally defined ([14]) for all $t > 0$ and for any $\delta > 0$, $\{\Pi_t(U_0, y_0) | t \geq \delta\}$ is precompact ([14]). Moreover, the ω-limit set $\omega(U_0, y_0)$ of $\Pi_t(U_0, y_0)$ $(t > 0)$ is compact, connected, and invariant ([13]), that is, Π_t restricted to $\omega(U_0, y_0)$ defines a usual skew-product (two-sided) flow. We call an invariant set $E \subset X \times Y$ of (1.3) a <u>minimal set</u> (or $(E, \mathbb{R})$ is a minimal subflow of (1.3)) if there is a $(U_0, y_0) \in E$ such that $\{\Pi_t(U_0, y_0) | t > 0\}$ is bounded and $E = \omega(U_0, y_0)$ is minimal in the usual sense.

It is proved in [33] that any minimal subflow $(E, \mathbb{R})$ of (1.3) is an <u>almost 1-1 extension</u> (or E is an almost 1-cover) of $(Y, \mathbb{R})$, that is, $Y_0 = \{y \in Y | card E \cap P^{-1}(y) = 1\}$ is a residual subset of Y, here $P : X \times Y \to Y, (x, y) \mapsto y$ is the natural projection. Of course, $(E, \mathbb{R})$ can be a 1-1 extension (1-cover) of $(Y, \mathbb{R})$ in many situations (see [32]). For example, if $(Y, \mathbb{R})$ is a periodic minimal flow, then it has shown in [3], [5] that any minimal subflow $(E, \mathbb{R})$ of (1.3) is a 1-1 extension of $(Y, \mathbb{R})$ hence a periodic minimal set. However, there are many examples, even in the case that $(Y, \mathbb{R})$ is almost periodic, that a minimal subflow $(E, \mathbb{R})$ of (1.3) is not a 1-1 (almost periodic) extension of $(Y, \mathbb{R})$ (see [16], [33] and examples in section 5 of the current paper). An almost 1-1 extension $(E, \mathbb{R})$ of an almost periodic minimal flow $(Y, \mathbb{R})$ is usually referred to as an <u>almost automorphic extension</u> of $(Y, \mathbb{R})$. Points $(U, y) = P^{-1}P(U, y) \subset E$ are <u>almost automorphic points</u> since each such point (U, y) corresponds to an (Bochner) <u>almost automorphic solution</u> $u(t, x, U, y)$ of $(1.1)_y - (1.2)_D$ or $(1.1)_y$-$(1.2)_N$. In fact, as shown in [33], almost automorphy is quite essential to the dynamics of scalar parabolic equations in one space dimension (which particularly include scalar ODEs if one considers Neumann boundary conditions).

The current paper is devoted to study the almost automorphic phenomena from a measure theoretical point of view. More precisely, we ask the following questions: 1) When can a minimal set E (or minimal subflow $(E, \mathbb{R})$) of (1.3) be uniquely ergodic? 2) In the case of unique ergodicity, what can one say about the flow $(E, \mathbb{R})$?

To partially answer the above questions, we obtain the following results.

1) A minimal set E of (1.3) is uniquely ergodic if and only if $(Y, \mathbb{R})$ is uniquely ergodic and $Y_0 = \{y \in Y | card E \cap P^{-1}(y) = 1\}$ has full measure.

2) If a minimal set of (1.3) is uniquely ergodic, then the flow $(E, \mathbb{R})$ is topologically conjugate to a skew-product subflow of $(\mathbb{R}^1 \times Y, \mathbb{R})$.

The result 1) is a generalization of the work of Johnson ([15]) in almost periodic linear scalar ODEs. It simply says that in order for a minimal set E of (1.3) to be uniquely ergodic, the set Y_0 is not only a topologically large set, it needs to be large in measure as well.

Result 2) indicates that in the case of unique ergodicity, dynamics on a minimal set E is expected to be simple since the flow $(E, \mathbb{R})$ in this case is essentially a scalar skew product flow. We refer to a minimal subflow $(E, \mathbb{R})$ of (1.3) as a <u>minimal PDE flow</u> if $(E, \mathbb{R})$ is not topologically conjugate to any subflow of $(\mathbb{R}^1 \times Y, \mathbb{R})$ simply because that the space variable x will play a role in such a flow. An immediate consequence of the above result 2) is as follows. If $(Y, \mathbb{R})$ is almost periodic minimal, then any minimal PDE flow of (1.3) is non-almost periodic almost automorphic since it is non-uniquely ergodic.

Ergodicity issue is important in studying dynamics of a minimal set of (1.3) in particular when $(Y, \mathbb{R})$ is almost periodic minimal. As shown in an example of section 5, a non-unique ergodic almost automorphic minimal set E of (1.3) ($(Y, \mathbb{R})$ is almost periodic minimal) may present certain complicated natures. This is certainly worthwhile for a future study.

The paper is organized as follows. In section 2, we summarize some preliminary materials such as zero number properties ([1], [4], [23]), Floquet theory ([8]), Sacker-Sell spectrum ([6], [22], [26], [27]) and invariant manifold theory ([7], [9], [41]). We also review invariant measure theory ([20], [24], [34]) and properties of almost automorphic functions ([37], [38], [39]). We characterize zero crossing numbers on invariant manifolds in section 3. Our main results are proved in section 4. In section 5, we discuss examples from [16], [17], [40] which show that both uniquely ergodic and non-uniquely ergodic (non-almost periodic) almost automorphic minimal sets exist in semiflow of type (1.3).

For simplicity, we only prove our main results for the case of Dirichlet boundary conditions. The case of Neumann boundary conditions can be proved similarly.

2. Preliminary

2.1. Invariant Measures

Let $(E, \mathbb{R})$ be a flow with compact metric phase space E. An invariant measure on E is a probability measure μ on E such that $\mu(A \cdot t) = \mu(A)$ for all Borel sets $A \subset E$ and all $t \in \mathbb{R}$, here $A \cdot t = \{\omega \cdot t | \omega \in A\}$. An invariant measure μ on E is ergodic if $\mu(A \Delta A \cdot t) = 0$ for all $t \in \mathbb{R}$ implies that $\mu(A) = 0$ or 1, where $A \Delta A \cdot t = (A \setminus A \cdot t) \cup (A \cdot t \setminus A)$. It is a consequence of Krylov-Bogoliubov theorem that invariant measures on E always exist ([24]). If E has only one invariant measure, that is, E is uniquely ergodic, then the unique invariant measure on E is an ergodic measure.

Let $C(E, \mathbb{R}^1)$ be the space of continuous functions $f : E \to \mathbb{R}^1$. By Riesz Representation theorem, there is an isomorphism between bounded positive linear functionals l on $C(E, \mathbb{R}^1)$ satisfying $l(1) = 1$ with the (regular, positive, Borel, probability) measures on E which is given by

$$l(f) = \int_E f(\omega)\mu(d\omega), \quad \forall f \in C(E, \mathbb{R}^1). \tag{2.1}$$

Now, a measure μ on E is invariant if and only if $l(f_t) = l(f)$ for all $f \in C(E, \mathbb{R}^1)$ and all $t \in \mathbb{R}$, where $f_t(\omega) = f(\omega \cdot t)$. Also μ is ergodic if and only if for $f \in L^1(E, \mathbb{R}^1)$ one has

$$f_t = f \quad for \quad all \quad t \in \mathbb{R} \iff f \equiv constant.$$

(See [20], [34]).

2.2. Almost Automorphic Minimal Set

Definition 2.1. *1) Let E be a complete metric space. A continuous function $f : \mathbb{R}^1 \to E$ is almost automorphic if given a sequence $\{t_n'\} \subset \mathbb{R}^1$, there is a subsequence $\{t_n\} \subset \{t_n'\}$ and a function $g : \mathbb{R}^1 \to E$ such that $f(t_n + t) \to g(t)$, $g(t - t_n) \to f(t)$ pointwise as $n \to \infty$.*

2) A flow $(E, \mathbb{R})$ with compact metric phase space E is said to be almost automorphic minimal if it contains a dense almost automorphic motion $\{x_0 \cdot t\}$. The point x_0 is referred to as an almost automorphic point.

3) Consider a homomorphism of minimal flows $p : (E, \mathbb{R}) \to (Y, \mathbb{R})$, where $(Y, \mathbb{R})$ is (Bohr) almost periodic minimal. $(E, \mathbb{R})$ is said to be an almost automorphic (almost periodic) extension of $(Y, \mathbb{R})$ if $(E, \mathbb{R})$ is an almost 1-1 (a 1-1) extension of $(Y, \mathbb{R})$.

Remark 2.1. 1) The notion of almost automorphic function was first introduced by Bochner ([2]) in his work in differential geometry. Subsequent studies were made by Veech ([37], [38], [39]). Flor ([12]), Reich ([25]), and Terras ([35], [36]) etc. Applications to differential equations were considered in works of Fink [11], Johnson [18], Veech [37], Shen and Yi [31], [32], [33] and others. An almost automorphic function can be viewed as a generalized almost periodic function since an (Bohr) almost periodic function is necessary an almost automorphic function but the converse is not true (see [38] for an example of non-almost periodic almost automorphic function and [16], [33] for examples of non-almost periodic almost automorphic solutions in almost periodic scalar ordinary and parabolic equations).

2) The connection between the almost automorphy and the almost periodicity was indicated in [37] as follows: a function is (Bohr) almost periodic if and only if its compact hull consists of almost automorphic functions, while the hull of an (non-almost periodic) almost automorphic function contains only residually many almost automorphic functions.

3) If $(E, \mathbb{R})$ contains a dense almost automorphic motion, then it is point distal ([40]), hence is necessary minimal (see [40] or [33] for discussions).

4) Let $p : (E, \mathbb{R}) \to (Y, \mathbb{R})$ be homomorphism of minimal flows, where $(Y, \mathbb{R})$ is (Bohr) almost periodic minimal. If $(E, \mathbb{R})$ is an almost automorphic extension of $(Y, \mathbb{R})$, then $(E, \mathbb{R})$ is almost automorphic minimal, moreover, almost automorphic points on E are precisely those $x = E \cap p^{-1}(px)$. In the case that $(Y, \mathbb{R})$ in (1.3) is almost periodic minimal, it follows from Shen and Yi ([33]) that any minimal set $E \subset X \times Y$ of (1.3) is an almost automorphic extension of Y, hence an almost automorphic minimal set.

