

STEADY STATES OF FOKKER-PLANCK EQUATIONS: III. DEGENERATE DIFFUSION

WEN HUANG, MIN JI, ZHENXIN LIU, AND YINGFEI YI

ABSTRACT. This is the third paper in a series concerning the study of steady states of a Fokker-Planck equation in a general domain in \mathbb{R}^n with $L_{loc}^{\bar{p}}$ drift term and $W_{loc}^{1,\bar{p}}$ diffusion term for any $\bar{p} > n$. In this paper, we give some existence results of stationary measures of the Fokker-Planck equation under Lyapunov conditions which allow the degeneracy of diffusion.

1. INTRODUCTION

Consider the *stationary Fokker-Planck equation* on \mathcal{U} :

$$(1.1) \quad \begin{cases} Lu(x) = 0, & x \in \mathcal{U}, \\ u(x) \geq 0, & \int_{\mathcal{U}} u(x) dx = 1, \end{cases}$$

where $\mathcal{U} \subset \mathbb{R}^n$ be a connected open set which can be bounded, unbounded, or the entire space \mathbb{R}^n , L is the *Fokker-Planck operator* on \mathcal{U} defined by

$$Lg(x) = \partial_{ij}^2(a^{ij}(x)g(x)) - \partial_i(V^i(x)g(x)), \quad g \in C^2(\mathcal{U})$$

with $V = (V^i)$ being the *drift field* and $(a^{ij}) \geq 0$ being the *diffusion matrix*, on \mathcal{U} .

In the above and also through the rest of the paper, we use short notations $\partial_i = \frac{\partial}{\partial x_i}$, $\partial_{ij}^2 = \frac{\partial^2}{\partial x_i \partial x_j}$, and we also adopt the usual summation convention on $i, j = 1, 2, \dots, n$ whenever applicable.

We make the following standard hypothesis:

A) $a^{ij} \in W_{loc}^{1,\bar{p}}(\mathcal{U})$, $V^i \in L_{loc}^{\bar{p}}(\mathcal{U})$ for all $i, j = 1, \dots, n$, where $\bar{p} > n$ is fixed.

A continuous function u on \mathcal{U} is said to be a *weak stationary solution* of the Fokker-Planck equation corresponding to L if it is a weak solution of the stationary Fokker-Planck equation (1.1), i.e.,

$$(1.2) \quad \begin{cases} \int_{\mathcal{U}} L^* f(x) u(x) dx = 0, & \text{for all } f \in C_0^\infty(\mathcal{U}), \\ u(x) \geq 0, & \int_{\mathcal{U}} u(x) dx = 1, \end{cases}$$

2000 *Mathematics Subject Classification.* 35Q84, 60J60, 37B25.

Key words and phrases. Fokker-Planck equation, Degenerate diffusion, Existence, Stationary solution, Stationary measure, Level set method.

The first author was partially supported by NSFC grant 11225105. The second author was partially supported by NSFC Innovation grant 10421101. The third author was partially supported by NSFC grant 11271151, SRF for ROCS, SEM, Chunmiao fund and the 985 program from Jilin University. The fourth author was partially supported by NSF grants DMS0708331, DMS1109201 and a Scholarship from Jilin University.

where

$$L^* = a^{ij} \partial_{ij}^2 + V^i \partial_i$$

is the adjoint Fokker-Planck operator and $C_0^\infty(\mathcal{U})$ denotes the space of C^∞ functions on \mathcal{U} with compact supports. More generally, a Borel probability measure μ on \mathcal{U} is said to be a *stationary measure* of the Fokker-Planck equation corresponding to L if it is a measure solution of the stationary Fokker-Planck equation (1.1), i.e.,

$$(1.3) \quad V^i \in L_{loc}^1(\mathcal{U}, \mu), \quad i = 1, 2, \dots, n, \text{ and,}$$

$$(1.4) \quad \int_{\mathcal{U}} L^* f(x) d\mu(x) = 0, \quad \text{for all } f \in C_0^\infty(\mathcal{U}).$$

A stationary measure μ of the Fokker-Planck equation corresponding to L is called *regular* if it admits a continuous density function u , i.e., $d\mu(x) = u(x)dx$. It is clear that such a density u is a weak stationary solution of the Fokker-Planck equation corresponding to L . Indeed, under the condition A), if (a^{ij}) is everywhere positive definite in \mathcal{U} , then it follows from a regularity theorem due to Bogachev-Krylov-Röckner ([5]) (see Theorem 2.5) that all stationary measures of the Fokker-Planck equation corresponding to L are regular with densities lying in $W_{loc}^{1,\bar{p}}(\mathcal{U})$. If $a^{ij} \in C_{loc}^{2,\alpha}(\mathcal{U})$, $V^i \in C_{loc}^{1,\alpha}(\mathcal{U})$, $i, j = 1, \dots, n$, for some $\alpha \in (0, 1)$, then it follows from the standard Schauder theory that the density functions become classical solutions of (1.1).

