Asymptotic Almost Periodicity of Scalar Parabolic Equations with Almost Periodic Time Dependence

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1. Introduction

This paper is devoted to the study of asymptotic almost periodicity of bounded solutions for the following time almost periodic one dimensional scalar parabolic equation:

$$\begin{cases} u_t = u_{xx} + f(t,x,u,u_x), \quad t>0, \quad 0<x<1, \\ u(t,0)=u(t,1)=0, \quad t>0, \end{cases} \tag{1.1}$$

where $f: \mathbb{R}^1 \times [0,1] \times \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1$ is a C^2 function, and $f(t,x,u,p)$ with all its partial derivatives (up to order 2) are (Bohr) almost periodic in t uniformly for other variables in compact subsets.

Denote by $H(f)$ the hull of f in compact open topology and let X^α be a fractional power space associated with the operator $u \to -u_{xx} : H_0^2(0,1) \to L^2(0,1)$ that satisfies $X^\alpha \hookrightarrow C^1[0,1]$ (that is, X^α is compact embedded in $C^1[0,1]$). Equation (1.1) generates a (local) skew product semiflow Π_t on $X^\alpha \times H(f)$ (see section 2) as follows:

$$\Pi_t(U,g) = (u(t,\cdot,U,g), g\cdot t), \tag{1.2}$$

where $g \cdot t$ is the flow on $H(f)$ defined by time translations ($H(f)$ is therefore almost periodic minimal under this flow, see [9], [20]), $u(t,\cdot,U,g)$ is the solution of

$$\begin{cases} u_t = u_{xx} + g(t,x,u,u_x), \quad t>0, \quad 0<x<1, \\ u(t,0)=u(t,1)=0, \quad t>0 \end{cases} \tag{1.3$_g$}$$

with $u(0,\cdot,U,g) = U(\cdot)$.

We shall study the asymptotic almost periodicity for a positively bounded motion $\Pi_t(U,g)$ of (1.2) by investigating its <u>ω-limit set</u> $\omega(U,g)$ (the set of all accumulation points of $\Pi_t(U,g)$ as t goes to infinite) since it has been shown in [22] that $\Pi_t(U,g)$ is asymptotically almost periodic if and only if $\omega(U,g)$ is an <u>almost periodic extension</u> of $H(f)$ (namely, a 1-cover of $H(f)$). We note that for (1.2), an ω-limit set is an almost periodic extension of $H(f)$ if and only if it is almost periodic minimal ([22]).

For a time periodic parabolic equation of form (1.1), it is known that any bounded solution is asymptotically periodic, that is, each ω-limit set of (1.2) is necessary a 1-cover of $H(f) \sim S^1$ (see [2], [5]). Nevertheless, similar results are false for the time almost periodic parabolic equation (1.1) since an ω-limit set of (1.2) may not be even minimal (see [18] and an example in section 5 of the current paper). In general, following the results of [22], even an ω-limit set of (1.2) is minimal, one expects that rather than an almost periodic extension it may be an almost automorphic extension (namely, an almost 1-cover of $H(f)$) or a proximal extension of $H(f)$ (proximal means that two trajectories of (1.2) starting on a same fibre clasp eventually following a time sequence). However, similar to [18] for the scalar ODE case, it is shown in [22] that if an ω-limit set of (1.2) is uniformly stable, then it is almost periodic minimal (see also [21], [23], [24] for different situations of almost periodic parabolic equations).

In this paper, we shall show that the hyperbolicity of $\omega(U, g)$ also implies its almost periodicity. More precisely, we have the following main results:

1) *Let $\omega(U_0, g_0) \subset X^\alpha \times H(f)$ be an ω-limit set of (1.2). Suppose that $\omega(U_0, g_0)$ is hyperbolic, that is, the linearized equation about the flow on $\omega(U_0, g_0)$ has an ED (exponential dichotomy) on $\omega(U_0, g_0)$ and the projections $P(y)$ $(y \in \omega(U_0, g_0))$ associated to the ED satisfy $ImP(y) \neq \{0\}$ for $y \in \omega(U_0, g_0)$ (see section 2). Then $\omega(U_0, g_0)$ is an almost periodic extension of $H(f)$, that is, $\Pi_t(U_0, g_0)$ is asymptotically almost periodic.*
2) *Suppose that $f(t, x, u, p) = F(\omega_1 t, \omega_2 t, \cdots, \omega_k t, x, u, p)$ is quasi-periodic in t and $F : T^k \times [0, 1] \times \mathbb{R}^2 \to \mathbb{R}^1$ is $C^{r,\gamma}$ $(r \geq 2,\ 0 < \gamma \leq 1)$ (that is, F and all its partial derivatives up to order r are locally Hölder continuous with Hölder exponent γ if $0 < \gamma < 1$, and are locally Lipschitz continuous if $\gamma = 1$). Then a hyperbolic ω-limit set $\omega(U_0, g_0)$ is $C^{r,\gamma}$ diffeomorphic to T^k and the flow on $\omega(U_0, g_0)$ is $C^{r,\gamma}$ conjugate to the twist flow on T^k.*

We remark that the above results are false for higher dimensional parabolic equations. There are examples even in autonomous two dimensional parabolic equations in which a hyperbolic invariant set may be rather chaotic ([17]). We also remark that if the flow on an ω-limit set $\omega(U_0, g_0)$ of (1.2) has an ED but the projections $P(y)$ associated to the ED satisfy $ImP(y) \equiv \{0\}$ for $y \in \omega(U_0, g_0)$, then $\omega(U_0, g_0)$ is uniformly stable. The above result 1) in this case then follows from [22] and the above result 2) follows from arguments in Theorem 4.9 of the current paper. Furthermore, we note that for a time almost periodic scalar ODE, similar results as above trivially hold since the hyperbolicity in the scalar ODE case is equivalent to either strong stability or strong instability. Following the arguments of the current paper and the Floquet theory ([7]), our main results also hold true in the case of Neumann boundary conditions.

The paper is organized as follows. In section 2, we summarize some of the preliminary materials such as the zero number property from [1], [16], invariant manifold theory due to [6], [11], [25] and Floquet theory developed in [7]. We also sketch the construction for the skew product semiflow (1.2). We discuss the zero crossing numbers on invariant manifolds in section 3 similar to [3], [4]. The main results are proved in section 4. An example with a nonhyperbolic ω-limit set is described in section 5.

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2. Preliminary

In this paper, the norm symbol $\|\cdot\|$ will have its obvious meaning unless specified otherwise.

1. Exponential Dichotomy (ED in short).

Consider

$$v' = A(y\cdot t)v, \quad t>0, \quad y\in Y, \quad v\in X, \tag{2.1}$$

where Y is a compact metric space, $y\cdot t$ is a flow in Y, X is a Banach space, $A(y):\mathcal{D}(A(y))\to X_0$ is a linear operator, here X_0 is a Banach space and $\mathcal{D}(A(y))\hookrightarrow X\hookrightarrow X_0$. Suppose that the evolution operator $\Phi(t,y): X\to X$ of (2.1) ($t\geq 0$, $y\in Y$) exists in the usually sense, that is, $\Phi(0,y)=I$ (the identity), and $\Phi(t,y)v\in\mathcal{D}(A(y\cdot t))$, $\Phi(t,y)v$ is differentiable in t with respect X_0 norm and satisfies (2.1) for $t>0$, moreover

$$\Phi(t+s,y)=\Phi(t,y\cdot s)\Phi(s,y), \quad t,s\in\mathbb{R}^+, y\in Y. \tag{2.2}$$

Definition 2.1. *Equation (2.1) is said to have an exponential dichotomy on Y if there exist $\beta>0$, $K>0$ and continuous projections $P(y): X\to X$ such that for any $y\in Y$, the following hold :*

1) $\Phi(t,y)P(y)=P(y\cdot t)\Phi(t,y), \quad t\in\mathbb{R}^+$;
2) $\Phi(t,y)|_{R(P(y))} : R(P(y))\to R(P(y\cdot t))$ *is an isomorphism for* $t\in\mathbb{R}^+$ *(hence* $\Phi(-t,y):=\Phi^{-1}(t,y\cdot -t): R(P(y))\to R(P(y\cdot -t))$ *is well defined for* $t\in\mathbb{R}^+$*);*

3)

$$\|\Phi(t,y)(I-P(y))\| \le Ke^{-\beta t}, \quad t \in \mathbb{R}^+, \qquad \|\Phi(t,y)P(y)\| \le Ke^{\beta t}, \quad t \in \mathbb{R}^-. \tag{2.3}$$

Remark 2.1. 1) (2.3) is equivalent to

$$\|\Phi(t-s, y\cdot s)(I-P(y\cdot s))\| \le Ke^{-\beta(t-s)}, \quad t \ge s, \quad t,s \in \mathbb{R}^1, \qquad \|\Phi(t-s, y\cdot s)P(y\cdot s)\| \le Ke^{\beta(t-s)}, \quad t \le s, \quad t,s \in \mathbb{R}^1 \tag{2.4}$$

for any $y \in Y$.

2)

$$\begin{aligned} R(P(y)) = \{v \in X | \Phi(t,y)v \quad &\textit{exists} \quad \textit{for} \quad t \in \mathbb{R}^1, \\ &\Phi(t,y)v \to 0 \quad \textit{exponentially} \quad \textit{as} \quad t \to -\infty\} \\ = \{v \in X | \Phi(t,y)v \quad &\textit{exists} \quad \textit{for} \quad t \in \mathbb{R}^1, \quad \Phi(t,y)v \to 0 \quad \textit{as} \quad t \to -\infty\}, \end{aligned}$$

and

$$\begin{aligned} R(I-P(y)) &= \{v \in X | \Phi(t,y)v \to 0 \quad \textit{exponentially} \quad \textit{as} \quad t \to \infty\} \\ &= \{v \in X | \Phi(t,y)v \to 0 \quad \textit{as} \quad t \to \infty\}. \end{aligned}$$

Definition 2.2. *$V^s(y) := R(I-P(y))$ and $V^u(y) := R(P(y))$ are referred to as the stable and unstable subspaces of (2.1) at $y \in Y$.*

2. Stable and Unstable Manifolds.

Consider

$$v' = A(y\cdot t)v + F(v, y\cdot t), \quad t > 0, \quad y \in Y, \quad v \in X, \tag{2.5}$$

where $F(\cdot,y) \in C^1(X, X_0)$, $F(v,\cdot) \in C^0(Y, X_0)$ $(v \in X)$, $F(v,y) = o(\|v\|)$, and A, X, X_0, Y, $y\cdot t$ are as in (2.1). We assume that the solution operator $\Lambda_t(\cdot,y)$ of (2.5) exists in usual sense (that is, $\Lambda_0(v,y) = v$, $\Lambda_t(v,y) \in \mathcal{D}(A(y\cdot t))$, $\Lambda_t(v,y)$ is differentiable in t with respect to X_0 norm and satisfies (2.5) for $t > 0$).