2.3. Zero Number Properties

For a given C^1 function $v : [0, 1] \to \mathbb{R}^1$, the zero number of v is defined as

$$Z(v(\cdot)) = \#\{x \in (0, 1) | v(x) = 0\}.$$

The following lemma can be found originally in [1], [23], and is improved in a recent work [4].

Lemma 2.1. *Consider the scalar linear parabolic equation:*

$$\begin{cases} v_t = a(t, x)v_{xx} + b(t, x)v_x + c(t, x)v, & t > 0, \quad x \in (0, 1), \\ v(t, 0) = 0, \quad v(t, 1) = 0, & t > 0, \end{cases} \tag{2.2}$$

where a, a_t, a_x, b, and c are bounded continuous functions, $a \geq \delta > 0$. Let $v(t, x)$ be a classical nontrivial solution of (2.2). Then, the following holds:

1) $Z(v(t, \cdot))$ is finite for $t > 0$ and is nonincreasing as t increases;

2) $Z(v(t, \cdot))$ can drop only at t_0 such that $v(t_0, \cdot)$ has a multiple zero in $[0, 1]$;

3) $Z(v(t, \cdot))$ can drop only finite times, and there exists a $t^* > 0$ such that $v(t, \cdot)$ has only simple zeros in $[0, 1]$ as $t \geq t^*$ (hence $Z(v(t, \cdot)) =$ constant as $t \geq t^*$).

2.4. Floquet Theory

Consider the following linear parabolic equation:

$$\begin{cases} w_t = w_{xx} + b(x, \omega \cdot t)w, & t > 0, \quad 0 < x < 1, \\ w(t, 0) = 0, \quad w(t, 1) = 0, & t > 0, \end{cases} \tag{2.3}$$

where $\omega \in \Omega$, $\omega \cdot t$ is a flow on a compact metric space Ω, $b : [0, 1] \times \Omega \to \mathbb{R}^1$ is continuous. Suppose that for any $w_0 \in L^2(0, 1)$, the solution $w(t, x, w_0, \omega)$ of (2.3) with $w(0, x, w_0, \omega) = w_0(x)$ exists. The following results are due to [8].

Theorem 2.2. *1) There is a sequence* $\{w_n\}_{n=1}^{\infty}$, $w_n : [0,1]\times\Omega \to \mathbb{R}^1$ $(n = 1,2,\cdots)$ *such that* $w_n(\cdot,\omega) \in C^{1,\gamma}[0,1]$ *for any* γ *with* $0 \le \gamma < 1$, $w_n(0,\omega) = w_n(1,\omega) = 0$, *and* $\|w_n(\cdot,\omega)\|_{L^2(0,1)} = 1$ *for any* $\omega \in \Omega$. $\{w_n(\cdot,\omega)\}_{n=1}^{\infty}$ *forms a (Floquet) basis of* $L^2(0,1)$ *and* $Z(w_n(\cdot,\omega)) = n-1$ *for all* $\omega \in \Omega$. *Let* $W_n(\omega) = span\{w_n(\cdot,\omega)\}$, $n = 1,2,\cdots$. *Then* $\bigoplus_{i=n_1}^{n_2} W_i(\omega) = \{w_0 \in L^2(0,1) | w(t,\cdot,w_0,\omega)$ *is exponentially bounded in* $L^2(0,1)$, *and* $n_1 - 1 \le Z(w(t,\cdot,w_0,\omega)) \le n_2 - 1$ *for all* $t \in \mathbb{R}^1\} \cup \{0\}$ *for any* n_1, n_2 *with* $n_1 \le n_2$.

2) Suppose $w_0(x) = \sum_{n=1}^{\infty} c_n^0 w_n(x,\omega)$ *(*c_n^0*'s are called Fourier coefficients). Then*

$$w(t,x,w_0,\omega) = \sum_{n=1}^{\infty} c_n(t) w_n(x,\omega\cdot t), \tag{2.4}$$

where

$$c_n^{'} = \mu_n(\omega\cdot t)c_n, \tag{2.5}$$

$c_n(0) = c_n^0$, $\mu_n(\omega\cdot t) = \int_0^1 [b(x,\omega\cdot t)w_n(x,\omega\cdot t)^2 - w_{nx}(x,\omega\cdot t)^2]dx$, $n = 1,2\cdots$. *Moreover, for each* $n \ge 1$, *there are* $T_n > 0$, $\kappa_n > 0$ *which are independent of* $\omega \in \Omega$ *such that*

$$\int_t^{t+T_n} \mu_{n+1}(\omega\cdot s)ds - \int_t^{t+T_n} \mu_n(\omega\cdot s)ds \le -\kappa_n, \tag{2.6}$$

for all $\omega \in \Omega$ *and* $t \in \mathbb{R}^1$.

3) Define $\Psi(\cdot) : \Omega \to L(L^2(0,1), l^2)$ *by* $\Psi(\omega)w_0 = \{c_n^0\}_{n=1}^{\infty}$, *where* $w_0(x) = \sum_{n=1}^{\infty} c_n^0 w_n(x,\omega)$. *Then* $\Psi(\omega\cdot t)w(t,x,w_0,\omega) = \{c_n(t)\}$, *here* $c_n(t)$*'s are given in (2.5). Moreover,* Ψ *is continuous,* $\Psi(\omega)$ *is an isomorphsim for each* $\omega \in \Omega$, *and there are positive constants* K_1, K_2 *which are independent of* ω *such that*

$$\|\Psi(\omega)\| \le K_1 \quad and \quad \|\Psi^{-1}(\omega)\| \le K_2.$$

2.5. Exponential Dichotomy (ED) and Sacker-Sell (S-S) Spectrum

Consider

$$\begin{cases} v_t = v_{xx} + a(x,\omega\cdot t)v_x + b(x,\omega\cdot t)v, \quad t > 0, \quad 0 < x < 1 \\ v(t,0) = 0, \quad v(t,1) = 0, \quad t > 0, \end{cases} \tag{2.7}$$

where $\omega \in \Omega$, $\omega\cdot t$ is a flow on compact metric space Ω, $a(x,\omega)$ and $b(x,\omega)$ are continuous in x,ω, and for any given $\omega \in \Omega$, $a(x,\omega\cdot t)$ is C^1 in x,t. Let X be a fractional power space associated to the operator $u \mapsto -u_{xx} : H_0^2(0,1) \to L^2(0,1)$ and $\Phi(t,\omega) : X \to X$ be the evolution operator generated by (2.7), that is, the evolution operator of the following equation:

$$v' = A(\omega \cdot t)v, \quad t > 0, \quad \omega \in \Omega, \quad v \in X, \tag{2.8}$$

where $A(\omega)v = v_{xx} + a(x,\omega)v_x + b(x,\omega)v$, $\omega \cdot t$ is as in (2.7).

Definition 2.2. *Equation (2.7) (or (2.8)) is said to have an* <u>*exponential dichotomy*</u> *on* Ω *if there exist* $\beta > 0$, $K > 0$ *and continuous projections* $P(\omega) : X \to X$ *such that for any* $\omega \in \Omega$*, the following holds:*

1) $\Phi(t,\omega)P(\omega) = P(\omega \cdot t)\Phi(t,\omega), \quad t \in \mathbb{R}^+$*;*

2) $\Phi(t,\omega)|_{R(P(\omega))} : R(P(\omega)) \to R(P(\omega \cdot t))$ *is an isomorphism for* $t \in \mathbb{R}^+$ *(hence* $\Phi(-t,\omega) := \Phi^{-1}(t, \omega \cdot -t) : R(P(\omega)) \to R(P(\omega \cdot -t))$ *is well defined for* $t \in \mathbb{R}^+$*);*

3)

$$\begin{aligned} \|\Phi(t,\omega)(I - P(\omega))\| &\le K e^{-\beta t}, \quad t \in \mathbb{R}^+, \\ \|\Phi(t,\omega)P(\omega)\| &\le K e^{\beta t}, \quad t \in \mathbb{R}^-. \end{aligned} \tag{2.9}$$

Remark 2.2. 1) (2.9) is equivalent to

$$\begin{aligned} \|\Phi(t-s, \omega \cdot s)(I - P(\omega \cdot s))\| &\le K e^{-\beta(t-s)}, \quad t \ge s, \quad t, s \in \mathbb{R}^1, \\ \|\Phi(t-s, \omega \cdot s)P(\omega \cdot s)\| &\le K e^{\beta(t-s)}, \quad t \le s, \quad t, s \in \mathbb{R}^1 \end{aligned} \tag{2.10}$$

for any $\omega \in \Omega$.

2)

$$\begin{aligned} R(P(\omega)) = \{v \in X | \Phi(t,\omega)v \quad &exists \quad for \quad t \in \mathbb{R}^1, \\ &\Phi(t,\omega)v \to 0 \quad exponentially \quad as \quad t \to -\infty\} \\ = \{v \in X | \Phi(t,\omega)v \quad &exists \quad for \quad t \in \mathbb{R}^1, \quad \Phi(t,\omega)v \to 0 \quad as \quad t \to -\infty\}, \end{aligned}$$

and

$$\begin{aligned} R(I - P(\omega)) &= \{v \in X | \Phi(t,\omega)v \to 0 \quad exponentially \quad as \quad t \to \infty\} \\ &= \{v \in X | \Phi(t,\omega)v \to 0 \quad as \quad t \to \infty\}. \end{aligned}$$

Now, for any given $\lambda \in \mathbb{R}^1$, consider

$$v' = (A(\omega \cdot t) - \lambda)v, \quad t > 0, \quad \omega \in \Omega, \quad v \in X, \tag{2.11$_\lambda$}$$

where $A(\omega)$ and $\omega \cdot t$ are as in (2.8).

Definition 2.3. *$\Sigma(\Omega) = \{\lambda \in \mathbb{R}^1 | (2.11)_\lambda$ has no ED on $\Omega\}$ is called the (Sacker-Sell) spectrum of (2.7) (or (2.8)).*

Remark 2.3. $\Sigma(\Omega)$ is of the following form: $\Sigma(\Omega) = \cup_{k=1}^{\infty} I_k$, where $I_k = [a_k, b_k]$ and $\{I_k\}$ is ordered from right to left, that is, $\cdots < a_n \leq b_n < a_{n-1} \leq b_{n-1} < \cdots < a_2 \leq b_2 < a_1 \leq b_1$ ([6], [22], [26], [27]).