When (a^{ij}) is everywhere positive definite in \mathcal{U} , the existence of stationary measures corresponding to L has been extensively studied. We refer the reader to [1]-[11], [18]-[21] and references therein for the case $\mathcal{U} = \mathbb{R}^n$. For a general domain \mathcal{U} , we have given several new existence, respectively, non-existence results for stationary measures corresponding to L in the previous two parts of the series ([15, 16]), by using Lyapunov-like, respectively anti-Lyapunov-like functions. We note that existence results in part I of the series ([15]) are still applicable to obtain the existence of a regular stationary measure corresponding to L when (a^{ij}) is degenerate on the boundary of \mathcal{U} , though the standard theory of elliptic equations fails to apply even when \mathcal{U} is bounded, due to the degeneracy on the boundary.

When (a^{ij}) is not necessarily everywhere positive definite in \mathcal{U} , some conditions on an anti-Lyapunov function are given in part II of the series ([16]) which ensure the non-existence of a regular stationary measure with positive density corresponding to L . Still, stationary measures which are not necessarily regular can exist. In [7], without assuming (a^{ij}) to be everywhere positive definite in \mathbb{R}^n , Bogachev-Röckner showed the existence of a stationary measure corresponding to L in \mathbb{R}^n when A) holds, V is continuous on \mathbb{R}^n , and there exists a non-negative function $U \in C^2(\mathbb{R}^n)$ with

$$(1.5) \quad \lim_{x \rightarrow \infty} U(x) = +\infty$$

and

$$(1.6) \quad \lim_{x \rightarrow \infty} L^* U(x) = -\infty.$$

In this paper, we will study the existence of stationary measures of the Fokker-Planck equation corresponding to L in a general domain \mathcal{U} without assuming (a^{ij}) to be everywhere positive definite in \mathcal{U} . Such a generality of the domain does allow a wide range of applications

as already remarked in part I of the series ([15]). Our result, extending the above result of [7] even in the case of \mathbb{R}^n , makes use of a *Lyapunov function* $U \in C^2(\mathcal{U})$ with respect to (1.1) which is a so-called *compact function* in \mathcal{U} (a notion generalizing condition (1.5), see Section 2.1 for details) satisfying the “dissipation” property that

$$(1.7) \quad \limsup_{x \rightarrow \partial\mathcal{U}} L^*U(x) = \limsup_{x \rightarrow \partial\mathcal{U}} (a^{ij}(x)\partial_{ij}^2U(x) + V^i(x)\partial_iU(x)) = -\gamma$$

for some constant $\gamma > 0$. In the above, the notion $\partial\mathcal{U}$ and limit $x \rightarrow \partial\mathcal{U}$ is defined through a unified topology which identifies the extended Euclidean space $\mathbb{E}^n = \mathbb{R}^n \cup \partial\mathbb{R}^n$ with the closed unit ball $\bar{\mathbb{B}}^n = \mathbb{B}^n \cup \partial\mathbb{B}^n$ in \mathbb{R}^n so that $\partial\mathbb{R}^n$, consisting of infinity elements of all rays, is identified with $\partial\mathbb{B}^n = \mathbb{S}^{n-1}$ (see [15] for details). Therefore, when $\mathcal{U} = \mathbb{R}^n$, the limit $x \rightarrow \partial\mathbb{R}^n$ is simply equivalent to $x \rightarrow \infty$.

More precisely, we will show the following result.

Theorem A. *Assume A) and that $V \in C(\mathcal{U}, \mathbb{R}^n)$. If, with respect to (1.1), there exists either a strong Lyapunov function or a Lyapunov function of the class $\mathcal{B}^*(A)$, then the Fokker-Planck equation corresponding to L admits a stationary measure.*

A strong Lyapunov function is a compact function U satisfying (1.7) with $\gamma = +\infty$ and the class $\mathcal{B}^*(A)$ contains functions U with controlled growth rates of $a^{ij}\partial_iU\partial_jU$ near $\partial\mathcal{U}$ (see Section 2 for details).