Theorem 2.1. *Consider (2.5) and assume that*

1) *Equation (2.1) has an ED on Y with ED constants K, β (that is, (2.4) holds);*
2) *The evolution operator $\Phi(t,y) : X \to X$ of (2.1) can be extended to X_0, and $\Phi(t,y)X_0 \subset X$ for $t > 0$;*
3) *There are constants $\rho, \widetilde{K}$ with $0 \le \rho < 1, \widetilde{K} > 0$ such that*

$$\begin{aligned} \|\Phi(t-s, y\cdot s)(I-P(y\cdot s))v\|_X &\le \widetilde{K}e^{-\beta(t-s)}(t-s)^{-\rho}\|v\|_{X_0}, \quad t>s, \quad t,s\in \mathbb{R}^1, \\ \|\Phi(t-s, y\cdot s)P(y\cdot s)v\|_X &\le \widetilde{K}e^{\beta(t-s)}\|v\|_{X_0}, \quad t\le s, \quad t,s\in \mathbb{R}^1 \end{aligned} \tag{2.6}$$

for any $v \in X_0$ *and* $y \in Y$*, where* Φ*,* P *are as in (2.4).*

Then, (2.5) possesses for each $y \in Y$ *a local stable manifold* $W^s(y)$ *and a local unstable manifold* $W^u(y)$ *which satisfy the following properties:*

1) *There are* $\delta^* > 0$*,* $M > 0$*, and bounded continuous functions* $h^{s,u} : \cup_{y\in Y} V^{s,u}(y) \times \{y\} \to \cup_{y\in Y} V^{u,s}(y)$ *with* $h^{s,u}(\cdot, y) : V^{s,u}(y) \to V^{u,s}(y)$ *being* C^1 *for each fixed* $y \in Y$*, and* $h^{s,u}(v, y) = o(\|v\|)$*,* $\|\frac{\partial h^{s,u}}{\partial v}(v, y)\| \le M$ *for all* $y \in Y$*,* $v \in V^{s,u}(y)$ *such that*

$$W^{s,u}(y) = \{v_0^{s,u} + h^{s,u}(v_0^{s,u}, y) | v_0^{s,u} \in V^{s,u}(y) \cap \{v \in X | \|v\| < \delta^*\}\}.$$

Moreover, $W^{s,u}(y)$ *are diffeomorphic to* $V^{s,u}(y) \cap \{v \in X | \|v\| < \delta^*\}$*, and* $W^{s,u}(y)$ *are tangent to* $V^{s,u}(y)$ *at* $0 \in X$ *for each* $y \in Y$*.*

2) $W^s(y)$ *and* $W^u(y)$ *are locally invariant in the sense that for* $v_{s,u} \in W^{s,u}(y)$ *there are intervals* $I^s = [0, \tau)$*,* $I^u = (-\tau, 0]$ *for some* $\tau > 0$ *such that* $\Lambda_t(v_{s,u}, y) \in W^{s,u}(y\cdot t)$ *for* $t \in I^{s,u}$*. They are also overflowing invariant in the sense that*

$$\Lambda_t(W^s(y), y) \subset W^s(y\cdot t) \quad for \quad t \gg 1,$$

$$\Lambda_t(W^u(y), y) \subset W^u(y\cdot t) \quad for \quad t \ll -1.$$

3) *There are* $\delta_1^*, \delta_2^* > 0$ *such that*

$$\begin{aligned} \{v \in X | \Lambda_t(v, y) \to 0 \quad as \quad t \to \infty, \|\Lambda_t(v, y)\| < \delta_1^* \quad for \quad all \quad t \ge 0\} \subset W^s(y) \\ \subset \{v \in X | \|v\| < \delta_2^*, \Lambda_t(v, y) \to 0 \quad as \quad t \to \infty\}, \end{aligned}$$

and

$$\begin{aligned} \{v \in X | \Lambda_t(v, y) \to 0 \quad as \quad t \to -\infty, \|\Lambda_t(v, y)\| < \delta_1^* \quad for \quad all \quad t \le 0\} \subset W^u(y) \\ \subset \{v | \|v\| < \delta_2^*, \Lambda_t(v, y) \to 0 \quad as \quad t \to -\infty\}. \end{aligned}$$

4) *There is a constant* $C > 0$ *such that for any* $y \in Y$ *and* $v_s \in W^s(y)$*,* $v_u \in W^u(y)$*, one has*

$$\|\Lambda_t(v_s, y)\| \le Ce^{-\frac{\beta}{2}t}\|v_s\| \quad for \quad t \ge 0,$$

$$\|\Lambda_t(v_u, y)\| \le Ce^{\frac{\beta}{2}t}\|v_u\| \quad for \quad t \le 0.$$

The proof of the theorem follows from the arguments in [6], [11], [25].

Remark 2.2. 1) By 3), if $v \in X$ is such that $\Lambda_t(v,y) \to 0$ as $t \to \infty$ $(-\infty)$, then there is a $T > 0$ such that $\Lambda_t(v,y) \in M^s(y \cdot t)$ for $t \geq T$ $(\Lambda_t(v,y) \in M^u(y \cdot t)$ for $t \leq -T)$.

2) For any $y \in Y$, $v_s \in W^s(y)$, and $v_u \in W^u(y)$, let $v_0^s(v_s, y) = (I - P(y))v_s$ and $v_0^u(v_u, y) = P(y)v_u$. Then,

$$\begin{aligned} &\Lambda_t(v_s,y) = v_0^s(\Lambda_t(v_s,y), y\cdot t) + h^s(v_0^s(\Lambda_t(v_s,y),y\cdot t), y\cdot t) \quad for \quad t \gg 1, \\ &\Lambda_t(v_u,y) = v_0^u(\Lambda_t(v_u,y), y\cdot t) + h^u(v_0^u(\Lambda_t(v_u,y),y\cdot t), y\cdot t) \quad for \quad t \ll -1, \end{aligned} \tag{2.7$_1$}$$

and

$$\begin{aligned} &\|v_0^s(\Lambda_t(v_s,y), y\cdot t)\| \leq \widetilde{C} e^{-\frac{\beta}{2}t}\|v_s\| \quad for \quad t \geq 0, \\ &\|v_0^u(\Lambda_t(v_u,y), y\cdot t)\| \leq \widetilde{C} e^{\frac{\beta}{2}t}\|v_u\| \quad for \quad t \leq 0, \end{aligned} \tag{2.7$_2$}$$

where $\widetilde{C}$ is a positive constant which is independent of y, v_s, and v_u.

3) If $A(y)$ in (2.5) is of form $A_0 + B(y)$, where $-A_0$ is a sectional operator in X_0, $B(y) : X \to X_0$ is uniformly bounded and $B(y \cdot t)$ is locally Hölder continuous in $t \in \mathbb{R}^1$ for any $y \in Y$, and if X $(X \supset \mathcal{D}(A_0),\ X \neq \mathcal{D}(A_0))$ is a fractional power space of $-A_0$, then the conditions 2), 3) in Theorem 2.1 are consequences of the condition 1) (see Lemma 7.6.2 of [11]).

4) Suppose that $A(\cdot)$ is as in 3). If $F(v, y\cdot t)$ is locally Hölder continuous in $t \in \mathbb{R}^1$ for any $v \in X$, $y \in Y$, then the solution operator $\Lambda_t(\cdot, y)$ of (2.5) exists ([11]). Let $\widetilde{X}$ be a fractional power space of $-A_0$ with $X \hookrightarrow \widetilde{X} \hookrightarrow X_0$. Assume that $F(\cdot,\cdot)$ can be extended to $\widetilde{X} \times Y$ with $F(\cdot, y) \in C^1(\widetilde{X}, X_0)$, $F(v,y) = o(\|v\|_{\widetilde{X}})$, and $F(v, y\cdot t)$ is locally Hölder continuous in $t \in \mathbb{R}^1$ for any $v \in \widetilde{X}$, $y \in Y$. Then $h^s(\cdot, y)$ and $h^u(\cdot, y)$ in the theorem can be extended to $(I - P(y))\widetilde{X}$ and $P(y)\widetilde{X}$ with $h^{s,u}(v,y) = o(\|v\|_{\widetilde{X}})$.

3. Zero Number Properties.

For a given C^1 function $v : [0,1] \to \mathbb{R}^1$, the zero number of v is defined as

$$Z(v(\cdot)) = \#\{x \in (0,1) | v(x) = 0\}.$$

The following properties can be found in [1] and [16].

Lemma 2.2. *Consider the scalar linear parabolic equation:*

$$\begin{cases} v_t = a(t,x)v_{xx} + b(t,x)v_x + c(t,x)v, \quad t > 0, \quad x \in (0,1), \\ v(t,0) = v(t,1) = 0, \quad t > 0, \end{cases} \tag{2.8}$$

where a, a_t, a_x, a_{xx}, b, b_t, b_x and c are bounded continuous functions, $a \geq \delta > 0$. Let $v(t,x)$ be a classical nontrivial solution of (2.8). Then, the following hold:

1) $Z(v(t,\cdot))$ is finite for $t > 0$ and is nonincreasing as t increases;

2) $Z(v(t,\cdot))$ can drop only at t_0 such that $v(t_0,\cdot)$ has a multiple zero in $[0,1]$;

3) $Z(v(t,\cdot))$ can drop only finite times, and there exists a $t^* > 0$ such that $v(t,\cdot)$ has only simple zeros in $[0,1]$ as $t \geq t^*$ (hence $Z(v(t,\cdot)) =$ constant as $t > t^*$).

4. Floquet Theory.

Consider the following linear parabolic equation:

$$\begin{cases} w_t = w_{xx} + b(x, y\cdot t)w, \quad t > 0, \quad 0 < x < 1, \\ w(t,0) = w(t,1) = 0, \quad t > 0, \end{cases} \tag{2.9}$$

where $y\cdot t$ is a flow on a compact metric space Y, $b : [0,1]\times Y \to \mathbb{R}^1$ is continuous. Suppose that for any $w_0 \in L^2(0,1)$, the solution $w(t,x,w_0,y)$ of (2.9) with $w(0,x,w_0,y) = w_0(x)$ exists. The following results are due to [7].