Suppose that $\Sigma(\Omega) = \cup_{k=1}^{\infty} I_k$ ($I_k = [a_k, b_k]$) is the spectrum of (2.7). For any given $0 < n_1 \leq n_2 \leq \infty$, if $n_2 \neq \infty$, let

$$\begin{aligned} V^{n_1,n_2}(\omega) = \{v \in X | \|\Phi(t,\omega)v\| = o(e^{a^- t}) \quad as \quad t \to -\infty, \\ \|\Phi(t,\omega)v\| = o(e^{b^+ t}) \quad as \quad t \to \infty\}, \end{aligned}$$

where a^-, b^+ are such that $\lambda_1 < a^- < a_{n_2} \leq b_{n_1} < b^+ < \lambda_2$ for any $\lambda_1 \in \cup_{k=n_2+1}^{\infty} I_k$ and $\lambda_2 \in \cup_{k=1}^{n_1-1} I_k$. If $n_2 = \infty$, let

$$V^{n_1,n_2}(\omega) = \{v \in X | \|\Phi(t,\omega)v\| = o(e^{b^+ t}) \quad as \quad t \to \infty\},$$

where b^+ is such that $b_{n_1} < b^+ < \lambda$ for any $\lambda \in \cup_{k=1}^{n_1-1} I_k$.

Definition 2.4. *$V^{n_1,n_2}(\omega)$ is called the invariant subspace of (2.7) (or (2.8)) associated to the spectrum set $\cup_{k=n_1}^{n_2} I_k$ at $\omega \in \Omega$.*

Remark 2.4. For given $\omega \in \Omega$, let $w(t,x) = exp(\frac{1}{2}\int_0^x a(s,\omega \cdot t)ds)v(t,x)$. Then (2.7) becomes

$$\begin{cases} w_t = w_{xx} + b^*(x,\omega \cdot t)w, \quad t > 0, \quad 0 < x < 1, \\ w(t,0) = 0, \quad , w(t,1) = 0, \quad t > 0, \end{cases} \tag{2.7$'$}$$

for some continuous function $b^* : [0,1] \times \Omega \to \mathbb{R}^1$, where $\omega \cdot t$ is as in (2.7). Moreover, one has

1) (2.7) has ED on Ω if and only if $(2.7)'$ has ED on Ω. It follows that if $\Sigma(\Omega)$ and $\Sigma'(\Omega)$ are the spectrum of (2.7) and $(2.7)'$ respectively, then $\Sigma(\Omega) = \Sigma'(\Omega)$.

2) Suppose that $\Sigma(\Omega) = \cup_{k=1}^{\infty} I_k$ and $\{w_n(\cdot,\omega)\}$ is the Floquet basis of $(2.7)'$. Then for any given $0 < n_1 \leq n_2$, $V^{n_1,n_2}(\omega) = exp(-\frac{1}{2}\int_0^x a(s,\omega)ds)V'^{n_1',n_2'}(\omega)$ (i.e. $V^{n_1,n_2}(\omega) = \{v(\cdot) | v(x) = exp(-\frac{1}{2}\int_0^x a(s,\omega)ds)w(x),\ w(\cdot) \in V'^{n_1',n_2'}\}$), where $V'^{n_1',n_2'}(\omega) = \bigoplus_{k=n_1'}^{n_2'} span\{w_n(\cdot,\omega)\}$, $n_1' = dim V^{1,n_1-1}(\omega) + 1$, $n_2' = dim V^{1,n_2}(\omega)$. Therefore, by Theorem 2.2, $N_1 \leq Z(v(\cdot)) \leq N_1 + N_2 - 1$ and $N_1 \leq Z(w(\cdot)) \leq N_1 + N_2 - 1$ for any

$v \in V^{n_1,n_2}(\omega) \setminus \{0\}$, $w \in V'^{n_1',n_2'}(\omega) \setminus \{0\}$, where $N_1 = n_1' - 1 = dim V^{1,n_1-1}(\omega)$, $N_2 = n_2' - n_1' = dim V^{n_1,n_2}(\omega)$.

3) Suppose that $0 \in \Sigma(\Omega)$ and n_0 is such that $0 \in I_{n_0} \subset \Sigma(\Omega)$. Then $V^s(\omega) = V^{n_0+1,\infty}(\omega)$, $V^{cs}(\omega) = V^{n_0,\infty}(\omega)$, $V^c(\omega) = V^{n_0,n_0}(\omega)$, $V^{cu}(\omega) = V^{1,n_0}(\omega)$, and $V^u(\omega) = V^{1,n_0-1}(\omega)$ are referred to as stable, center stable, center, center unstable, and unstable subspaces of (2.7) (or (2.8)) at $\omega \in \Omega$ respectively.

2.6. Invariant Manifolds

Consider

$$v' = A(\omega \cdot t)v + F(v, \omega \cdot t), \quad t > 0, \quad \omega \in \Omega, \quad v \in X, \tag{2.12}$$

where $\omega \cdot t$ and $A(\omega \cdot t)$ are as in (2.8), $F(\cdot, \omega) \in C^1(X, X_0)$, $F(v, \cdot) \in C^0(Y, X_0)$ $(v \in X)$, $F(v, y) = o(\|v\|)$ $(X_0 = L^2(0,1))$. It is well known that the solution operator $\Lambda_t(\cdot, \omega)$ of (2.12) exists in usual sense (that is, for any $v \in X$, $\Lambda_0(v, \omega) = v$, $\Lambda_t(v, \omega) \in \mathcal{D}(A(\omega \cdot t))$, $\Lambda_t(v, \omega)$ is differentiable in t with respect to X_0 norm and satisfies (2.12) for $t > 0$) ([14]).

Now suppose that $\Sigma(\Omega) = \cup_{k=1}^{\infty} I_k$ is the spectrum of (2.8). The following theorem can be proved using arguments of [7], [9], [21], [30].

Theorem 2.3. *There is a $\delta_0 > 0$ such that for any $0 < \delta^* < \delta_0$ and $0 < n_1 \le n_2 \le \infty$, (2.12) possess for each $\omega \in \Omega$ a local invariant manifold $W^{n_1,n_2}(\omega, \delta^*)$ which satisfies the following properties:*

1) There are $M > 0$, and bounded continuous function $h^{n_1,n_2} : \cup_{\omega \in \Omega}\big(V^{n_1,n_2}(\omega) \times \{\omega\}\big) \to \cup_{\omega \in \Omega}\big(V^{n_2+1,\infty}(\omega) \bigoplus V^{1,n_1-1}(\omega)\big)$ with $h^{n_1,n_2}(\cdot, \omega) : V^{n_1,n_2}(\omega) \to V^{n_2+1,\infty}(\omega) \bigoplus V^{1,n_1-1}(\omega)$ being C^1 for each fixed $\omega \in \Omega$, and $h^{n_1,n_2}(v, \omega) = o(\|v\|)$, $\|\dfrac{\partial h^{n_1,n_2}}{\partial v}(v, \omega)\| \le M$ for all $\omega \in \Omega$, $v \in V^{n_1,n_2}(\omega)$ such that

$$W^{n_1,n_2}(\omega, \delta^*) = \big\{v_0^{n_1,n_2} + h^{n_1,n_2}(v_0^{n_1,n_2}, \omega) | v_0^{n_1,n_2} \in V^{n_1,n_2}(\omega) \cap \{v \in X | \|v\| < \delta^*\}\big\}.$$

Moreover, $W^{n_1,n_2}(\omega, \delta^*)$ are diffeomorphic to $V^{n_1,n_2}(\omega) \cap \{v \in X | \|v\| < \delta^*\}$, and $W^{n_1,n_2}(\omega, \delta^*)$ are tangent to $V^{n_1,n_2}(\omega)$ at $0 \in X$ for each $\omega \in \Omega$.

2) $W^{n_1,n_2}(\omega, \delta^*)$ is locally invariant in the sense that for any $v \in W^{n_1,n_2}(\omega, \delta^*)$, there is a $\tau > 0$ such that $\Lambda_t(v, \omega) \in W^{n_1,n_2}(\omega \cdot t, \delta^*)$ for any $t \in \mathbb{R}^1$ with $0 < t < \tau$.

Remark 2.5. 1) The existence of δ_0 in the above theorem which is independent of n_1 and n_2 is due to the increasing of the gaps between the spectrum intervals I_n and I_{n+1} as n increases (see [7], [8]).

2) Note that as usual $W^{n_1,n_2}(\omega,\delta^*)$ is constructed in terms of appropriate rate conditions for the solutions of (2.12) by replacing F by a cut-off function $\tilde{F}$ (see [7]). It then follows that for any $n_1 \le n_2 \le n_3 \le \infty$ and $\omega \in \Omega$, $W^{n_1,n_2}(\omega,\delta^*) \subset W^{n_1,n_3}(\omega,\delta^*)$, and for any $u \in W^{n_1,\infty}(\omega,\delta^*)$, there are $u_n \in W^{n_1,n}(\omega,\delta^*)$ $(n_1 \le n < \infty)$ such that $u_n \to u$ as $n \to \infty$.

3) For any $0 < n_1 \le n_2 < \infty$ and $\omega \in \Omega$, there are $\delta_1^* < \delta_0$, $\tau > 0$ such that $\Lambda_t W^{n_1,n_2}(\omega,\delta_1^*) \subset W^{n_1,n_2}(\omega \cdot t,\delta^*)$ for any t with $|t| < \tau$.