Proof of Theorem A above uses the level set method in particular the integral identity which we derived in [14] (see also Theorem 2.2). It also follows the approach in [7] for perturbing the diffusion matrix via a family of non-degenerate ones then taking weak*-limits of the corresponding stationary measures. More generally, consider a family of adjoint Fokker-Planck operators:

$$L_{V,A}^* = a^{ij}\partial_{ij}^2 + V^i\partial_i, \quad V = (V^i) \in \mathcal{V}, \quad A = (a^{ij}) \in \mathcal{A},$$

where \mathcal{V} is a subset of vector-valued functions in $C(\mathcal{U}, \mathbb{R}^n)$ under the compact-open topology and \mathcal{A} is a subset of $n \times n$ matrix-valued, everywhere positive definite functions on \mathcal{U} of the class $W_{loc}^{1,\bar{p}}$. Then one can speak of a *uniform Lyapunov* or a *strong uniform Lyapunov function* with respect to the family $\{L_{V,A}^* : V \in \mathcal{V}, A \in \mathcal{A}\}$ and the *class $\mathcal{B}^*(\mathcal{A})$* with respect to the family \mathcal{A} (see Section 2.2 for details). For such a uniform Lyapunov function, one can also define its *essential lower bound* in the same way as that for a Lyapunov function (see Section 2).

For each $V \in \mathcal{V}$, $A \in \mathcal{A}$, let $\mathcal{M}_{V,A}$ denote the set of stationary measures of the Fokker-Planck equations with drift field V and diffusion matrix A . The proof of Theorem A will rely on the following result on the upper semi-continuity of the stationary measure set $\mathcal{M}_{V,A}$ in V, A which is of interests on its own rights.

Theorem B. *Assume that there is a uniform Lyapunov function U with respect to the family $\{L_{V,A}^* : V \in \mathcal{V}, A \in \mathcal{A}\}$ such that either U is a uniform strong Lyapunov function with*

$$\sup_{V \in \mathcal{V}, A \in \mathcal{A}} (|V|_{C(\bar{\Omega}_{\rho_m})} + |A|_{C(\bar{\Omega}_{\rho_m})}) < \infty,$$

or U is of the class $\mathcal{B}^(\mathcal{A})$, where ρ_m is the essential lower bound of U and Ω_{ρ_m} denotes the ρ_m -sublevel set of U . Then the following holds:*

- a) The set $\mathcal{M} = \bigcup_{V \in \mathcal{V}, A \in \mathcal{A}} \mathcal{M}_{V,A}$ is relatively sequentially compact in the space $M(\mathcal{U})$ of Borel probability measures on \mathcal{U} ;
- b) For any $(V_0, A_0) \in \bar{\mathcal{V}} \times \bar{\mathcal{A}}$ and any sequence $\{(V_k, A_k)\} \subset \mathcal{V} \times \mathcal{A}$ with $(V_k, A_k) \rightarrow (V_0, A_0)$, there is a subsequence (V_{k_i}, A_{k_i}) such that $\mu_{V_{k_i}, A_{k_i}}$ converges to a stationary measure $\mu_0 \in \mathcal{M}_{V_0, A_0}$. In particular, $\mathcal{M}_{V_0, A_0} \neq \emptyset$.

We remark that if the equation (1.1) is defined on $\mathcal{U} \times M$, where $\mathcal{U} \subset \mathbb{R}^n$ is a connected open set and M is a smooth, compact manifold without boundary, then one can modify the definitions of (uniform) Lyapunov, strong (uniform) Lyapunov functions in Section 2 in an obvious way by replacing the domain $\mathcal{U} \subset \mathbb{R}^n$ with $\mathcal{U} \times M$. Then the proofs in later sections can be modified accordingly so that Theorems A, B still hold with respect to such a generalized domain.

This paper is organized as follows. Section 2 is a preliminary section in which we will recall notions of compact and Lyapunov functions and define new notions of uniform Lyapunov and strong uniform Lyapunov functions. We will also recall the Prokhorov's theorem on tightness and sequential compactness of probability measures, main ingredients of the level set method from [14], and a regularity result for stationary measures from [5]. In Section 3, we study the upper-semi-continuity of stationary measures on drift and diffusions in a general domain. Theorem B will be proved. In Section 4, we show Theorem A in a general domain and also give a corollary in \mathbb{R}^n involving more explicit conditions.

Through the rest of the paper, for simplicity, we will use the same symbol $|\cdot|$ to denote absolute value of a number, and norm of a vector or a matrix.