Theorem 2.3. *1) There is a sequence $\{w_n\}_{n=1}^{\infty}$, $w_n : [0,1]\times Y \to \mathbb{R}^1$ $(n = 1,2,\cdots)$ such that $w_n(\cdot,y) \in C^{1,\gamma}[0,1]$ for any γ with $0 \leq \gamma < 1$, $w_n(0,y) = w_n(1,y) = 0$, and $\|w_n(\cdot,y)\|_{L^2(0,1)} = 1$ for any $y \in Y$. $\{w_n(\cdot,y)\}_{n=1}^{\infty}$ forms a (Floquet) basis of $L^2(0,1)$ and $Z(w_n(\cdot,y)) = n-1$ for all $y \in Y$. Let $W_n(y) = span\{w_n(\cdot,y)\}$, $n = 1,2,\cdots$. Then $\bigoplus_{i=n_1}^{n_2} W_i(y) = \{w_0 \in L^2(0,1) | w(t,\cdot,w_0,y)$ is exponentially bounded in $L^2(0,1)$, and $n_1 - 1 \leq Z(w(t,\cdot,w_0,y)) \leq n_2 - 1$ for all $t \in \mathbb{R}^1\} \cup \{0\}$ for any n_1, n_2 with $n_1 \leq n_2$.*

2) Suppose $w_0(x) = \sum_{n=1}^{\infty} c_n^0 w_n(x,y)$ (c_n^0's are called Fourier coefficients). Then

$$w(t,x,w_0,y) = \sum_{n=1}^{\infty} c_n(t) w_n(x, y\cdot t), \tag{2.10}$$

where

$$c_n^{'} = \mu_n(y\cdot t)c_n, \tag{2.11}$$

$c_n(0) = c_n^0$, $\mu_n(y\cdot t) = \int_0^1 [b(x,y\cdot t)w_n(x,y\cdot t)^2 - w_{nx}(x,y\cdot t)^2]dx$, $n = 1,2\cdots$. Moreover, for each $n \geq 1$, there are $T_n > 0$, $\kappa_n > 0$ which are independent of $y \in Y$ such that

$$\int_t^{t+T_n} \mu_{n+1}(y\cdot s)ds - \int_t^{t+T_n} \mu_n(y\cdot s)ds \leq -\kappa_n, \tag{2.12}$$

for all $y \in Y$ and $t \in \mathbb{R}^1$.

3) Define $\Psi(\cdot) : Y \to L(L^2(0,1), l^2)$ *by* $\Psi(y)w_0 = \{c_n^0\}_{n=1}^{\infty}$*, where* $w_0(x) = \sum_{n=1}^{\infty} c_n^0 w_n(x,y)$*. Then* $\Psi(y \cdot t)w(t,x,w_0,y) = \{c_n(t)\}$*, here* $c_n(t)$*'s are given in (2.10). Moreover,* Ψ *is continuous,* $\Psi(y)$ *is an isomorphsim for each* $y \in Y$*, and there are positive constants* K_1*,* K_2 *which are independent of* y *such that*

$$\|\Psi(y)\| \leq K_1 \quad and \quad \|\Psi^{-1}(y)\| \leq K_2.$$

5. Skew Product Flow.

Consider

$$\begin{cases} u_t = u_{xx} + f(t,x,u,u_x), \quad t > 0, \quad 0 < x < 1, \\ u(t,0) = u(t,1) = 0, \quad t > 0, \end{cases} \tag{2.13}$$

where $f : \mathbb{R}^1 \times [0,1] \times \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1$ is a C^2 function and $f(t,x,u,p)$ as well as all its partial derivatives up to order 2 are (Bohr) almost periodic in t uniformly for (x,u,p) in compact sets of $[0,1] \times \mathbb{R}^1 \times \mathbb{R}^1$.

Let $C := C(\mathbb{R}^1 \times [0,1] \times \mathbb{R}^1 \times \mathbb{R}^1, \mathbb{R}^1)$ be the space of continuous functions $P : \mathbb{R}^1 \times [0,1] \times \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1$. Give C the compact open topology. This topology is metrizable. To be more precise, for any $P, Q \in C$, define

$$d(P,Q) = \sum_{n=1}^{\infty} \frac{\min\{1, \|P-Q\|_n\}}{2^n},$$

where

$$\|P-Q\|_n = \sup_{t \in \mathbb{R}^1, x \in [0,1], |u| \leq n, |p| \leq n} |P(t,x,u,p) - Q(t,x,u,p)|.$$

Then (C,d) is a metric space and the time translation $(P,t) \to P \cdot t$, $P \cdot t(s,x,u,p) = P(t+s,x,u,p)$ defines a flow on C ([20]). Let $H(f) = cl\{f \cdot t | t \in \mathbb{R}^1\}$ be the hull of f. Then $H(f) \subset C$ is an almost periodic minimal set under the translation flow (see [20]). Define $F : H(f) \times [0,1] \times \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1$ by $F(g,x,u,p) := g(0,x,u,p)$. Then $F(g \cdot t,x,u,p) \equiv g(t,x,u,p)$. Since each $g \in H(f)$ is C^2 ([12]), $F(g \cdot t,x,u,p)$ is C^2 in t, x, u and p. Moreover, for $(g_i,x,u,p) \in H(f) \times [0,1] \times \mathbb{R}^2$ $(i = 1,2)$, it is easy to see that

$|F(g_1,x,u,p)-F(g_2,x,u,p)| \le d(g_1,g_2)$, and $d(g_1\cdot t, g_2\cdot t) \le d(g_1,g_2)$ ([25]). By the above construction, equation (2.13) gives rise to a family of equations on $H(f)$:

$$\begin{cases} u_t = u_{xx} + F(g\cdot t, x, u, u_x), \quad t>0, \quad 0<x<1, \\ u(t,0) = u(t,1) = 0, \quad t>0, \end{cases} \tag{2.14$_g$}$$

where $g \in H(f)$. Note that $(2.14)_f$ coincides with (2.13).

Let X^α be a fractional power space associated with the operator $A_0 : H_0^2(0,1) \to X_0 \equiv L^2(0,1)$: $A_0 u = -u_{xx}$, that satisfies $X^\alpha \hookrightarrow C^1[0,1]$, $X^\alpha \supset \mathcal{D}(A_0)$, $X^\alpha \ne \mathcal{D}(A_0)$. Define $\widetilde{F} : X^\alpha \times H(f) \to X_0$ by $\widetilde{F}(u,g)(x) = F(g,x,u,u_x)$. Then $\tilde{F}$ is Lipschitz in g and C^2 in u ([11]), and $(2.14)_g$ gives rise to an ODE in X^α:

$$u' = A_0 u + \widetilde{F}(u, g\cdot t), \quad u \in X^\alpha, \quad g \in H(f). \tag{2.15}$$

Let $u(t,x,U_0,g)$ $(u(t,U_0,g))$ be the solution of $(2.14)_g$ ((2.15)) with $u(0,x,U_0,g) = U_0(x)$ $(u(0,U_0,g) = U_0 \in X^\alpha)$. This solution locally exists and also continuously depends on $U_0 \in X^\alpha$ and $g \in H(f)$ ([11]). Thus, $(2.14)_g$ or (2.15) generates a (local) skew product semiflow Π_t on $X^\alpha \times H(f)$:

$$\Pi_t(U_0,g) := (U_0,g)\cdot t := (u(t,U_0,g), g\cdot t) = (u(t,\cdot,U_0,g), g\cdot t), t>0. \tag{2.16}$$

Moreover, following [11] and the standard a priori estimates for parabolic equations, if a motion $(U,g)\cdot t$ is bounded for t in its existence interval, then it is globally defined, and for any $\delta > 0$, $\{(U,g)\cdot t | t \ge \delta > 0\}$ is relatively compact both in $X^\alpha \times H(f)$ and in $H^2(0,1)\times H(f)$, hence, the ω-limit set $\omega(U,g)|_{X^\alpha \times H(f)} = \omega(U,g)|_{H^2(0,1)\times H(f)}$. Furthermore, u has continuous derivatives u_{tx}, u_{xxx} in $(0,\infty)\times[0,1]$. Note also that the skew product semiflow (2.15) has a unique continuous backward extension on $\omega(U,g)$([10]), that is, it is in fact a usual skew product flow on $\omega(U,g)$.

Now, let $Y \subset X^\alpha \times H(f)$ be a compact invariant set of (2.16). For each $y = (U,g) \in Y$, the linearized equation of $(2.14)_g$ about $y\cdot t = (U,g)\cdot t$ reads:

$$\begin{cases} v_t = v_{xx} + a(x, y\cdot t)v_x + b(x, y\cdot t)v, \quad t>0, \quad 0<x<1, \\ v(t,0) = v(t,1) = 0, \quad t>0, \end{cases} \tag{2.17}$$

where $a(x,y) = F_p(g,x,U,U_x) = g_p(0,x,U,U_x)$, $b(x,y) = F_u(g,x,U,U_x) = g_u(0,x,U,U_x)$. Denote $A(y) = \dfrac{\partial^2}{\partial x^2} + a(\cdot,y)\dfrac{\partial}{\partial x} + b(\cdot,y)$. Then (2.17) becomes an ODE:

$$v' = A(y\cdot t)v, \quad v \in X^\alpha. \tag{2.18}$$

Definition 2.3. *Equation (2.17) is said to have an ED on Y if equation (2.18) has an ED on Y. We say a compact invariant set $Y \subset X^\alpha \times H(f)$ of (2.16) is hyperbolic if (2.17) or (2.18) has an ED on Y and $ImP(y) \neq \{0\}$ for all $y \in Y$, here $P(y)$, $y \in Y$, are projections associated to the ED.*

3. Numbers of Zeros on Invariant Manifolds

Consider (2.16) and let $Y \subset X^\alpha \times H(f)$ be a connect and compact invariant set of (2.16). For any $y = (U_0, g_0) \in Y$, recall $y\cdot t = (u(t,\cdot,U_0,g_0), g_0\cdot t)$. Let $v = u - u(t,\cdot,U_0,g_0)$ in (2.14). Then v satisfies

$$v' = A(y\cdot t)v + G(v, y\cdot t), \tag{3.1}$$

where $G(v,y) = \widetilde{F}(v+U_0,g_0) - \widetilde{F}(U_0,g_0) - B(y)v = O(\|v\|^2)$, $B(y) = A(y) - \dfrac{\partial^2}{\partial x^2}$, $\widetilde{F}(v,y)$ and $A(y)$ are as in (2.15) and (2.18). We now assume Y is hyperbolic, that is,

$$v' = A(y\cdot t)v, \quad v \in X^\alpha, \tag{3.2}$$

or

$$\begin{cases} v_t = v_{xx} + a(x, y\cdot t)v_x + b(x,y\cdot t)v, \quad t>0, \quad 0<x<1, \\ v(t,0) = v(t,1) = 0, \quad t > 0 \end{cases} \tag{3.3}$$

has an ED on Y and $ImP(y) \neq \{0\}$ for all $y \in Y$, where a and b are defined as in (2.17), $P(y)$, $y \in Y$, are projections associated to the ED. Let $X_0 = L^2(0,1)$. Note that $G(\cdot,y) \in C^2(X^\alpha, X_0)$, $G(v,\cdot) \in C^{0,1}(Y, X_0)$ $(v \in X^\alpha)$. By Theorem 2.1 and Remark 2.2 3), 4), equation (3.1) possess local stable manifolds $W^s(y)$ and unstable manifolds $W^u(y)$ which satisfy properties stated in Theorem 2.1. Now for each $y = (U,g) \in Y$, define

$$\begin{aligned} M^s(y) &= \{u \in X^\alpha | u - U \in W^s(y)\}, \\ M^u(y) &= \{u \in X^\alpha | u - U \in W^u(y)\}. \end{aligned} \tag{3.4}$$

Then $M^s(y)$ and $M^u(y)$ are overflowing invariant to (2.15), that is, $u(t, M^s(y), g) \subset M^s(y\cdot t)$ for $t \gg 1$, and $u(t, M^u(y), g) \subset M^u(y\cdot t)$ for $t \ll -1$, where $u(t,\cdot,g)$ is the solution operator of (2.15). We note also that $dimM^u(y) = dimV^u(y)$ is a positive integer which is independent of $y \in Y$ ([15], [19]), here $V^u(y)$ is the unstable subspace of (3.2) or (3.3).