Definition 2.5. *Suppose that $0 \in \Sigma(\Omega)$ and $0 \in I_{n_0} = [a_{n_0}, b_{n_0}] \subset \Sigma(\Omega)$. Then $W^s(\omega,\delta^*) = W^{n_0+1,\infty}(\omega,\delta^*)$, $W^{cs}(\omega,\delta^*) = W^{n_0,\infty}(\omega,\delta^*)$, $W^c(\omega,\delta^*) = W^{n_0,n_0}(\omega,\delta^*)$, $W^{cu}(\omega,\delta^*) = W^{1,n_0}(\omega,\delta^*)$, and $W^u(\omega,\delta^*) = W^{1,n_0-1}(\omega,\delta^*)$ are referred to as local stable, center stable, center, center unstable, and unstable manifolds of (2.12) at $\omega \in \Omega$ respectively.*

Remark 2.6. 1) $W^s(\omega,\delta^*)$ and $W^u(\omega,\delta^*)$ are overflowing invariant in the sense that if δ^* is sufficiently small, then

$$\Lambda_t(W^s(\omega,\delta^*),\omega) \subset W^s(\omega \cdot t,\delta^*) \quad for \quad t \gg 1,$$

and

$$\Lambda_t(W^u(\omega,\delta^*),\omega) \subset W^u(\omega \cdot t,\delta^*) \quad for \quad t \ll -1.$$

Moreover, one has

$$\Lambda_t(v,\omega) \to 0 \quad as \quad t \to \infty, \quad for \quad any \quad v \in W^s(\omega,\delta^*),$$

and

$$\Lambda_t(v,\omega) \to 0 \quad as \quad t \to -\infty, \quad for \quad any \quad v \in W^u(\omega,\delta^*).$$

2) By the invariant foliation theory ([7], [9]), one has that for any $\omega \in \Omega$,

$$W^{cs}(\omega,\delta^*) = \cup_{u_c \in W^c(\omega,\delta^*)} \bar{W}_s(u_c,\omega,\delta^*),$$

where $\bar{W}_s(u_c,\omega,\delta^*)$ is the so called stable leaf of (2.12) at u_c, and it is invariant in the sense that if $\tau > 0$ is such that $\Lambda_t(u_c,\omega) \in W^c(\omega \cdot t,\delta^*)$ and $\Lambda_t(u,\omega) \in W^{cs}(\omega,\delta^*)$ for all $0 \le t < \tau$, where $u \in \bar{W}_s(u_c,\omega,\delta^*)$, then $\Lambda_t(u,\omega) \subset \bar{W}_s(\Lambda_t(u_c,\omega),\omega \cdot t,\delta^*)$ for $0 \le t < \tau$. Moreover, if $dim V^c(\omega) = 1$, then by the constructions of [7], [9], there are $M, \rho > 0$

such that for any $u \in \bar{W}_s(u_c, \omega, \delta^*)$ $(u_c \neq 0)$ and $\tau > 0$ with $\Lambda_t(u, \omega) \in W^{cs}(\omega \cdot t, \delta^*)$, $\Lambda_t(u_c, \omega) \in W^c(\omega \cdot t, \delta^*)$ for $0 \leq t < \tau$, one has that

$$\frac{\|\Lambda_t(u, \omega) - \Lambda_t(u_c, \omega)\|}{\|\Lambda_t(u_c, \omega)\|} \leq M e^{-\rho t} \frac{\|u - u_c\|}{\|u_c\|}$$

for $0 \leq t < \tau$.

3. Zero Numbers on Invariant Manifolds

Let $\Omega = E \subset X \times Y$ be a compact invariant set of (1.3). For any $\omega = (U, y) \in \Omega$, denote $\omega \cdot t = \Pi_t(U, y)$. Let $v = u - u(t, \cdot, U, y)$ in $(1.1)_y - (1.2)_D$. Then v satisfies the following equation

$$\begin{cases} v_t = v_{xx} + a(x, \omega \cdot, t) v_x + b(x, \omega \cdot t) v + \tilde{f}(v, v_x, x, \omega \cdot t), t > 0, 0 < x < 1, \\ v(t, 0) = 0, \quad v(t, 1) = 0, \quad t > 0, \end{cases} \tag{3.1}$$

where $\tilde{f}(v, v_x, x, \omega) = f(y, x, v + U, v_x + U_x) - f(y, x, U, U_x) - a(x, \omega) v_x - b(x, \omega) v$, $a(x, \omega) = f_p(y, x, U, U_x)$, $b(x, \omega) = f_u(y, x, U, U_x)$.

Denote $A(\omega) = \dfrac{\partial^2}{\partial x^2} + a(\cdot, \omega) \dfrac{\partial}{\partial x} + b(\cdot, \omega)$, $F(v, \omega) = \tilde{f}(v, v_x, \cdot, \omega)$. Then (3.1) can be written as

$$v' = A(\omega \cdot t) v + F(v, \omega \cdot t). \tag{3.2}$$

Suppose that $\Sigma(\Omega) = \cup_{k=1}^{\infty} I_k$ ($\{I_k\}$ is ordered from right to the left) is the Sacker-Sell spectrum of the linear equation associated to (3.2):

$$v' = A(\omega \cdot t) v, \quad t > 0, \quad \omega \in \Omega, \quad v \in X. \tag{3.3}$$

For any given $0 < n_1 \leq n_2 \leq \infty$, let $V^{n_1, n_2}(\omega)$ be the invariant subspace of (3.3) associated to the spectrum set $\cup_{k=n_1}^{n_2} I_k$ at $\omega \in \Omega$.

Lemma 3.1. *For given* $0 < n_1 \leq n_2 \leq \infty$, $N_1 \leq Z(v(\cdot)) \leq N_1 + N_2 - 1$ *for any* $v \in V^{n_1, n_2}(\omega)$, *where* $N_1 = dim V^{1, n_1 - 1}(\omega)$, $N_2 = dim V^{n_1, n_2}(\omega)$.

Proof. It directly follows from Remark 2.4 2). ■

For given $0 < n_1 \leq n_2 \leq \infty$ and $\omega = (U, y) \in \Omega$, by Theorem 2.3, there is a well defined local invariant manifold $W^{n_1, n_2}(\omega, \delta^*)$ of (3.2) (or (3.1)). Let

$$M^{n_1, n_2}(\omega, \delta^*) = \{u \in X | u - U \in W^{n_1, n_2}(\omega, \delta^*)\}. \tag{3.4}$$

$M^{n_1,n_2}(\omega,\delta^*)$ is referred to as a <u>local invariant manifold</u> of $(1.1)_y$-$(1.2)_D$ (or (1.3)) at (U,y).

Suppose that $0 \in \Sigma(\Omega)$ and n_0 is such that $I_{n_0} = [a_0, b_0] \subset \Sigma$ with $a_0 \leq 0 \leq b_0$. For given $\omega = (U,y) \in \Omega$, $\delta^* > 0$, let

$$\begin{aligned}
M^s(\omega,\delta^*) &= M^{n_0+1,\infty}(\omega,\delta^*),\\
M^{cs}(\omega.\delta^*) &= M^{n_0,\infty}(\omega,\delta^*),\\
M^c(\omega,\delta^*) &= M^{n_0,n_0}(\omega,\delta^*),\\
M^{cu}(\omega,\delta^*) &= M^{1,n_0}(\omega,\delta^*),\\
M^u(\omega,\delta^*) &= M^{1,n_0-1}(\omega,\delta^*).
\end{aligned}$$

Then $M^s(\omega,\delta^*)$, $M^{cs}(\omega,\delta^*)$, $M^c(\omega,\delta^*)$, $M^{cu}(\omega,\delta^*)$, and $M^u(\omega,\delta^*)$ are continuous in $\omega \in \Omega$ and are referred to as <u>local stable</u>, <u>center stable</u>, <u>center</u>, <u>center unstable</u>, and <u>unstable manifolds</u> of (1.1) (or (1.3)) at $\omega = (U,y) \in \Omega$ respectively.

<u>Remark 3.1.</u> By Remark 2.6 2), for any $\omega = (U,y) \in \Omega$, one has that

$$M^{cs}(\omega,\delta^*) = \cup_{u_c \in M^c(\omega,\delta^*)} \bar{M}_s(u_c,\omega,\delta^*),$$

where $\bar{M}_s(u_c,\omega,\delta^*) = \{u \in X | u - U \in \bar{W}^s(u_c - U,\omega,\delta^*)$ and $\bar{M}_s(U,\omega,\delta^*) = M^s(\omega,\delta^*)$. Moreover, if $dim V^c(\omega) = 1$, then there are $M, \rho > 0$ such that for any $u^* \in \bar{M}_s(u_c,\omega,\delta^*)$, $u_c \neq U$, and $\tau > 0$ with $u(t,\cdot,u^*,y) \in M^{cs}(\omega\cdot t,\delta^*)$, $u(t,\cdot,u_c,y) \in M^c(\omega,\delta^*)$ for any $0 \leq t < \tau$, one has that

$$\frac{\|u(t,\cdot,u^*,y) - u(t,\cdot,u_c,y)\|}{\|u(t,\cdot,u_c,y) - u(t,\cdot,U,y)\|} \leq M e^{-\rho t} \frac{\|u^* - u_c\|}{\|u_c - U\|}$$

for $0 \leq t < \tau$.

<u>Theorem 3.2.</u> *Let δ_0 be as in Theorem 2.3 and is sufficiently small. For any given $0 < n_1 \leq n_2 < \infty$, let N_1 and N_2 be as in Lemma 3.1. Then the following holds.*

1) If $n_2 < \infty$, then there is a $\delta^*_{n_1,n_2} < \delta_0$ such that $N_1 \leq Z(u(\cdot) - U(\cdot)) \leq N_1 + N_2 - 1$ for any $u \in M^{n_1,n_2}(\omega,\delta^*_{n_1,n_2}) \setminus \{U\}$ $(\omega = (U,y) \in \Omega)$.

2) If $n_1 > 0$ is such that $I_{n_1} \subset \mathbb{R}^- = \{\lambda \in \mathbb{R}^1 | \lambda < 0\}$, then for any $0 < \delta^* < \delta_0$, $n_1 \leq n_2 \leq \infty$, $u \in M^{n_1,n_2}(\omega,\delta^*) \setminus \{U\}$ $(\omega = (U,y) \in \Omega)$, $Z(u(\cdot) - U(\cdot)) \geq N_1$.

3) If $n_1 > 0$ is such that $0 \in I_{n_1}$ and $dim V^{n_1,n_1}(\omega) = 1$, then there is a $0 < \delta^*_{n_1,\infty} < \delta_0$ such that $Z(u(\cdot) - U(\cdot)) \geq N_1$ for any $u \in M^{n_1,\infty}(\omega,\delta^*_{n_1,\infty}) \setminus \{U\}$ $(\omega = (U,y) \in \Omega)$.