Now, consider a transformation $w(x,t) = exp\big(\frac{1}{2}\int_0^x a(s, y\cdot t)ds\big)v(t,x)$ to (3.3). Then w satisfies an equation of the following form:

$$\begin{cases} w_t = w_{xx} + b^*(x, y\cdot t)w, \quad t>0, \quad 0<x<1, \\ \\ w(t,0) = w(t,1) = 0, \quad t>0. \end{cases} \tag{3.5}$$

Lemma 3.1. *Equation (3.5) has an ED on* Y *with the ED projections* $\widetilde{P}(y)$: $\widetilde{P}(y)w(x) = [exp\frac{1}{2}\int_0^x a(s,y)ds]P(y)[exp - \frac{1}{2}\int_0^x a(s,y)ds]w(x)$ *for any* $y \in Y$, $w \in X^\alpha$.

Proof. This is straightforward. ■

Lemma 3.2. *Consider (3.5). Let* $\{w_n(\cdot,y)\}_{n=1}^\infty$ *be the Floquet basis of* $L^2(0,1)$ *defined in Theorem 2.3 for equation (3.5). Let* $\widetilde{V}^s(y)$ *and* $\widetilde{V}^u(y)$ *be stable and unstable subspaces associated to the ED of (3.5). If for some* n, $w_n(\cdot,y) \in \widetilde{V}^s(y)$ ($w_n(\cdot,y) \in \widetilde{V}^u(y)$ *respectively), then* $w_k(\cdot,y) \in \widetilde{V}^s(y)$ *for all* $k \geq n$ ($w_k(\cdot,y) \in \widetilde{V}^u(y)$ *for all* $k \leq n$ *respectively).*

Proof. Let $\{\mu_n\}_{n=1}^\infty$ be given in (2.11) associated to the Floquet basis $\{w_n\}_{n=1}^\infty$ of (3.5). Suppose $w_n(\cdot,y) \in \widetilde{V}^s(y)$ for some n. Then $e^{\int_0^t \mu_n(y\cdot s)ds} \to 0$ as $t \to \infty$, that is, $\int_0^t \mu_n(y\cdot s)ds \to -\infty$ as $t \to \infty$. Now for $t > 0$, it follows from (2.12) that

$$\begin{aligned} &\int_0^t \mu_{n+1}(y\cdot s)ds - \int_0^t \mu_n(y\cdot s)ds \\ &= \sum_{k=1}^{[t/T_n]} \int_{(k-1)T_n}^{kT_n} (\mu_{n+1}(y\cdot s) - \mu_n(y\cdot s))ds + \int_{[t/T_n]T_n}^t (\mu_{n+1}(y\cdot s) - \mu_n(y\cdot s))ds \\ &\leq -\kappa_n[t/T_n] + \int_{[t/T_n]T_n}^t (\mu_{n+1}(y\cdot s) - \mu_n(y\cdot s))ds \\ &= -\kappa_n[t/T_n] + O(1). \end{aligned}$$

Hence, $\int_0^t \mu_{n+1}(y\cdot s)ds \to -\infty$ as $t \to \infty$. This implies that $\|w(t,\cdot,w_{n+1}(\cdot,y),y)\|_{L^2(0,1)} = \|c_{n+1}(t)w_{n+1}(\cdot, y\cdot t)\|_{L^2(0,1)} \to 0$ as $t \to \infty$. It then follows from the standard a priori estimates for parabolic equations that $\|c_{n+1}(t)w_{n+1}(\cdot, y\cdot t)\|_{X^\alpha} \to 0$ as $t \to \infty$, that is, $w_{n+1}(\cdot,y) \in \widetilde{V}^s(y)$. By induction, $w_k(\cdot,y) \in \widetilde{V}^s(y)$ for all $k \geq n$. Similarly, if $w_n(\cdot,y) \in \widetilde{V}^u(y)$ for some n, then $w_k(\cdot,y) \in \widetilde{V}^u(y)$ for all $k \leq n$. ■

Lemma 3.3 *Consider (3.5) and let* $\{w_n(\cdot,y)\}_{n=1}^{\infty}$, $\widetilde{V}^s(y)$, $\widetilde{V}^u(y)$ *be defined as in Lemma 3.2. Denote* $W_n(y) = span\{w_n(\cdot,y)\}$ *for* $n = 1,2,\cdots$, *and denote* $N = dim\widetilde{V}^u(y)$. *Then* $\widetilde{V}^u(y) = \bigoplus_{n=1}^{N} W_n(y)$, $\widetilde{V}^s(y) = cl \cup_{m\geq N+1} \bigoplus_{n=N+1}^{m} W_n(y)$ *under* X^α*-norm.*

Proof. Take $w_0 \in \widetilde{V}^u(y)\setminus\{0\}$. Then $w_0 = \sum_{n=1}^{\infty} c_n^0 w_n(\cdot,y)$ in $L^2(0,1)$ for some $\{c_n^0\} \in l^2$. Since $w_0 \neq 0$, there is a n_0 such that $c_{n_0}^0 \neq 0$. Let $w(t,x,w_0,y)$ be the solution of (3.5) with $w(0,x,w_0,y) = w_0(x)$. By Theorem 2.3, $w(t,\cdot,w_0,y) = \sum_{n=1}^{\infty} c_n(t) w_n(\cdot, y\cdot t)$, where $c_n(t) = c_n^0 e^{\int_0^t \mu_n(y\cdot s)ds}$, μ_n's are given by (2.11) with respect to (3.5). By Theorem 2.3 3), there is a constant K which is independent of $y \in Y$ such that

$$\begin{aligned} |c_{n_0}(t)| &\leq \{\sum_{k=1}^{\infty} c_k(t)^2\}^{\frac{1}{2}} \\ &\leq K\|w(t,\cdot,w_0,y)\|_{L^2} \\ &\leq \widetilde{K}\|w(t,\cdot,w_0,y)\|_{X^\alpha}, \end{aligned} \tag{3.6}$$

where $\widetilde{K}$ is a constant. It follows that $c_{n_0}(t) \to 0$ as $t \to -\infty$ and then $\|w(t,\cdot,w_{n_0}(\cdot,y),y)\|_{X^\alpha} = \|c_{n_0}(t)w_{n_0}(\cdot,y)\|_{X^\alpha} \to 0$ as $t \to -\infty$. Thus, $w_{n_0}(\cdot,y) \in \widetilde{V}^u(y)$. By Lemma 3.2, $w_n(\cdot,y) \in \widetilde{V}^u(y)$ for all $n \leq n_0$. Now suppose $\widetilde{V}^u(y)$ contains only $n_0 < N$ many $w_n(\cdot,y)$'s for $n = 1,2,\cdots,n_0$. Then there is a nonzero $w_0 \in \widetilde{V}^u(y) \setminus \bigoplus_{n=1}^{n_0} W_n(y)$ and w_0 will have a nonzero coefficient c_{n_1} with $n_1 > n_0$. Repeat the above argument, one would have that $\widetilde{V}^u(y)$ contains $w_{n_1}(\cdot,y)$, a contradiction. Hence $\widetilde{V}^u(y) = \bigoplus_{n=1}^{N} W_n(y)$.

Now take any $w_0 \in \widetilde{V}^s(y)$. We claim that $w_0 = \sum_{n=N+1}^{\infty} c_n^0 w_n(\cdot,y)$ in $L^2(0,1)$. For otherwise, a similar inequality as (3.6) would show that $w_{n_0}(\cdot,y) \in \widetilde{V}^s(y)$ for some $n_0 \leq N$. The rest of the proof are just elaborations of the above arguments, that is, we first show that there is a $n_0 \geq N+1$ such that $w_{n_0}(\cdot,y) \in \widetilde{V}^s(y)$, hence $w_n(\cdot,y) \in \widetilde{V}^s(y)$ for $n \geq n_0$ by Lemma 3.2, next we argue that if $\widetilde{V}^s(y)$ only contains $\{w_n(\cdot,y)\}_{n=n_0}^{\infty}$ for some $n_0 > N+1$, then this leads to a contradiction. Therefore, $\{w_n(\cdot,y)\}_{n=N+1}^{\infty} \subset \widetilde{V}^s(y)$ and $\widetilde{V}^s(y) = cl \cup_{m\geq N+1} \bigoplus_{n=N+1}^{m} W_n(y)$ under X^α-norm. We omitt the details. ■

Corollary 3.4. *Consider (3.3). Let* $V^s(y)$, $V^u(y)$ *be the stable and unstable subspaces associated to the ED of (3.3) and denote* $N = dimV^u(y)$. *Then for any* $y \in Y$, $Z(U(\cdot)) \leq N-1$ *for any* $U \in V^u(y) \setminus \{0\}$, *and* $Z(U(\cdot)) \geq N$ *for any* $U \in V^s(y) \setminus \{0\}$.

Proof. This is because of Lemma 3.2, Lemma 3.3, and Theorem 2.3 1). ■

Recall the skew product semiflow Π_t on $X^\alpha \times H(f)$ generated by (2.15) is

$$\Pi_t(U,g) = (u(t,\cdot,U,g), g\cdot t), \quad t>0.$$

Theorem 3.5. *Let $M^s(y)$, $M^u(y)$ be the local stable and unstable manifolds of (2.15) at $y \in Y$ defined in (3.4). Denote $N = dimM^u(y)$. Then for any $y_0 = (u_0, g_0) \in Y$, any $u^s \in M^s(y_0) \setminus \{u_0\}$, and any $u^u \in M^u(y_0) \setminus \{u_0\}$, one has that $Z(u^s - u_0) \geq N$, and $Z(u^u - u_0) \leq N-1$.*

Proof. We only prove the result for $M^s(y)$. First, for integers $1 \leq m \leq k \leq n < \infty$ and any $y \in Y$, define $\bar{W}_k(y)(x) = \{exp(\frac{1}{2}\int_0^x a(s,y)ds)w(x) | w \in W_k(y)\}$ and $V^{m,n}(y) = \oplus_{k=m}^n \bar{W}_k(y)$, $V^{m,\infty}(y) = cl \cup_{n\geq m} V^{m,n}(y)$. For each positive integer n, denote $I_n = [a_n, b_n]$ as the Sacker-Sell spectrum of (2.11) associated to (3.5), that is, $a_n = \inf_{y\in Y} min\{\bar{\lambda}_n^-(y), \bar{\lambda}_n^+(y)\}$, $b_n = \sup_{y\in Y} max\{\tilde{\lambda}_n^-(y), \tilde{\lambda}_n^+(y)\}$, where $\bar{\lambda}_n^\pm(y) = \liminf_{t\to\pm\infty} \frac{1}{t}\int_0^t \mu_n(y\cdot s)ds$, $\tilde{\lambda}_n^\pm(y) = \limsup_{t\to\pm\infty} \frac{1}{t}\int_0^t \mu_n(y\cdot s)ds$ (see [7], [15]). Since Y is hyperbolic and $dimM^u(y) = N$, one has that $a_N > 0$ and $b_{N+1} < 0$. Let $n_0 = N+1$ and $n_0 < n_1 < n_2 < \cdots$ be such that $I_{n_i} \cap I_{n_i+1} = \emptyset$ for any $i = 1, 2, \cdots$. Then, for each n_i, the gap between I_{n_i} and I_{n_i+1} is away from zero ([7], [15]), and $V^{n_0,n_i}(y) \subset V^s(y)$. Similar to Theorem 2.1, there is $\delta^* > 0$ such that (3.1) possesses for each n_i and $y \in Y$ a local invariant manifold $W^{n_0,n_i}(y)$ of form $W^{n_0,n_i}(y) = \{v_0^{n_0,n_i} + h^{n_0,n_i}(v_0^{n_0,n_i}, y) | v_0^{n_0,n_i} \in V^{n_0,n_i}(y) \cap \{v \in X^\alpha | \|v\| < \delta^*\}\}$, where $h^{n_0,n_i}(\cdot,y) : V^{n_0,n_i}(y) \to V^{1,n_0-1}(y) \oplus V^{n_i+1,\infty}(y)$ satisfies the same properties as $h^{s,u}(\cdot,y)$ in Theorem 2.1. Moreover, $W^{n_0,n_i}(y) \subset W^s(y)$, and for any $v \in W^s(y)$, there are $v_{n_i} \in W^{n_0,n_i}(y)$ such that $v_{n_i} \to v$ as $n_i \to \infty$.