Proof. 1) Suppose that the theorem is not true for some $0 < n_1 \leq n_2 < \infty$. Then there are $\delta_n^* \to 0$, $\omega_n = (U_n, y_n) \in \Omega$, and $u_n \in W^{n_1,n_2}(\omega_n, \delta_n^*) \setminus \{U_n\}$ such that $Z(u_n(\cdot) - U_n(\cdot)) < N_1$ or $Z(u_n(\cdot) - U_n(\cdot)) \geq N_1 + N_2$. Let $v_n(\cdot) = \dfrac{u_n(\cdot) - U_n(\cdot)}{\|u_n - U_n\|}$. Since $u_n - U_n \in W^{n_1,n_2}(\omega_n, \delta_n^*)$, $n_2 < \infty$ and Ω is compact, $\{v_n(\cdot)\}$ is relatively compact. Without loss of generality, we assume that $v_n(\cdot) \to v^*$ and $\omega_n = (U_n, y_n) \to \omega^* = (U^*, y^*)$ as $n \to \infty$. Then $v^* \in V^{n_1,n_2}(\omega^*)$. Moreover, by Remark 2.5 3), $v^*(t, \cdot) = \lim_{n\to\infty} \dfrac{u(t, \cdot, u_n, y_n) - u(t, \cdot, U_n, y_n)}{\|u_n - U_n\|}$ satisfies (3.3) with $\omega = \omega^*$ for $|t| \ll 1$. Now by Lemma 3.1, $N_1 \leq Z(v(t, \cdot)) \leq N_1 + N_2 - 1$ for $|t| \ll 1$. Suppose that $t_1 > 0$, $t_2 < 0$ ($|t_1|, |t_2| \ll 1$) are such that $v(t_1, \cdot)$, $v(t_2, \cdot)$ have only simple zeros in $[0, 1]$. Then $Z(u(t_1, \cdot, u_n, y_n) - u(t_1, \cdot, U_n, y_n)) = Z(v(t_1, \cdot)) \geq N_1$ and $Z(u(t_2, \cdot, u_n, y_n) - u(t_2, \cdot, U_n, y_n)) = Z(v(t_2, \cdot)) \leq N_1 + N_2 - 1$ as $n \gg 1$. By Lemma 2.1, $N_1 \leq Z(u_n(\cdot) - U_n(\cdot)) \leq N_1 + N_2 - 1$, a contradiction.

2) Suppose that n_1 is such that $I_{n_1} \subset \mathbb{R}^-$. We first prove that 2) is true for any $n_2 < \infty$. In fact, by Remark 2.6 1), when δ_0 is sufficiently small, for any $n_1 \leq n_2 < \infty$, $0 < \delta^* < \delta_0$, and $u^* \in M^{n_1,n_2}(\omega, \delta^*) \setminus \{U\}$, $u(t, \cdot, u^*, y) \in M^{n_1,n_2}(\omega \cdot t, \delta^*_{n_1,n_2}) \setminus \{u(t, \cdot, U, y)\}$ as $t \gg 1$. Then by 1), one has that $Z(u(t, \cdot, u^*, y) - u(t, \cdot, U, y)) \geq N_1$ for $t \gg 1$. It then follows from Lemma 2.1 that $Z(u^*(\cdot) - U(\cdot)) \geq N_1$.

Next we prove that 2) also holds when $n_2 = \infty$. Let $u^* \in M^{n_1,\infty}(\omega, \delta^*) \setminus \{U\}$. By Remark 2.5 2), there are $u_n \in M^{n_1,n}(\omega, \delta^*) \setminus \{U\}$ ($n_1 \leq n < \infty$) such that $u_n \to u^*$ as $n \to \infty$. Now suppose that $0 < t_0 \ll 1$ is such that $u(t_0, \cdot, u^*, y) - u(t_0, \cdot, U, y)$ has only simple zeros in $[0, 1]$. Then $Z(u(t_0, \cdot, u^*, y) - u(t_0, \cdot, U, y)) = Z(u(t_0, \cdot, u_n, y) - u(t_0, \cdot, U, y)) \geq N_1$ as $n \gg 1$. This implies that $Z(u^*(\cdot) - U(\cdot)) \geq N_1$.

3) Suppose that n_1 is such that $0 \in I_{n_1}$ and $dim V^{n_1,n_1}(\omega) = 1$.

First, we note that by 1), $Z(u(\cdot) - U(\cdot)) = N_1$, and by Theorem 2.3 1),

$$u - U = v_0^{n_1,n_1} + h^{n_1,n_1}(v_0^{n_1,n_1}, \omega) \tag{3.5}$$

for any $u \in M^{n_1,n_1}(\omega, \delta^*_{n_1,n_1}) \setminus \{U\}$, where $0 \neq v_0^{n_1,n_1} \in V^{n_1,n_1}(\omega)$, $h^{n_1,n_1}(v, \omega) = o(\|v\|)$.

Next, by Theorem 2.2, $V^{n_1,n_1}(\omega) = span\{w_{N_1}(\cdot, \omega)\}$. Since $w_{N_1}(\cdot, \omega)$ has only simple zeros in $[0, 1]$ and Ω is compact, there is a $\delta_c > 0$ such that for any $v \in X$ with $\|v\| \leq \delta_c$, and $\omega \in \Omega$,

$$Z(w_{N_1}(\cdot, \omega)) = Z\big(\frac{w_{N_1}(\cdot, \omega)}{\|w_{N_1}(\cdot, \omega)\|} + v(\cdot)\big) = N_1, \tag{3.6}$$

and for any $u \in M^{n_1,n_1}(\omega, \delta^*_{n_1,n_1}) \setminus \{U\}$,

$$\frac{\|h^{n_1,n_1}(v_0^{n_1,n_1}, \omega)\|}{\|v_0^{n_1,n_1}\|} < \frac{\delta_c}{2} \tag{3.7}$$

when $\delta^*_{n_1,n_1}$ is sufficiently small, where $v_0^{n_1,n_1}$ is as in (3.5).

Now, note that $M^{n_1,\infty}(\omega, \delta^*) = M^{cs}(\omega, \delta^*)$ and $M^{n_1,n_1}(\omega, \delta^*) = M^c(\omega, \delta^*)$. Fix a δ^* with $0 < \delta^* < \delta_0$. By Remark 3.1, $M^{n_1,\infty}(\omega, \delta^*) = \cup_{u_c \in M^c(\omega,\delta^*)} \bar{M}_s(u_c, \omega, \delta^*)$, and there is a $\delta^*_{n_1,\infty} > 0$ such that for any $u^* \in \bar{M}_s(u_c, \omega, \delta^*_{n_1,\infty}) \setminus \{u_c\}$ and $u_c \in M^c(\omega, \delta^*_{n_1,\infty}) \setminus \{U\}$, the following holds.

i) If $\tau > 0$ is such that $u(t, \cdot, u_c, y) \in M^c(\omega \cdot t, \delta^*_{n_1,n_1}) \setminus \{u(t, \cdot, U, y)\}$ for $0 \le t < \tau$, then $u(t, \cdot, u^*, y) \in M^{n_1,\infty}(\omega \cdot t, \delta^*)$ for $0 \le t < \tau$.

ii) If $\tau > 0$ is such that $u(t, \cdot, u_c, y) \in M^c(\omega \cdot t, \delta^*_{n_1,n_1}) \setminus \{u(t, \cdot, U, y)\}$ for $0 \le t < \tau$ and $u(\tau, \cdot, u_c, y) \notin M^c(\omega \cdot \tau, \delta^*_{n_1,n_1}) \setminus \{u(\tau, \cdot, U, y)\}$, then

$$\frac{\|u(\tau, \cdot, u^*, y) - u(\tau, \cdot, u_c, y)\|}{\|u(\tau, \cdot, u_c, y) - u(\tau, \cdot, U, y)\|} \le \frac{\delta_c}{2}. \tag{3.8}$$

iii) If for any $t > 0$, $u(t, \cdot, u_c, y) \in M^c(\omega \cdot t, \delta^*_{n_1,n_1}) \setminus \{u(t, \cdot, U, y)\}$, then

$$\frac{\|u(t, \cdot, u^*, y) - u(t, \cdot, u_c, y)\|}{\|u(t, \cdot, u_c, y) - u(t, \cdot, U, y)\|} < \frac{\delta_c}{2} \tag{3.9}$$

when $t \gg 1$.

Now let $u^* \in M^{n_1,\infty}(\omega, \delta^*_{n_1,\infty}) \setminus \{U\}$. If $u \in M^{n_1+1,\infty}(\omega, \delta^*_{n_1,\infty}) \setminus \{U\}$, then by 2), $Z(u^*(\cdot) - U(\cdot)) \ge N_1 + 1 > N_1$. If $u \in M^{n_1,\infty}(\omega, \delta^*_{n_1,\infty}) \setminus M^{n_1+1,\infty}(\omega, \delta^*_{n_1,\infty})$, then by Remark 3.1, there is a $u_c \in M^{n_1,n_1}(\omega, \delta^*_{n_1,\infty}) \setminus \{U\}$ such that $u^* \in \bar{M}_s(u_c, \omega, \delta^*_{n_1,\infty}) \setminus \{U\}$. Therefore, by (3.5),

$$\begin{aligned}
&u(t, \cdot, u^*, y) - u(t, \cdot, U, y)\\
&= u(t, \cdot, u_c, y) - u(t, \cdot, U, y) + u(t, \cdot, u^*, y) - u(t, \cdot, u_c, y)\\
&= c(t) w_{N_1}(\cdot, \omega \cdot t) + h^{n_1,n_1}(c(t) w_{N_1}(\cdot, \omega \cdot t), \omega \cdot t) + u(t, \cdot, u^*, y) - u(t, \cdot, u_c, y)\\
&= c(t)\|w_{N_1}(\cdot, \omega \cdot t)\| \Big[\frac{w_{N_1}(\cdot, \omega \cdot t)}{\|w_{N_1}(\cdot, \omega \cdot t)\|} + \frac{h^{n_1,n_1}(c(t) w_{N_1}(\cdot, \omega \cdot t), \omega \cdot t)}{c(t)\|w_{N_1}(\cdot, \omega \cdot t)\|}\\
&\qquad + \frac{u(t, \cdot, u^*, y) - u(t, \cdot, u_c, y)}{c(t)\|w_{N_1}(\cdot, \omega \cdot t)\|} \Big]
\end{aligned}$$

for some non-zero scalar function $c(t)$. By (3.7) and (3.8) or (3.9),

$$\frac{\|h^{n_1,n_1}(c(t)w_{N_1}(\cdot,\omega\cdot t),\omega\cdot t)\|}{|c(t)|\cdot\|w_{N_1}(\cdot,\omega\cdot t)\|}+\frac{\|u(t,\cdot,u^*,y)-u(t,\cdot,u_c,y)\|}{|c(t)|\cdot\|w_{N_1}(\cdot,\omega\cdot t)\|}\le\delta_c$$

for some $t>0$. Hence, by (3.6), one has that

$$Z(u(t,\cdot,u^*,y)-u(t,\cdot,U,y))=Z(w_{N_1}(\cdot,\omega\cdot t))=N_1$$

for some $t>0$. This implies that $Z(u^*(\cdot)-U(\cdot))\ge N_1$. 3) is proved. ■