Now, for any $y_0 = (u_0, g_0) \in Y$ and $u^s \in M^s(y_0) \setminus \{u_0\}$, let $v(t,x) = u(t,x,u^s,g_0) - u(t,x,u_0,g_0)$. Then $v(t,x)$ satisfies the following linear parabolic equation:

$$\begin{cases} v_t = v_{xx} + a_0(t,x)v_x + b_0(t,x)v, & t>0, \quad 0<x<1, \\ v(t,0) = v(t,1) = 0, & t>0, \end{cases} \tag{3.7}$$

where $a_0(t,x) = \int_0^1 g_{0p}(t,x,u(t,x,u^s,g_0), \tau u_x(t,x,u^s,g_0) + (1-\tau)u_x(t,x,u_0,g_0))d\tau$, $b_0(t,x) = \int_0^1 g_{0u}(t,x,\tau u(t,x,u^s,g) + (1-\tau)u(t,x,u_0,g_0), u_x(t,x,u_0,g_0))d\tau$. By Lemma 2.2, we may assume that $v(0,\cdot)$ has only simple zeros in $[0,1]$. Since $u^s - u_0 \in W^s(y_0)$, there are $u^{n_i} \in X^\alpha$ such that $u^{n_i} - u_0 \in W^{n_0,n_i}(y_0)$ and $u^{n_i} \to u^s$ as $n_i \to \infty$. Therefore,

$$Z(u^{n_i}(\cdot) - u_0(\cdot)) = Z(u^s(\cdot) - u_0(\cdot)) \quad for \quad n_i \gg 1. \tag{3.8}$$

Fix $n_i \gg 1$. Let $v^{n_i}(t,x) = u(t,x,u^{n_i},g_0) - u(t,x,u_0,g_0)$. Similar to Remark 2.2 2),

$$v^{n_i}(t,\cdot) = v_0^{n_i}(v^{n_i}(t,\cdot), y_0 \cdot t) + h^{n_0,n_i}(v_0^{n_i}(v^{n_i}(t,\cdot), y_0 \cdot t), y_0 \cdot t) \quad for \quad t \gg 1,$$

where $v_0^{n_i}(v^{n_i}(t,\cdot), y_0 \cdot t) \in V^{n_0,n_i}(y_0 \cdot t)$. Moreover, $v_0^{n_i}(v^{n_i}(t,\cdot), y_0 \cdot t) \to 0$ as $t \to \infty$. Since $dim V^{n_0,n_i}(y_0 \cdot t) = const < \infty$ for all $t \in \mathbb{R}$, there is $t_n \to \infty$ such that $y_0 \cdot t_n \to y^* = (u^*, g^*)$, and

$$\lim_{n\to\infty} \frac{v^{n_i}(t_n,\cdot)}{\|v^{n_i}(t_n,\cdot)\|} = \lim_{n\to\infty} \frac{v_0^{n_i}(v^{n_i}(t_n,\cdot), y_0 \cdot t_n)}{\|v_0^{n_i}(v^{n_i}(t_n,\cdot), y_0 \cdot t_n)\|} \equiv w^*(\cdot) \in V^{n_0,n_i}(y^*) \tag{3.9}$$

as $n \to \infty$. Let Φ be the evolution operator of (3.2). Then $\Phi(t,y^*)w^*$ is a solution of (3.3) with $y = y^*$. Furthermore, $\frac{v^{n_i}(t+t_n,\cdot)}{\|v^{n_i}(t+t_n,\cdot)\|} \to \Phi(t,y^*)w^*$ ($t \in \mathbb{R}$) as $n \to \infty$. Suppose that $t_0 > 0$ is such that $\Phi(t_0,y^*)w^*$ has only simple zeros in $[0,1]$. Then $Z(v^{n_i}(t_0+t_n,\cdot)) = Z(\Phi(t_0,y^*)w^*)$ as $n \gg 1$. Since $\Phi(t_0,y^*)w^* \in V^{n_0,n_i}(y^* \cdot t_0) \subset V^s(y^* \cdot t_0)$, by Lemma 2.2 and Corollary 3.4, one has that

$$Z(v^{n_i}(0,\cdot)) \geq Z(v^{n_i}(t_0+t_n,\cdot)) = Z(\Phi(t_0,y^*)w^*) \geq N \quad for \quad n \gg 1. \tag{3.10}$$

By (3.8) and (3.10), $Z(u^s(\cdot) - u_0(\cdot)) \geq N$. ■

4. Hyperbolic ω-limit set

Definotion 4.1. *A set $Y \subset X^\alpha \times H(f)$ is said to be an almost periodic extension of $H(f)$ if $card(Y \cap P^{-1}(g)) = 1$ for all $g \in H(f)$, where $P: X^\alpha \times H(f) \to H(f)$, $(U,g) \mapsto g$ is the natural projection.*

We shall show in this section that any hyperbolic ω-limit set of (2.16) is an almost periodic extension of $H(f)$.

Lemma 4.1. *Let $Y \subset X^\alpha \times H(f)$ be a connect and compact hyperbolic invariant set of (2.16). Again, denote by $V^s(y)$, $V^u(y)$ ($y \in Y$) the stable and unstable subspaces associated to ED of (3.2). Take (u_1,g), $(u_2,g) \in Y$. If $\|u_1 - u_2\|$ is sufficiently small, then $V^s(u_1,g) \bigoplus V^u(u_2,g) = X^\alpha$, and therefore, $(u_1 + V^s(u_1,g)) \cap (u_2 + V^u(u_2,g)) \neq \emptyset$.*

Proof. Let $P(y)$ be the projections associated with ED of (3.2). Since $P: Y \to L(X^\alpha, X^\alpha)$ is continuous, $V^s(y) = (I - P(y))X^\alpha$, $V^u(y) = P(y)X^\alpha$ vary continuously in y. For (u_1,g), $(u_2,g) \in Y$ such that $\|u_1 - u_2\|$ is sufficiently small, it is easy to see that $V^s(u_1,g) \cap V^u(u_2,g) = \{0\}$. Since $dim V^u(u_1,g) = dim V^u(u_2,g) < \infty$, $V^s(u_1,g) \bigoplus V^u(u_1,g) = X^\alpha$, one has that $V^s(u_1,g)\bigoplus V^u(u_2,g) = X^\alpha$. Thus, there is a unique $u_1^s \in V^s(u_1,g)$ and a unique $u_2^u \in V^u(u_2,g)$ such that $u_1 - u_2 = u_2^u - u_1^s$, that is, $u_1 + u_1^s = u_2 + u_2^u$. ■

Lemma 4.2. *Let Y be as in Lemma 4.1 and let $M^s(y)$, $M^u(y)$ $(y \in Y)$ be the local stable and unstable manifolds of (2.15) defined in (3.4). For (u_1,g), $(u_2,g) \in Y$ such that $\|u_1 - u_2\| \ll 1$, one has $M^s(u_1,g) \cap M^u(u_2,g) \neq \emptyset$.*

Proof. By Theorem 2.1, for each $y_0 = (u_0,g) \in Y$, there are C^1 functions $h^s(\cdot,y_0)$: $V^s(y_0) \to V^u(y_0)$, $h^u(\cdot,y_0) : V^u(y_0) \to V^s(y_0)$ such that $h^s(u,y_0)$, $h^u(u,y_0) = o(\|u\|)$, $|\dfrac{\partial h^{s,u}}{\partial u}(u,y_0)| < M$ for $u \in V^{s,u}(y_0)$, and

$$
\begin{aligned}
M^s(u_0,g) &= \big\{u_0 + u^s + h^s(u^s,u_0,g) | u^s \in V^s(y_0) \cap \{u \in X^\alpha | \|u - u_0\| < \delta^*\}\big\}, \\
M^u(u_0,g) &= \big\{u_0 + u^u + h^u(u^u,u_0,g) | u^u \in V^u(y_0) \cap \{u \in X^\alpha | \|u - u_0\| < \delta^*\}\big\}.
\end{aligned}
\tag{4.1}
$$

Now, for any (u_1,g), $(u_2,g) \in Y$, consider the mapping $Q(u_1,u_2,g) : X^\alpha = V^s(u_1,g)\bigoplus V^u(u_2,g) \to X^\alpha : (u_1^s,u_2^u) \mapsto u_2^u - u_1^s$. By Lemma 4.1, there is a $\delta_1 > 0$ such that if $\|u_1 - u_2\| < \delta_1$, then $V^s(u_1,g)\bigoplus V^u(u_2,g) = X^\alpha$, that is, $Q(u_1,u_2,g)$ is an isomorphism. Define $\widetilde{Q}(u_1,u_2,g) : X^\alpha = V^s(u_1,g)\bigoplus V^u(u_2,g) \to X^\alpha$, $(u_1^s,u_2^u) \mapsto Q(u_1,u_2,g)(u_1^s,u_2^u) + h^u(u_2^u,u_2,g) - h^s(u_1^s,u_1,g)$. Then it is not difficult to see that there are $\delta_2,\delta_3 > 0$ such that if $\|u_1 - u_2\| < \delta_2$, then $\widetilde{Q}(u_1,u_2,g)(u_1^s,u_2^u) = u$ has a unique solution $(u_1^s,u_2^u) \in \big(V^s(u_1,g) \cap \{u \in X^\alpha | \|u\| < \delta^*\}\big) \bigoplus (V^u(u_2,g) \cap \{u \in X^\alpha | \|u\| < \delta^*\})$ for any $u \in X^\alpha$ with $\|u\| < \delta_3$, in particular, there is a unique $u_1^s \in V^s(u_1,g) \cap \{u \in X^\alpha | \|u\| < \delta^*\}$ and a unique $u_2^u \in V^u(u_2,g) \cap \{u \in X^\alpha | \|u\| < \delta^*\}$ such that $u_1 - u_2 = \widetilde{Q}(u_1^s,u_2^u)$, that is, $u_1 + u_1^s + h^s(u_1^s) = u_2 + u_2^u + h^u(u_2^u)$, provided $\|u_1 - u_2\| < \min\{\delta_2,\delta_3\}$.■