<u>**Corollary 3.3.**</u> *Suppose that* $0\in\Sigma(\Omega)$ *and* n_0 *is such that* $0\in I_{n_0}$, $dimV^c(\omega)(=dimV^{n_0,n_0}(\omega))=1$. *Let* $N_u=dimV^u(\omega)(=dimV^{1,n_0-1}(\omega))$. *Then for* δ^* *sufficiently small, one has*

$$Z(u(\cdot)-U(\cdot))\ge N_u+1\quad for\quad u\in M^s(\omega,\delta^*)\setminus\{U\},$$

$$Z(u(\cdot)-U(\cdot))\ge N_u\quad for\quad u\in M^{cs}(\omega,\delta^*)\setminus\{U\},$$

$$Z(u(\cdot)-U(\cdot))=N_u\quad for\quad u\in M^{c}(\omega,\delta^*)\setminus\{U\},$$

$$Z(u(\cdot)-U(\cdot))\le N_u\quad for\quad u\in M^{cu}(\omega,\delta^*)\setminus\{U\},$$

and

$$Z(u(\cdot)-U(\cdot))\le N_u-1\quad for\quad u\in M^{u}(\omega,\delta^*)\setminus\{U\},$$

where $\omega=(U,y)\in\Omega$.

4. Ergodicity of Minimal Sets

<u>**Lemma 4.1.**</u> *Let* E *be a minimal set of (1.3). Then one has*

1) E *is an almost* 1*-cover of* Y*, that is,* $Y_0=\{y\in Y|card(E\cap P^{-1}(y))=1\}$ *is a residual subset of* Y*;*

2) If E *is hyperbolic, that is, the linearized equation*

$$\begin{cases} v_t=v_{xx}+a(x,\omega\cdot t)v_x+b(x,\omega\cdot t)v, & t>0,\quad 0<x<1\\ v(t,0)=0,\quad v(t,1)=0, & t>0\end{cases}\tag{4.1}$$

has an exponential dichotomy on E*, where* $\omega=(U,y)\in E$, $a(x,\omega)=f_p(y,x,U(x),U_x(x))$, $b(x,\omega)=f_u(y,x,U(x),U_x(x))$*, then* E *is a* 1*-cover of* Y*.*

Proof. See [32], [33]. ■

Let E be a minimal invariant set of (1.3), and $\Sigma(E)$ be the S-S spectrum of (4.1).

Lemma 4.2. *Suppose that $0 \in \Sigma(E)$. Then for any $(U_1, y), (U_2, y) \in E$ with $\|U_1 - U_2\| \ll 1$, one has $V^s(U_1, y) \bigoplus V^{cu}(U_2, y) = X$, and $V^{cs}(U_1, y) \bigoplus V^u(U_2, y) = X$. Consequently, $(U_1 + V^s(U_1, y)) \cap (U_2 + V^{cu}(U_2, y)) \neq \emptyset$, and $(U_1 + V^{cs}(U_1, y)) \cap (U_2 + V^u(U_2, y)) \neq \emptyset$, where $V^s(U_i, y)$, $V^{cs}(U_i, y)$, $V^{cu}(U_i, y)$, and $V^u(U_i, y)$ are the stable, center stable, center unstable, and unstable subspaces of (4.1) at $(U_i, y) \in E$ $(i = 1, 2)$ respectively.*

Proof. It follows from the same arguments as in Lemma 4.1 of [32]. ■

Lemma 4.3. *Let E be a minimal set of (1.3) and suppose that $0 \in \Sigma(E)$. Then for any $(U_1, y), (U_2, y) \in E$ with $\|U_1 - U_2\| \ll 1$, one has that $M^s(U_1, y, \delta^*) \cap M^{cu}(U_2, y, \delta^*) \neq \emptyset$, and $M^{cs}(U_1, y, \delta^*) \cap M^u(U_2, y, \delta^*) \neq \emptyset$, where $M^s(U_i, y, \delta^*)$, $M^{cs}(U_i, y, \delta^*)$, $M^{cu}(U_i, y, \delta^*)$, and $M^u(U_i, y, \delta^*)$ are the local stable, center stable, center unstable, and unstable manifolds of (1.3) at $\omega = (U_i, y) (i = 1, 2)$ respectively.*

Proof. It follows from the above Lemma 4.2 and the arguments in Lemma 4.2 of [32]. ■

Lemma 4.4. *Let E be a minimal set of (1.3) and assume that E is not a precise 1-cover of Y. If $dim V^c(\omega) = 1$, then $Z(U_1(\cdot) - U_2(\cdot)) = N_u$ for any $(U_1, y), (U_2, y) \in E$ with $U_1 \neq U_2$, where N_u are as in Corollary 3.3.*

Proof. Since E is not a precise 1-cover of Y, $0 \in \Sigma(E)$. Recall $N_u = dim V^u(\omega)$, where $V^u(\omega)$ is the unstable subspace of (4.1) at $\omega = (U, y) \in E$. Now for given $\omega_1 = (U_1, y), \omega_2 = (U_2, y) \in E$ with $U_1 \neq U_2$, by Lemma 4.1 1), one easily sees that $U_1 \cdot t_n - U_2 \cdot t_n \to 0$ and $U_1 \cdot s_n - U_2 \cdot s_n \to 0$ as $n \to \infty$ for some sequences $t_n \to \infty$ and $s_n \to -\infty$. Without loss of generality, we assume that $U_1(\cdot) - U_2(\cdot)$ has only simple zeros in $[0, 1]$. Then there is $\epsilon > 0$ such that for any $v \in X$ with $\|v\| < \epsilon$,

$$Z(U_1(\cdot) - U_2(\cdot) + v(\cdot)) = Z(U_1(\cdot) - U_2(\cdot)). \tag{4.2}$$

Now, by Lemma 4.3, for $n \gg 1$, $W^{cs}(\omega_1 \cdot t_n, \delta^*) \cap W^u(\omega_2 \cdot t_n, \delta^*) \neq \emptyset$ and $W^s(\omega_1 \cdot s_n, \delta^*) \cap W^{cu}(\omega_2 \cdot t_n, \delta^*) \neq \emptyset$. Let $w_n^+ \in W^{cs}(\omega_1 \cdot t_n, \delta^*) \cap W^u(\omega_2 \cdot t_n, \delta^*)$ and $w_n^- \in W^s(\omega_1 \cdot s_n, \delta^*) \cap W^{cu}(\omega_2 \cdot t_n, \delta^*)$. Then $u(t, \cdot, w_n^+, y_n \cdot t_n)$ exists for $t < 0$ and $u(t, \cdot, w_n^-, y_n \cdot s_n)$ exists for $t > 0$. Moreover $\|u(-t_n, \cdot, w_n^+, y_n \cdot t_n) - u(-t_n, \cdot, \omega_2 \cdot t_n)\| = \|u(-t_n, \cdot, w_n^+, y_n \cdot t_n) -$

$U_2(\cdot)\| \to 0$ as $n \to \infty$ and $\|u(-s_n, \cdot, w_n^-, y_n \cdot s_n) - u(-s_n, \cdot, w_1 \cdot s_n)\| = \|u(-s_n, \cdot, w_n^-, y_n \cdot s_n) - U_1(\cdot)\| \to 0$ as $n \to \infty$. Therefore, by (4.2) and Corollary 3.3, when $n \gg 1$,

$$\begin{aligned} Z(U_1(\cdot) - U_2(\cdot)) &= Z(U_1(\cdot) - u(-t_n, \cdot, w_n^+, y_n \cdot t_n)) \\ &= Z(u(-t_n, \cdot, \omega_1 \cdot t_n) - u(-t_n, \cdot, w_n^+, y_n \cdot t_n)) \\ &\geq Z(u(t_n, \cdot, U_1, y) - w_n^+(\cdot)) \\ &\geq N_u, \end{aligned}$$

and

$$\begin{aligned} Z(U_1(\cdot) - U_2(\cdot)) &= Z(u(-s_n, \cdot, w_n^-, y_n \cdot s_n) - U_2(\cdot)) \\ &= Z(u(-s_n, \cdot, w_n^-, y_n \cdot s_n) - u(-s_n, \cdot, \omega_2 \cdot s_n)) \\ &\leq Z(w_n^-(\cdot) - u(s_n, \cdot, U_2, y_n)) \\ &\leq N_u. \end{aligned}$$

This proves the lemma. ■

Lemma 4.5. *Let E be a minimal set of (1.3). If E is uniquely ergodic, then the S-S spectrum $\Sigma(E)$ of (4.1) consists of pure points, that is, $\Sigma(E) = \{\lambda_1, \lambda_2, \cdots\}$, and $dim V^{n,n}(U, y) = 1$ for any n $((U, y) \in E)$.*

Proof. First of all, for any given $\omega = (U, y) \in E$, let $w(t, x) = exp(\frac{1}{2}\int_0^x a(s, \omega \cdot t)ds)v(t, x)$. Then (4.1) becomes

$$\begin{cases} w_t = w_{xx} + b^*(x, \omega \cdot t)w, & t > 0, \quad 0 < x < 1, \\ w(t, 0) = 0, \quad w(t, 1) = 0, & t > 0 \end{cases} \tag{4.3}$$

for some continuous function $b^* : [0, 1] \times E \to \mathbb{R}^1$.

Now, by Floquet theory (Theorem 2.2), (4.3) can be decomposed into infinitely many scalar ODEs:

$$c_n' = \mu_n(\omega \cdot t)c_n, \tag*{$(4.4)_n$}$$

where $\mu_n(\omega \cdot t) = \int_0^1 [b^*(x, \omega \cdot t)w_n(x, \omega \cdot t)^2 - w_{nx}(x, \omega \cdot t)^2]ds$, $(n = 1, 2, 3, \cdots)$. Suppose that $\Sigma_n(E)$ is the S-S spectrum of $(4.4)_n$. Then $\sigma(E) = \cup_{n=1}^{\infty}\Sigma_n(E)$ ([8]).