Lemma 4.3. *Let (u_i,g), $(u_i^*,g^*) \in X^\alpha \times H(f)$ $(i = 1,2)$ be such that $u_1 \neq u_2$, $u_1^* \neq u_2^*$ and let $\Pi_t(u_i,g)$, $\Pi_t(u_i^*,g^*)$ $(i = 1,2)$ be defined for $t \in \mathbb{R}^1$. If there is a sequence $\{t_n\}$ $(\{s_n\})$ with $t_n \to \infty$ $(s_n \to -\infty)$ as $n \to \infty$ such that $\Pi_{t_n}(u_i,g) \to (u_i^*,g^*)$, $(\Pi_{s_n}(u_i,g) \to (u_i^*,g^*))$ $(i = 1,2)$ as $n \to \infty$, then $Z(u(t,\cdot,u_1^*,g^*) - u(t,\cdot,u_2^*,g^*)) \equiv$ constant for all $t \in \mathbb{R}^1$.*

Proof. This is just Lemma 2.2 of [22]. ■

Definition 4.2. *Suppose that $Y \subset X^\alpha \times H(f)$ is a compact invariant set of (2.16). A pair (u_1,g), $(u_2,g) \in Y$ is said to be two sided proximal if $\inf_{t\in\mathbb{R}^+} \|u(t,\cdot,u_1,g) - u(t,\cdot,u_2,g)\| = 0$ and $\inf_{t\in\mathbb{R}^-} \|u(t,\cdot,u_1,g) - u(t,\cdot,u_2,g)\| = 0$. Y is said to be a proximal extension of $H(f)$ if any (u_1,g), $(u_2,g) \in Y$ forms a two sided proximal pair.*

Remark 4.1. It is easy to see that the above definition is equivalent to the usual definition of proximal extension ([8], [14], [22]).

Lemma 4.4. *Let $Y \subset X^\alpha \times H(f)$ be as in Lemma 4.1. Then Y does not contain any two sided proximal pair.*

Proof. Suppose that there is a two sided proximal pair $\{(u_1, g_0), (u_2, g_0)\} \in Y$. Let $t_0 \in \mathbb{R}^1$ be such that $u(t_0, \cdot, u_1, g_0) - u(t_0, \cdot, u_2, g_0)$ has only simple zeros in $[0, 1]$ (such a t_0 exists due to Lemma 2.2). Then there is $\epsilon_0 > 0$ such that for any $v \in X^\alpha$ with $\|v\| < \epsilon_0$, $u(t_0.\cdot, u_1, g_0) - u(t_0, \cdot, u_2, g_0) + v(\cdot)$ has only simple zeros in $[0, 1]$, and

$$Z(u(t_0.\cdot, u_1, g_0) - u(t_0, \cdot, u_2, g_0) + v(\cdot)) = Z(u(t_0.\cdot, u_1, g_0) - u(t_0, \cdot, u_2, g_0)). \tag{4.2}$$

Let $\{t_n\}$, $\{s_n\}$ with $t_n \to \infty$, $s_n \to -\infty$ as $n \to \infty$ be such that

$$\begin{aligned} \|u(t_n, \cdot, u_1, g_0) - u(t_n, \cdot, u_2, g_0)\| \to 0, \\ \|u(s_n, \cdot, u_1, g_0) - u(s_n, \cdot, u_2, g_0)\| \to 0 \end{aligned} \tag{4.3}$$

as $n \to \infty$. Then by Lemma 4.2, for $n \gg 1$, there are $u_+^n \in M^s(u(t_n, \cdot, u_1, g_0), g_0 \cdot t_n) \cap M^u(u(t_n, \cdot, u_2, g_0), g_0 \cdot t_n)$, and $u_-^n \in M^s(u(s_n, \cdot, u_1, g_0), g_0 \cdot s_n) \cap M^u(u(s_n, \cdot, u_2, g_0), g_0 \cdot s_n)$. By Theorem 2.1 4), one has

$$\begin{aligned} \|u(s, \cdot, u_+^n, g_0 \cdot t_n) - u(s, \cdot, u(t_n, \cdot, u_2, g_0), g_0 \cdot t_n)\| \le Ce^{\frac{\beta}{2}s}\|u_+^n - u(t_n, \cdot, u_2, g_0)\|, \\ \|u(t, \cdot, u_-^n, g_0 \cdot s_n) - u(t, \cdot, u(s_n, \cdot, u_1, g_0), g_0 \cdot s_n)\| \le Ce^{-\frac{\beta}{2}t}\|u_-^n - u(s_n, \cdot, u_1, g_0)\| \end{aligned} \tag{4.4}$$

for any $s \le 0$, $t \ge 0$. Note that

$$\begin{aligned} u(t_0 - t_n, \cdot, u(t_n, \cdot, u_2, g_0), g_0 \cdot t_n) = u(t_0, \cdot, u_2, g_0) \\ u(t_0 - s_n, \cdot, u(s_n, \cdot, u_1, g_0), g_0 \cdot s_n) = u(t_0, \cdot, u_1, g_0). \end{aligned} \tag{4.5}$$

By (4.4), (4.5), there is n_0 such that

$$\begin{aligned} \|u(t_0 - t_{n_0}, \cdot, u_+^{n_0}, g_0 \cdot t_{n_0}) - u(t_0, \cdot, u_2, g_0)\| < \epsilon_0, \\ \|u(t_0 - s_{n_0}, \cdot, u_-^{n_0}, g_0 \cdot s_{n_0}) - u(t_0, \cdot, u_1, g_0)\| < \epsilon_0. \end{aligned} \tag{4.6}$$

Now, by (4.2), (4.6) and Theorem 3.5, one has

$$\begin{aligned} Z(u(t_0, \cdot, u_1, g_0) - u(t_0, \cdot, u_2, g_0)) &= Z(u(t_0, \cdot, u_1, g_0) - u(t_0 - t_{n_0}, \cdot, u_+^{n_0}, g_0 \cdot t_{n_0}) \\ &\quad + u(t_0 - t_{n_0}, \cdot, u_+^{n_0}, g_0 \cdot t_{n_0}) - u(t_0, \cdot, u_2, g_0)) \\ &= Z(u(t_0, \cdot, u_1, g_0) - u(t_0 - t_{n_0}, \cdot, u_+^{n_0}, g_0 \cdot t_{n_0}) \\ &\ge N, \end{aligned}$$

and

$$\begin{aligned}Z(u(t_0,\cdot,u_1,g_0)-u(t_0,\cdot,u_2,g_0)) &= Z(u(t_0,\cdot,u_1,g_0)-u(t_0-s_{n_0},\cdot,u_-^{n_0},g_0\cdot s_{n_0})\\ &\quad + u(t_0-s_{n_0},\cdot,u_-^{n_0},g_0\cdot s_{n_0})-u(t_0,\cdot,u_2,g_0))\\ &= Z(u(t_0-s_{n_0},\cdot,u_-^{n_0},g_0\cdot s_{n_0})-u(t_0,\cdot,u_2,g_0))\\ &\le N-1.\end{aligned}$$

This is a contradiction. Hence Y does not contain any two sided proximal pair. ■

Lemma 4.5. *Let $Y \subset X \times H(f)$ be a minimal set of (2.16). Then Y is a proximal extension of $H(f)$.*

Proof. This is Theorem 3.3 of [22]. ■

Corollary 4.6. *Let $Y \subset X \times H(f)$ be a hyperbolic minimal set of (2.16). Then Y is an almost periodic extension of $H(f)$.*

Proof. It follows directly from Lemma 4.4 and 4.5. ■

Lemma 4.7. *Let $(u_0, g_0) \in X \times H(f)$ be such that the motion $\Pi_t(u_0, g_0)$ $(t > 0)$ is bounded. Then one of the following is true.*

1) $\omega(u_0, g_0)$ is minimal;

2) $\omega(u_0, g_0) = E_1 \cup E_{11}$, where E_1, E_{11} are disjoint, E_1 is minimal. Moreover, for any $(u, g) \in E_{11}$, one has $\omega(u, g) \cap E_1 \neq \emptyset$, and $\alpha(u, g) \cap E_1 \neq \emptyset$, where α is referred to as α-limit set;

3) $\omega(u_0, g_0) = E_1 \cup E_2 \cup E_{12}$, where E_1, E_2, E_{12} are disjoint, and E_1, E_2 are minimal. Moreover, for any $(u, g) \in E_{12}$, one has $\omega(u, g) \cap (E_1 \cup E_2) \neq \emptyset$, and $\alpha(u, g) \cap (E_1 \cup E_2) \neq \emptyset$. Furthermore, there is an integer $N_0 > 0$ such that $Z(u_1(\cdot) - u_2(\cdot)) = N_0$ for all $(u_1, g) \in E_1 \cap P^{-1}(g)$, $(u_2, g) \in E_2 \cap P^{-1}(g)$, and all $g \in H(f)$.

Proof. See Lemma 2.4 and Theorem 2.6 of [22]. ■

Theorem 4.8. *Consider an ω-limit set $\omega(u_0, g_0) \subset X \times H(f)$ of $(u_0, g_0) \in X \times H(f)$. If $\omega(u_0, g_0)$ is hyperbolic, then it is an almost periodic extension of $H(f)$.*

Proof. We shall show that $\omega(u_0, g_0)$ must be minimal. Then by Corollary 4.6, it is an almost periodic extension of $H(f)$.

Suppose $\omega(u_0, g_0)$ is not minimal. Then it is the either case 2) or the case 3) of Lemma 4.7.

Let $\omega(u_0, g_0) = E_1 \cup E_{11}$, where E_1 and E_{11} are as in 2) of Lemma 4.7. By Corollary 4.6, E_1 is an almost periodic extension of $H(f)$. Fix $y_0 = (u_{11}, g) \in E_{11}$. Let $(u_1, g) =$

$E_1 \cap P^{-1}(g)$. Then, by Lemma 4.7 2), (u_1, g), (u_{11}, g) is a two sided proximal pair. This is impossible by Lemma 4.4.

Now, we let $\omega(u_0, g_0) = E_1 \cup E_2 \cup E_{12}$ with E_1, E_2, E_{12} being defined as in Lemma 4.7 3). E_1, E_2 are almost periodic extensions of $H(f)$ by Lemma 4.5.

For any $y_0 = (u_{12}, g) \in E_{12}$, let $(u_i, g) = E_i \cap P^{-1}(g)$ $(i = 1, 2)$. Then, by Lemma 4.4, both $\{(u_1, g), (u_{12}, g)\}$ and $\{(u_2, g), (u_{12}, g)\}$ do not form two sided proximal pairs. Therefore, without loss of generality, we may assume following Lemma 4.7 3) that $\omega(u_{12}, g) \cap E_1 \neq \emptyset$, $\alpha(u_{12}, g) \cap E_2 \neq \emptyset$. Since $E_i \cap P^{-1}(g)$, $i = 1, 2$, are singletons for all $g \in H(f)$, it is easily seen that $\Pi_t(u_{12}, g) - \Pi_t(u_1, g) \to 0$ as $t \to \infty$, and $\Pi_t(u_{12}, g) - \Pi_t(u_2, g) \to 0$ as $t \to -\infty$. It then follows from Remark 2.2 3) that $\Pi_t(u_{12}, g) \in M^s(\Pi_t(u_1, g))$ for $t \gg 1$, and $\Pi_t(u_{12}, g) \in M^u(\Pi_t(u_2, g))$ for $t \ll -1$. By Theorem 3.5 and Lemma 2.2, one has

$$Z(u(t, \cdot, u_{12}, g) - u(t, \cdot, u_1, g)) \geq N \quad for \quad t \in \mathbb{R}^1, \tag{4.7}$$

and

$$Z(u(t, \cdot, u_{12}, g) - u(t, \cdot, u_2, g)) \leq N - 1 \quad for \quad t \in \mathbb{R}^1, \tag{4.8}$$

where $N = dim M^u(u, g)$ for $(u, g) \in \omega(u_0, g_0)$. We claim that there is a $\delta_0 > 0$ such that

$$|u(t, x, u_1, g) - u(t, x, u_2, g)| + |u_x(t, x, u_1, g) - u_x(t, x, u_2, g)| \geq \delta_0 \tag{4.9}$$

for any $x \in [0, 1]$ and $t \in \mathbb{R}^1$. If this is not true, then there is a sequence $\{\tilde{x}_n\} \subset [0, 1]$, and a sequence $\{\tilde{t}_n\}$ with $|\tilde{t}_n| \to \infty$ as $n \to \infty$ such that $|u(\tilde{t}_n, \tilde{x}_n, u_1, g) - u(\tilde{t}_n, \tilde{x}_n, u_2, g)| + |u_x(\tilde{t}_n, \tilde{x}_n, u_1, g) - u_x(\tilde{t}_n, \tilde{x}_n, u_2, g)| \to 0$ as $n \to \infty$. Without loss of generality, we assume that $\tilde{t}_n \to \infty$, $\tilde{x}_n \to \tilde{x} \in [0, 1]$, $g \cdot \tilde{t}_n \to \tilde{g}$, and $u(\tilde{t}_n, \cdot, u_1, g) \to \tilde{u}_1$, $u(\tilde{t}_n, \cdot, u_2, g) \to \tilde{u}_2$ in X^α as $n \to \infty$. Then $(\tilde{u}_1, \tilde{g}) \in E_1$, $(\tilde{u}_2, \tilde{g}) \in E_2$, therefore, $\tilde{u}_1 \neq \tilde{u}_2$ and $\tilde{u}_1(\cdot) - \tilde{u}_2(\cdot)$ has a multiple zero at $x = \tilde{x}$. But, by Lemma 4.3, $\tilde{u}_1(\cdot) - \tilde{u}_2(\cdot)$ has only simple zeros, a contradiction. By (4.7), (4.8), and (4.9), one has

$$Z(u(t, \cdot, u_2, g) - u(t, \cdot, u_1, g)) = Z(u(t, \cdot, u_2, g) - u(t, \cdot, u_{12}, g)) \leq N - 1 \quad for \quad t \gg 1,$$

and

$$Z(u(t, \cdot, u_2, g) - u(t, \cdot, u_1, g)) = Z(u(t, \cdot, u_{12}, g) - u(t, \cdot, u_1, g)) \geq N \quad for \quad t \ll -1.$$

But by Lemma 4.7 3),

$$Z(u(t,\cdot,u_2,g)-u(t,\cdot,u_1,g)) = constant \quad for \quad t\in \mathbb{R}^1,$$

a contradiction. ■

Remark 4.2. Suppose that for some $U_0 \in X^\alpha$, $\omega(U_0,f)$ is hyperbolic. By the above theorem, $\omega(U_0,f)$ is an almost periodic extension of $H(f)$. Let $(U^*,f)=\omega(U_0,f)\cap P^{-1}(f)$. Then $u(t,\cdot,U^*,f)$ is an almost periodic solution of (1.1).

We now consider in particular the quasi-periodic time dependent case. Consider equation (2.13) and assume that f is quasi-periodic in t with $k(\geq 2)$ frequences $\omega_1,\cdots,\omega_k$, that is, $f(t,x,u,p)\equiv F(\omega_1 t,\omega_2 t\cdots,\omega_k t,x,u,p)$. We further assume that $F: T^k\times[0,1]\times\mathbb{R}^1\times\mathbb{R}^1\to\mathbb{R}^1$ is $C^{r,\gamma}(r\geq 1, 0<\gamma\leq 1)$. In this case, the hull $H(f)$ is $C^{r,\gamma}$ diffeomorphic to the standard torus T^k and equations over the hull can be written as

$$\begin{cases} u_t = u_{xx}+F(\theta\cdot t,x,u,u_x), \quad t>0, \quad 0<x<1, \\ u(t,0)=u(t,1)=0, \quad t>0, \end{cases} \tag{4.10$_\theta$}$$

where $\theta\cdot t=\theta+\omega t$, $\omega=(\omega_1,\omega_2,\cdots,\omega_k)^\top$, $\theta\in T^k$.

As usual, $(4.10)_\theta$ gives rise to an ODE on $X^\alpha\times T^k$,

$$u' = A_0 u+\widetilde{F}(u,\theta\cdot t)\equiv G(u,\theta\cdot t) \tag{4.11}$$

in the same way as (2.15), where $\widetilde{F}: X^\alpha\times T^k\to X_0$ is $C^{r,\gamma}$. This equation again generates a (local) skew product semiflow Π_t on $X^\alpha\times T^k$:

$$\Pi_t(u_0,\theta_0)\equiv(u_0,\theta_0)\cdot t\equiv(u(t,\cdot,u_0,\theta_0),\theta_0\cdot t), \tag{4.12}$$

where $u(t,\cdot,u_0,\theta_0)$ is the solution of $(4.10)_{\theta_0}$ with $u(0,\cdot,u_0,\theta_0)=u_0(\cdot)$.

Theorem 4.9. *Suppose that* $F: T^k\times[0,1]\times\mathbb{R}^1\times\mathbb{R}^1\to\mathbb{R}^1$ *is* $C^{r,\gamma}$ $(r\geq 1,\ 0<\gamma\leq 1)$*. Let* $Y\subset X^\alpha\times T^k$ *be a hyperbolic invariant set of (4.12). Asume that* Y *is an almost periodic extension of* T^k*. Then* Y *is* $C^{r,\gamma}$ *diffeomorphic to* T^k *and the flow on* Y *is* $C^{r,\gamma}$ *conjugate to the twist flow on* T^k*.*

Before we prove the theorem, let us recall a basic property of exponential dichotomy.

Lemma 4.10. *Consider*

$$v' = A(y\cdot t)v+F(y\cdot t), \quad v\in X^\alpha, \quad y\in Y, \tag{4.13}$$

where Y is a compact metric space, $y \cdot t$ is a flow on Y, $A(y)$ is of form in Remark 2.2 3), and the function $F(y \cdot t)$ is bounded and locally Hölder continuous in $t \in \mathbb{R}^1$. If

$$v' = A(y \cdot t)v, \quad v \in X^\alpha \tag{4.14}$$

has an ED on Y, then (4.13) has for each y a unique bounded solution $v_0(y \cdot t)$. Moreover, there is a constant M which is independent of y such that

$$\|v_0(y \cdot t)\|_{X^\alpha} \leq M \sup_{t \in \mathbb{R}^1} \|F(y \cdot t)\|_{X_0}. \tag{4.15}$$

Proof. In fact, the unique bounded solution $v_0(y \cdot t)$ satisfies

$$v_0(y \cdot t) = \int_{-\infty}^{t} \Phi(t-s, y \cdot s)(I - P(y \cdot s))F(y \cdot s)ds - \int_{t}^{\infty} \Phi(t-s, y \cdot s)P(y \cdot s)F(y \cdot s)ds,$$

where Φ is the evolution operator of (4.14), $P(y)$, $y \in Y$, are the projections associated to ED. The Lemma then follows from Theorem 2.1 3), Remark 2.2 3) and a simple estimation. ■

Proof of Theorem 4.9. Since Y is an almost periodic extension of T^k, Y is a graph over T^k, namely, $Y = \{(u(\theta), \theta) | \theta \in T^k\}$, for some function $u : T^k \to X$. By invariance of Y, $u(\theta \cdot t)$ is a solution of (4.11) and by minimality of Y, u is continuous (hence uniform continuous). Thus, $u(\theta \cdot t)$ is a quasi-periodic solution of (4.11) for each $\theta \in T^k$.

Suppose $r = 1$. We first claim u is Lipschitz continuous. Fix a $\theta_0 \in T^k$. By continuity of u, for any $\epsilon > 0$, there is $\delta > 0$ such that $\|u(\theta \cdot t) - u(\theta_0 \cdot t)\|_{X^\alpha} < \epsilon$ if $|\theta - \theta_0| < \delta$. Let $v(t) = u(\theta \cdot t) - u(\theta_0 \cdot t)$. Then $v(t)$ satisfies

$$v' = A(\theta_0 \cdot t)v + G_1(t), \tag{4.16}$$

where $A(\theta) = G_u(u(\theta), \theta)$, and

$$\begin{aligned} G_1(t) &= G(u(\theta \cdot t), \theta \cdot t) - G(u(\theta_0 \cdot t), \theta_0 \cdot t) - G_u(u(\theta_0 \cdot t), \theta_0 \cdot t) \cdot (u(\theta \cdot t) - u(\theta_0 \cdot t)) \\ &= [G(u(\theta \cdot t), \theta_0 \cdot t) - G(u(\theta_0 \cdot t), \theta_0 \cdot t) - G_u(u(\theta_0 \cdot t), \theta_0 \cdot t) \cdot (u(\theta \cdot t) - u(\theta_0 \cdot t))] \\ &\quad + [G(u(\theta \cdot t), \theta \cdot t) - G(u(\theta \cdot t), \theta_0 \cdot t)]. \end{aligned}$$

Since $G : X^\alpha \times T^k \to X_0$ is $C^{1,\gamma}$, one has

$$\|G_1(t)\|_{X_0} = o(\|u(\theta \cdot t) - u(\theta_0 \cdot t)\|_{X^\alpha}) + O(|\theta - \theta_0|).$$

Thus, as $|\theta - \theta_0| < \delta$, one has that

$$\|G_1(t)\|_{X_0} \leq \epsilon M_1 \|v(t)\|_{X^\alpha} + M_2|\theta - \theta_0|, \tag{4.17}$$

for some constants $M_1, M_2 > 0$.