Next, by the unique ergodicity of E and the relation between Lyapunov exponents of $(4.4)_n$ and its S-S spectrum ([20]), we have $\Sigma_n(E) = \{\lambda_n\}$, where $\lambda_n = \lim_{t\to\infty} \frac{1}{t}\int_0^t \mu_n(\omega \cdot t)dt$ for $\nu-$ a.e. $\omega \in E$, ν is the unique ergodic measure of E. Hence, $\Sigma(E) = \{\lambda_1, \lambda_2, \lambda_3, \cdots\}$.

Finally, by the relative dichotomy relation (2.6), we have $\lambda_1 > \lambda_2 > \lambda_3 > \cdots$, and $\lambda_n \to -\infty$ as $n \to \infty$. Therefore, for any given n, $dimV^{n,n}(\omega) = 1$. This proves the lemma. ■

Lemma 4.6. *Let E be a minimal set of (1.3). Define*

$$\tilde{E} = \{(U_x(0), y) | (U, y) \in E\} \tag{4.5}$$

If for any $(U_1, y), (U_2, y) \in E$ with $U_1 \neq U_2$, $Z(u(t, \cdot, U_2, y) - u(t, \cdot, U_2, y))$ is a constant for all $t \in \mathbb{R}^1$, then the following holds:

1) $h : E \to \tilde{E}$, $(U, y) \mapsto (U_x(0), y)$ is a homomorphism, and

$$\tilde{\Pi}_t h(U, y) = (u_x(t, 0, U, y), y \cdot t) = h(u(t, \cdot, U, y), y \cdot t) \tag{4.6}$$

defines a flow on $\tilde{E}$;

2) If $(U_1, y), (U_2, y)$ are such that $h_y U_1 > h_y U_2$, then $h_{y \cdot t} u(t, 0, U_1, y) > h_{y \cdot t} u(t, 0, U_2, y)$ for all $t \in \mathbb{R}^1$, where $h_y U = U_x(0)$.

proof. 1) h is clearly onto and continuous. Now let $(U_1, y), (U_2, y) \in E$ be such that $U_1 \neq U_2$. By our condition and Lemma 2.1, $U_1(\cdot) - U_2(\cdot)$ has only simple zeros in $[0, 1]$. In particular, $h_y U_1 \neq h_y U_2$. It can be easily verified that $\tilde{\Pi}_t$ defines a flow on $\tilde{E}$.

2) We only note that $\tilde{\Pi}_t$ on $\tilde{E}$ is a scalar skew-product subflow. ■

The following is our main result 2) stated in section 1.

Corollary 4.7. *Let E be a uniquely ergodic minimal set of (1.3). The flow on E is topologically conjugated to the scalar skew-product flow (4.6). Consequently, if flow on E is not topologically conjugate to any scalar skew-product flow, then E is non-almost periodic almost automorphic.*

Proof. The corollary obviously holds if E is a 1-cover of Y. The purely almost 1-cover case of E is an immediate consequence of Lemma 4.4, Lemma 4.5, and Lemma 4.6.

We note that in the case of Neumann boundary conditions, $\tilde{E}$ in (4.5), (4.6) should be defined as

$$\tilde{E} = \{(U(0), y) | (U, y) \in E\}. ■$$

We now prove our main result 1) stated in section 1. Many arguments of our proof follows from [15].

<u>Theorem 4.8.</u> *Let E be a minimal set of (1.3) and denote $Y_0 = \{y \in Y | cardE \cap P^{-1}(y) = 1\}$. Then E is uniquely ergodic if and only if $(Y, \mathbb{R})$ is uniquely ergodic and $\nu(Y_0) = 1$, where ν is the ergodic measure on Y.*

Proof. We note that if E is precisely a 1-cover of Y, then the theorem holds automatically. Thus, without loss of generality, we shall assume that E is not a 1-cover of Y. It then follows from Lemma 4.1 2) that E is not hyperbolic.

First, we prove the 'only if' part. It is clear that Y must be uniquely ergodic since the projection $P : E \to Y$ defines a flow homomorphism. Let ν be the ergodic measure on Y and let $Y_0 = \{y \in Y | cardE \cap P^{-1}(y) = 1\}$. By Lemma 4.1 1), $Y_0 \subset Y$ is residual and invariant. Now if $\nu(Y_0) \neq 1$, then $\nu(Y_0) = 0$, that is, $Y_0^c = Y \setminus Y_0$ is invariant and $\nu(Y_0^c) = 1$. Define for each $y \in Y$,

$$A(y) = \{U_x(0) | (U, y) \in E \cap P^{-1}(y)\}, \tag{4.7}$$

and

$$a_1(y) = \max A(y), \quad a_2(y) = \min A(y). \tag{4.8}$$

It is easy to see that $a_1 : Y \to \mathbb{R}^1$ is upper semi-continuous and $a_2 : Y \to \mathbb{R}^1$ is lower semi-continuous. It follows from Corollary 4.7 that $a_1(y) > a_2(y)$ for $y \in Y_0^c$ and $\tilde{\Pi}_t(a_i(y), y) = (a_i(y \cdot t), y \cdot t)$ $(i = 1, 2)$, $t \in \mathbb{R}^1$, where $\tilde{\Pi}_t$ is defined in (4.6). Now consider linear functional $l_i : C(E, \mathbb{R}^1) \to \mathbb{R}^1$,

$$l_i(F) = \int_Y F(h^{-1}(a_i(y), y))\nu(dy), \tag{4.9}$$

$i = 1, 2$, where h is as in Lemma 4.6. It is clear that l_i $(i = 1, 2)$ define invariant measures on E (see section 2). Assume that $l_1 = l_2$. Take $F \in C(E, \mathbb{R}^1)$: $F(U, y) = U_x(0)$ for $(U, y) \in E$. It is clear by (4.9) that $a_1(y) \equiv a_2(y)$ for $\nu-$ a. e. $y \in Y$, a contradiction. Thus, l_1, l_2 defines distinct invariant measures on E. This contradicts with the unique ergodicity of E. Therefore, $\nu(Y_0) = 1$.

To prove the 'if' part of the theorem, we consider the set $E_0 = \{(U, y) \in E | (U, y) = E \cap P^{-1}(y)\}$. Then the natural projection P defines a flow isomorphism between E_0 and Y_0. Now let ν be the unique ergodic measure on Y and let μ be an invariant measure on E. One has that $P(\mu) = \nu$, that is, $\mu(D) = \nu(P(D))$ for all Borel sets $D \subset E_0 \subset E$. Note that $\mu(E_0) = \nu(Y_0) = 1$. Therefore, E is uniquely ergodic. ■

5. Examples

We discuss examples of almost automorphic minimal sets of skew-product flows which are generated from certain scalar almost periodic ODEs (note that solutions of a scalar ODE are spatial homogeneous solutions of a scalar parabolic equation with the Neumann boundary conditions).

5.1. Uniquely Ergodic Minimal Set

A linear almost periodic scalar ODE

$$x' = A(t)x + B(t) \tag{5.1}$$

is constructed by Johnson in [16] satisfying following properties:

a) $A(t)$, $B(t)$ are uniform limits of 2^n-periodic functions $A_n(t)$, $B_n(t)$ respectively;

b) $\int_0^t A(s)ds \to \infty$ as $t \to \infty$;

c) If $x_0(t)$ is the solution of (5.1) satisfying $x_0(0) = 0$, then $|x_0(t)| \leq 1$, and

$$x_0(2^n) = \begin{cases} \frac{1}{5}, & n \geq 4, \quad n \quad odd \\ 0, & n \quad even. \end{cases} \tag{5.2}$$

The solution $x_0(t)$ can not be almost periodic. If not, the frequence module of $x_0(\cdot)$ must be contained in that of $A(\cdot)$ and $B(\cdot)$. Note that $\lim_{n\to\infty} A(2^n)$ and $\lim_{n\to\infty} B(2^n)$ exist. It follows that $\lim_{n\to\infty} x_0(2^n)$ also exists (see [11]), which contradicts to (5.2).

Let $Y = Hull(A, B)$. One can define functions $a, b : Y \to \mathbb{R}^1$ such that if $y_0 = (A, B)$, then $a(y_0 \cdot t) \equiv A(t)$, $b(y_0 \cdot t) \equiv B(t)$ (see [15], [42]). Thus, equations

$$x' = a(y \cdot t)x + b(y \cdot t) \tag*{$(5.3)_y$}$$

generate a skew product flow $\Pi_t : \mathbb{R}^1 \times Y \to \mathbb{R}^1 \times Y$,

$$\Pi_t(x_*, y_*) = (x(t, x_*, y_*), y_* \cdot t), \tag{5.4}$$

where $x(t, x_*, y_*)$ is the solution of $(5.3)_{y_*}$ with initial value x_*.

Since $\lim_{t\to\infty} \int_0^t A(s)ds = \infty$, all nontrivial solution of

$$x' = A(t)x \tag{5.5}$$

are unbounded. It follows that $x_0(t)$ is the only bounded solution of (5.1), that is $E = \omega(0, y_0)$ is the only minimal set of Π_t, and moreover, E is a non-almost periodic almost automorphic minimal set. It is shown in Johnson [15] that this minimal set E is uniquely ergodic.

5.2 Non-uniquely Ergodic Minimal Set

An idea of constructing such an example is given by Johnson ([17], [18]) as follows. Consider linear ODE system

$$z^{'} = A(y \cdot t)z, \tag{5.6}$$

where $z \in \mathbb{R}^2$, $y \in Y$, Y is an almost periodic minimal set. Let $\theta = Argz$. Then θ satisfies a scalar equation

$$\theta^{'} = f(\theta, y \cdot t), \tag{5.7}$$

where f is periodic in θ with period π. Thus, (5.7) induces a skew-product flow $\tilde{\Pi}_t$ on $P^1 \times Y$, where P^1 is the real project 1-space. By S-S spectrum theory ([26], [27]), the spectrum of (5.6) is either a single point, two single points, or a nondegenerate closed interval. Now, suppose that the S-S spectrum Σ of (5.7) is a nondegenerate closed interval. It is shown in [19], [26] that $\tilde{\Pi}_t$ contains a unique minimal set $\tilde{E}$, and in [18] that there are exactly two ergodic measures on $\tilde{E}$. We then lift the flow $\tilde{\Pi}_t$ to Π_t on $\mathbb{R}^1 \times Y$, that is, Π_t is generated by

$$x^{'} = f(x, y \cdot t), \quad x \in \mathbb{R}^1. \tag{5.8}$$

It follows that Π_t has a minimal set with exactly two ergodic measures (it therefore can not be almost periodic).