By Lemma 4.10,

$$\|v(t)\|_{X^\alpha} \leq \epsilon \widetilde{M_1} \sup_{t \in \mathbb{R}^1} \|v(t)\|_{X^\alpha} + \widetilde{M_2}|\theta - \theta_0|$$

for some constants $\widetilde{M_1} > 0$, $\widetilde{M_2} > 0$, that is,

$$\|u(\theta) - u(\theta_0)\|_{X^\alpha} = \|v(0)\|_{X^\alpha} \leq \sup_{t \in \mathbb{R}^1} \|v(t)\|_{X^\alpha} \leq \frac{\widetilde{M_2}}{1 - \epsilon \widetilde{M_1}} |\theta - \theta_0|. \tag{4.18}$$

Next, for any given $h \in T^k$, let $\tilde{v}_0(\theta_0 \cdot t, h)$ be the unique bounded solution of

$$v' = A(\theta_0 \cdot t)v + G_\theta(u(\theta_0 \cdot t), \theta_0 \cdot t)h. \tag{4.19}$$

Since $\tilde{v}_0(\theta_0 \cdot t, h)$ is linear in $h \in T^k$, there is a $v_0 : T^k \to L(T^k, X^\alpha)$, $\theta \mapsto v_0(\theta)$ such that $v_0(\theta)h = \tilde{v}_0(\theta, h)$. Define $w(t) = u(\theta \cdot t) - u(\theta_0 \cdot t) - v_0(\theta_0 \cdot t)(\theta - \theta_0)$. Then $w(t)$ is the unique bounded solution of

$$w' = A(\theta_0 \cdot t)w + G_2(t) \tag{4.20}$$

where

$$\begin{aligned} G_2(t) &= G(u(\theta \cdot t), \theta \cdot t) - G(u(\theta_0 \cdot t), \theta_0 \cdot t) \\ &\quad - G_u(u(\theta_0 \cdot t), \theta_0 \cdot t) \cdot (u(\theta \cdot) - u(\theta_0 \cdot t)) - G_\theta(u(\theta_0 \cdot t), \theta_0 \cdot t)(\theta - \theta_0) \\ &= [G(u(\theta \cdot t), \theta_0 \cdot t) - G(u(\theta_0 \cdot t), \theta_0 \cdot t) - G_u(u(\theta_0 \cdot t), \theta_0 \cdot t)(u(\theta \cdot t) - u(\theta_0 \cdot t))] \\ &\quad + [G(u(\theta \cdot t), \theta \cdot t) - G(u(\theta \cdot t), \theta_0 \cdot t) - G_\theta(u(\theta \cdot t), \theta_0 \cdot t)(\theta - \theta_0)] \\ &\quad + [(G_\theta(u(\theta \cdot t), \theta_0 \cdot t) - G_\theta(u(\theta_0 \cdot t), \theta_0 \cdot t))(\theta - \theta_0)]. \end{aligned}$$

Since $G : X^\alpha \times T^k \to X_0$ is $C^{1,\gamma}$ and u is Lipschitz, one has

$$\|G_2(t)\|_{X_0} \leq \epsilon M_3 |\theta - \theta_0|$$

as $|\theta - \theta_0| \ll 1$, where M_3 is a constant. Thus, by Lemma 4.10,

$$\|w(0)\|_{X^\alpha} \le \sup_{t\in \mathbb{R}^1} \|w(t)\|_{X^\alpha} \le \epsilon \widetilde{M}_3 |\theta - \theta_0| \tag{4.21}$$

for some $\widetilde{M}_3 > 0$, that is, $u(\theta)$ is differentiable with $u_\theta(\theta) = v_0(\theta)$. Now, for fixed $\theta_0, h \in T^k$, $\bar{v}(t) \equiv u_\theta(\theta \cdot t)h - u_\theta(\theta_0 \cdot t)h \equiv v_0(\theta \cdot t)h - v_0(\theta_0 \cdot t)h$ satisfies

$$\bar{v}^{'} = A(\theta_0 \cdot t)\bar{v} + G_3(t) \tag{4.22}$$

where

$$G_3(t) = [G_u(u(\theta \cdot t), \theta \cdot t) - G_u(u(\theta_0 \cdot t), \theta_0 \cdot t)]v_0(\theta \cdot t)h + [G_\theta(u(\theta \cdot t), \theta \cdot t) - G_\theta(u(\theta_0 \cdot t), \theta_0 \cdot t)]h.$$

Since $G : X^\alpha \times T^k \to X_0$ is $C^{1,\gamma}$ and u is Lipschitz continuous, there is a $M_4 > 0$ such that

$$\|G_3(t)\|_{X_0} \le M_4 |\theta - \theta_0|^\gamma \cdot |h|. \tag{4.23}$$

Applying Lemma 4.10 again, one has that

$$\|u_\theta(\theta) - u_\theta(\theta_0)\|_{X^\alpha} = \|\bar{v}(0)\|_{X^\alpha} \le \widetilde{M}_4 |\theta - \theta|^\gamma$$

for some constant $\widetilde{M}_4 > 0$, that is, u_θ is $C^{0,\gamma}$ continuous.

We have shown in the above that if $F : T^k \times [0,1] \times \mathbb{R}^1 \times \mathbb{R}^1$ is $C^{1,\gamma}$, then $u : T^k \to X^\alpha$ is $C^{1,\gamma}$, the rest of the proof can be carried over by induction. ■

Corollary 4.11. *Suppose $F : T^k \times [0,1] \times \mathbb{R}^1 \times \mathbb{R}^1$ is $C^{r,\gamma}$ ($r \ge 2$, $0 < \gamma \le 1$). Let $\omega(u_*, \theta_*) \subset X^\alpha \times T^k$ be a hyperbolic ω-limit set of (4.12). Then $\omega(u_*, \theta_*)$ is $C^{r,\gamma}$ diffeomorphic to T^k and the flow on $\omega(u_*, \theta_*)$ is $C^{r,\gamma}$ conjugate to the twist flow on T^k.*

Proof. By Theorem 4.8, $\omega(u_*, g_*)$ is an almost periodic extension of T^k. The Corollary then follows from Theorem 4.9. ■

5. An Example

We modify an example of [18] to explain that if an ω-limit set of (2.16) does not possess hyperbolicity, then it may not even be a minimal set.

Consider

$$\begin{cases} u_t = u_{xx} + (f(t) - \lambda_1)u, \quad 0 < x < 1, \quad t > 0, \\ u(t,0) = u(t,1) = 0, \quad t > 0, \end{cases} \tag{5.1}$$

where $f(t)$ is almost periodic which has a Fourier expansion $f(t) = -\sum_{k=1}^{\infty} 2^k \pi \sin(2^{-k}\pi t)$, λ_1 is the first eigenvalue of $\frac{\partial^2}{\partial x^2} : H_0^2(0,1) \to L^2(0,1)$. Let $H(f)$ be the hull of f. Consider the skew product semiflow Π_t on $X^\alpha \times H(f)$ generated from (5.1),

$$\Pi_t(U, g) = (u(t, \cdot, U, g), g \cdot t), \tag{5.2}$$

where $u(t, x, U, g)$ is the solution of

$$\begin{cases} u_t = u_{xx} + (g(t) - \lambda_1)u, \quad 0 < x < 1, \quad t > 0, \\ u(t, 0) = u(t, 1) = 0, \quad t > 0 \end{cases} \tag*{$(5.3)_g$}$$

with $u(0, x, U, g) = U(x)$, $g \in H(f)$. Let U_1 be the first eigenfunction of $\frac{\partial^2}{\partial x^2} : H_0^2(0,1) \to L^2(0,1)$. It is easy to see that $u(t, \cdot, U_1, f) = e^{\int_0^t f(s)ds} U_1$ is a solution of (5.1). By discussions in [18], $\phi(t) = e^{\int_0^t f(s)ds}$ satisfies the following properties:

1) $\phi(t)$ is bounded for $t \geq 0$;
2) There is a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that $\phi(t_n) \to 0$ as $n \to \infty$, and $\phi(2^n) \geq e^{-2\pi - 2}$ for $n = 1, 2, \cdots$;
3) For any sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that $\lim_{n \to \infty} \phi(t + t_n) = \phi^*(t)$ exists, $\phi^*(t)$ is not almost periodic if it is nonzero.

Now, consider $\omega(U_1, g)$. By the property 2) above, $\{0\} \times H(f)$ is a proper subset of $\omega(U_1, f)$. Since $\{0\} \times H(f)$ is minimal (in fact, it is the only minimal set contained in $\omega(U_1, f)$), it follows that $\omega(U_1, g)$ is not a minimal set, hence is not hyperbolic by Theorem 4.8.

We now investigate flows on $\omega(U, f)$. Let $\Omega^+(\phi, f)$ be the positive limit set of $\{\phi(t), f \cdot t\}$ in $\mathbb{R}^1 \times H(f)$. Then $\omega(U_1, g) = \{(\phi_g^* U_1, g) | (\phi_g^*, g) \in \Omega^+(\phi, f)\}$, that is, $\omega(U_1, f) \cap P^{-1}(g)$ is a compact set of $span\{U_1\} \times \{g\}$ for each $g \in H(f)$, here $P : X^\alpha \times H(f) \to H(f)$ is the natural projection. For any $(\phi_g^*, g) \in \Omega^+(\phi, f)$, it is clear that $\Pi_t(\phi_g^* U_1, g) = (\phi_g^* e^{\int_0^t g(s)ds} U_1, g \cdot t) \in \omega(U_1, f)$ for all t. Moreover, if $\phi_g^* \neq 0$, then it follows from Lemma 4.7 that there are sequences $\{t_n\}$ and $\{s_n\}$ with $t_n \to +\infty$ and $s_n \to -\infty$ such that $u(t_n, \cdot, \phi_g^* U_1, g) \to 0$ and $u(s_n, \cdot, \phi_g^* U_1, g) \to 0$ as $n \to \infty$. One also observes that there is a residual subset $H_0(f) \subset H(f)$ such that $\omega(U_1, f) \cap P^{-1}(g) = (0, g)$ for all $g \in H_0(f)$ (note that $card\omega(U_1, f) \cap P^{-1}(g) \not\equiv 1$ for $g \in H(f)$). To see this, we note that $\int_0^t g(s)ds$ is unbounded and $\lim_{|t|\to\infty} \frac{1}{t}\int_0^t g(s)ds = 0$ for all $g \in H(f)$ (if $\lim_{|t|\to\infty} \frac{1}{t}\int_0^t f(s)ds \neq 0$, then $\omega(U_1, f) = \{0\} \times H(f)$ by [9]). It follows from [13] that $H_0(f) = \{g \in H(f) | \limsup_{t\to\infty} \int_0^t g(s)ds = +\infty, \liminf_{t\to\infty} \int_0^t g(s)ds = -\infty\}$ is residual. Now take $g \in H_0(f)$, if there is $(\phi_g^* U_1, g) \in \omega(U_1, f) \cap P^{-1}(g)$ with $\phi_g^* \neq 0$, then $\phi_g^* e^{\int_0^t g(s)ds}$ is unbounded, but $\Pi_t(\phi_g^* U_1, g) = (\phi_g^* e^{\int_0^t g(s)ds} U_1, g \cdot t) \subset \omega(U_1, f)$, a contradiction.

6. References

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