A typical such system is the equation constructed by Vinograd ([41]) as follows.

Consider

$$x^{'} = \begin{pmatrix} 0 & 1 + a(y \cdot t) \\ 1 - a(y \cdot t) & 0 \end{pmatrix} x, \tag{5.9}$$

where $y \in T^2$, $y \cdot t = (y_1 + t, y_2 + \alpha t)$, α is irrational.

Let $\theta = argx$. Then θ satisfies

$$\theta^{'} = -a(y \cdot t) + \cos 2\theta. \tag{5.10}$$

The equation (5.9) has the following properties.

1) $a(y)$ is the limit of a nondecreasing sequence $\{a_n(y)\}$ and $a_n(y) \geq 0$.

2) For $y_0 = (0,0) \in T^2$, the equation

$$\theta' = -a_n(y_0 \cdot t) + \cos 2\theta \tag{5.11$_n$}$$

has for each n two solutions $\{\theta_1^n(t)\}$, $\{\theta_2^n(t)\}$ such that

$$-\frac{\pi}{4} < \theta_1^n(t) < \theta_1^{n+1}(t) < \theta_2^{n+1}(t) < \theta_2^n(t) < \frac{\pi}{4} \quad (n \geq 1), \tag{5.12}$$

$$0 < \inf_t(\theta_1^n(t), \theta_2^n(t)) \equiv \gamma_n \to 0. \tag{5.13}$$

3) The equation

$$x' = \begin{pmatrix} 0 & 1 + a_n(y \cdot t) \\ 1 - a_n(y \cdot t) & 0 \end{pmatrix} x \tag{5.14$_n$}$$

has two Lyapunov exponents β_n, $-\beta_n$ with $\beta_n > \frac{1}{2}$.

We now review some properties of (5.9), (5.10) studied in Johnson [17].

1) The S-S spectrum of $(5.14)_n$ is $\Sigma_n = \{-\beta_n, \beta_n\}$ and the S-S spectrum Σ of (5.9) is a nondegenerate interval containing $[-\frac{1}{2}, \frac{1}{2}]$.

2) Sets $E_1^n = cl\{\theta_1^n(t), y_0 \cdot t)\}$, $E_2^n = cl\{(\theta_2^n(t), y_0 \cdot t)\}$ are disjoint almost periodic minimal sets of the flow Π_t^n on $P^1 \times T^2$ which is generated by $(5.11)_n$, that is E_1^n, E_2^n are 1-cover of T^2.

3) Let $E_1^n = \{(g_n(y), y) | y \in T^2\}$, $E_2^n = \{(h_n(y), y) | y \in T^2\}$. Then

$$-\frac{\pi}{4} < g_n(y) \leq g_{n+1}(y) < h_{n+1}(y) \leq h_n(y) < \frac{\pi}{4}.$$

4) Let $g(y) = \lim_{n\to\infty} g_n(y)$, $h(y) = \lim_{n\to\infty} h_n(y)$. Then $Y_0 = \{y \in T^2 | g(y) = h(y)\}$ is a residual subset of T^2. Let $E = clg(y_0 \cdot t)$, $y_0 \in Y$ and $\tilde{E} = \{(\theta, y) \in P^1 \times T^2 | g(y) \leq \theta \leq h(y)\}$. Then $E \subset \tilde{E}$ is the unique almost automorphic minimal set of Π_t which has exactly two ergodic measures.

5) $\tilde{E}$ is an isolated invariant set. $\tilde{E}$ has the following complicated nature: a) $\tilde{E}$ is connected; b) $\tilde{E}$ is locally connected at all points where $g(y) = h(y)$; c) $\tilde{E}$ is not locally connected at all points.

This example simply shows that in the case of scalar almost periodic ODE (thus in scalar parabolic PDE in one space dimension with the Neumann boundary conditions) if minimal set E of the generated skew product flow is almost automorphic but not uniquely ergodic, then one may expect a complicated topological and dynamical nature on E or in vincinity of E (using theory of [28], one can obtain more information about $\tilde{E}$ and E, e.g., points $g(y)$ and $h(y)$ are 'expansion points').

6. References

[1] S. B. Angenent, The Zero Set of a Solution of a Parabolic Equation, *J. reine angew. Math.* **390** (1988).

[2] S. Bochner, A New Approach to Almost Periodicity, *Proc. National Academy Sciences* **48** (1962).

[3] P. Brunovský, P. Poláčik and B. Sandstede, Convergence in General Periodic Parabolic Equations in One space Dimension, *Nonlinear Analysis, Theory, Method and Applications* **18** (1992).

[4] X.-Y. Chen, in Preparation.

[5] X.-Y. Chen and H. Matano, Convergence, Asymptotic Periodicity, and Finite-Point Blow-Up in One-Dimensional Semilinear Heat Equations, *J. Diff. Eq.* **78** (1989).

[6] S.-N. Chow and H. Leiva, Dynamical Spectrum for Time Dependent Linear Systems in Banach Spaces, to appear in *JJIAM.*

[7] S.-N. Chow, X.-B. Lin and K. Lu, Smooth Invariant Foliations in Infinite Dimensional Spaces, *J. Diff. Eq.* **94** (1991).

[8] S.-N. Chow, K. Lu and J. Mallet-Paret, Floquet Bundles for Scalar Parabolic Equations, Preprint.

[9] S.-N. Chow and Y. Yi, Center Manifold and Stability for Skew-product Flows, *J. Dyn. and Diff. Eq.* **6** (1994).

[10] R. Ellis, Lectures on Topological Dynamics, Benjamin, New York (1969).

[11] A. M. Fink, Almost Periodic Differential Equations, Lecture Notes in Mathematics, **377**, Springer-Verlay Berlin Herdelberg New York (1974).

[12] P. Flor, Rhythmische Abbildungen Abelscher Gruppen II, *Zeit. Wahrsch.* **7** (1967).

[13] J. K. Hale, Asymptotic Behavior of Dissipative Systems, in Mathematical Surveys and Monographs, *Amer. Math. Soc.*, Providence, RI (1988).

[14] D. Henry, Geometric Theory of Semilinear Parabolic Equations, *Lecture Notes in Math.*, **840**, Springer-Verlay, Berlin (1981).

[15] R. A. Johnson, Bounded Solutions of Scalar Almost Periodic Linear Equations, *Illinois J. Math.* **25** (1981).

[16] R. A. Johnson, A Linear Almost Periodic Equations with an Almost automorphic solution, *proc. Amer. Math. Soc.* **82** (1981).

[17] R. A. Johnson, On Almost-periodic Linear Differential Systems of Millionščikov and Vinograd, *J. of Math. Anal. Appl.* **85** (1982).

[18] R. A. Johnson, On a Floquet Theory for Almost Periodic Two-dimensional Linear Systems, *J. Diff. Eq.* **37** (1980).

[19] R. A. Johnson, Linear Differential Systems with Interval Spectrum, *Proc. Amer. Math. Soc.* **80** (1980).

[20] R.A. Johnson, K. J. Palmer and G. R. Sell, Ergodic Properties of Linear Dynamical Systems (1984).

[21] R. A. Johnson, Concerning a Theorem of Sell, *J. Diff. Eq.* **30** (1987).

[22] L.T. Magalhães, The Spectra of Linearizations of Dissipative Infinite Dimensional Dynamical Systems, Preprint.

[23] H. Matano, Nonincrease of the Lap-number of a Solution for a one-dimensional Semilinear Parabolic Equation, *J. Fac. Sci. Univ. Tokyo* Sect. **IA 29** (1982).

[24] V. Nemytskii and V. Stepanov, Qualitative Theory of Differential Equations, Princeton Univ. Press (1960).

[25] A. Reich, Präkompakte Gruppen und Fastperiodizität, *Math z.* **116** (1970).

[26] R. J. Sacker and G. R. Sell, A Spectral Theory for Linear Differential Systems, *J. Diff. Equ.* **27** (1978).

[27] R. J. Sacker and G. R. Sell, Dichotomies for Linear Evolutionary Equations in Banach Spaces, Preprint (1991).

[28] J. Selgrade, Isolated Invariant Sets for Flows on Vector Bundles, *Trans. Amer. Math. Soc.* **203** (1975).

[29] G. R. Sell, Topological Dynamics and Ordinary Differential Equations, Van Norstand Reinhold Company (1971).

[30] G. R. Sell, The Structure of a Flow in the Vicinity of an Almost Periodic Motion. *J. Diff. Eq.* **27** (1978).

[31] W. Shen and Y. Yi, Dynamics of Almost Periodic Scalar Parabolic Equations, to appear in *J. Diff. Eq.*

[32] W. Shen and Y. Yi, Asymptotic Almost Periodicity of Scalar Parabolic Equations with Almost Periodic Time Dependence, to appear in *J. Diff. Eq.*

[33] W. Shen and Y. Yi, On Minimal Sets of Scalar Parabolic Equations with Skew-product Structures, to appear in *Trans. Amer. Math. Soc.*.

[34] Ya. G. Sinai (ed) Dynamical Systems II, Springer-Verlag (1989).

[35] R. Terras, Almost Automorphic Functions on Topological Groups, *Indiana Univ. Math. J.* **21** (1972).

[36] R. Terras, On Almost Periodic and Almost Automorphic Differences of Functions, *Duke Math. J.* **40** (1973).

[37] W. A. Veech, Almost Automorphic Functions on Groups, *Amer. J. Math.* **87** (1965).

[38] W. A. Veech, On a Theorem of Bochner, *Ann. of Math. (2)* **86** (1967).

[39] W. A. Veech, Almost Automorphic Functions, *Proceedings of the National Academy of Sciences, U.S.A.* **49** (1963).

[40] W. A. Veech, Point-Distal Flows, *Amer. J. Math.* **92** (1970).

[41] R. E. Vinograd, A Problem Suggested by N. P. Erugin, *Diff. Equ.* **11** (1975).

[42] Y. Yi, Stability of Integral Manigolds and Orbital Attraction of Quasi-periodic Motions, *J. Diff. Eq.* **103) (1993).**