2. PRELIMINARY

In this section, we will recall notions of compact and Lyapunov functions and define new notions of uniform Lyapunov and strong uniform Lyapunov functions. We will also recall the Prokhorov's theorem on tightness and sequential compactness of probability measures, the integral identity and the derivative formula from [14] which are of fundamental importance to the level set method to be adopted in this paper, and a regularity result for stationary measures from [5].

2.1. Compact functions. Recall from [15] that a non-negative continuous function U in \mathcal{U} is a *compact function* if

- i) $U(x) < \rho_M$, $x \in \mathcal{U}$; and
- ii) $\lim_{x \rightarrow \partial \mathcal{U}} U(x) = \rho_M$,

where $\rho_M = \sup_{x \in \mathcal{U}} U(x)$ is called the *essential upper bound of U* . For each $\rho \in [0, \rho_M)$, we denote by Ω_ρ the ρ -sublevel set of U .

When \mathcal{U} is unbounded, the notion $\partial \mathcal{U}$ and the limit $x \rightarrow \partial \mathcal{U}$ in ii) above should be understood under the topology which is defined through a fixed homeomorphism between the extended Euclidean space $\mathbb{E}^n = \mathbb{R}^n \cup \partial \mathbb{R}^n$ and the closed unit ball $\bar{\mathbb{B}}^n = \mathbb{B}^n \cup \partial \mathbb{B}^n$ in \mathbb{R}^n which identifies \mathbb{R}^n with B^n and $\partial \mathbb{R}^n$ with \mathbb{S}^{n-1} , and in particular, identifies each $x_* \in \mathbb{S}^{n-1}$ with the infinity element $x_*^\infty \in \partial \mathbb{R}^n$ of the ray through x_* (see [15] for details). Consequently,

if $\mathcal{U} = \mathbb{R}^n$, then $x \rightarrow \partial\mathbb{R}^n$ under this topology simply means $x \rightarrow \infty$ in the usual sense, and it is easy to see that the following result holds.

Proposition 2.1. *An unbounded, non-negative function $U \in C(\mathcal{U})$ is a compact function in \mathcal{U} iff*

$$\lim_{x \rightarrow \partial\mathcal{U}} U(x) = +\infty.$$

We recall from [16] the following definition of class $\mathcal{B}^*(A)$ of compact functions.

Definition 2.1. Let $A = (a^{ij})$ be a everywhere positive semi-definite, $n \times n$ matrix-valued function on \mathcal{U} . A compact function $U \in C^1(\mathcal{U})$ with essential upper bound ρ_M is said to be of the class $\mathcal{B}^*(A)$ if there exist a $\rho_m \in (0, \rho_M)$ and a non-negative measurable function H defined on $[\rho_m, \rho_M)$ such that

$$(2.1) \quad \nabla U(x) \neq 0, \quad \forall x \in U^{-1}(\rho) \text{ for a.e. } \rho \in [\rho_m, \rho_M),$$

$$(2.2) \quad a^{ij}(x) \partial_i U(x) \partial_j U(x) \leq H(\rho), \quad x \in \partial\Omega_\rho, \quad \rho \in [\rho_m, \rho_M), \text{ and}$$

$$(2.3) \quad \int_{\rho_0}^{\rho_M} \frac{1}{H(\rho)} d\rho = +\infty, \quad \forall \rho_0 \in (\rho_m, \rho_M).$$

Remark 2.1. As remarked in [16], (2.1) says that the set of regular values of the function U is of full Lebesgue measure in $[\rho_m, \rho_M)$. By Sard's theorem ([13]), if $U \in C^n(\mathcal{U})$, then the set of regular values of U is of full Lebesgue measure in $[0, \rho_M)$. Consequently, (2.1) is satisfied by any compact function $U \in C^n(\mathcal{U})$.

The following proposition proved in [16] gives some useful sufficient conditions for a C^2 function in \mathbb{R}^n to be of the class $\mathcal{B}^*(A)$.

Proposition 2.2. *Let $U \in C^2(\mathbb{R}^n)$ be a function such that the Hessian matrix D^2U is bounded under the sup-norm and uniformly positive definite in $\{x \in \mathbb{R}^n : |x| \geq r_0\}$ for some $r_0 > 0$. Then the following holds.*

- a) *There is a constant $c \geq 0$ such that $U + c$ is an unbounded compact function in \mathbb{R}^n .*
- b) *$U + c$ is of the class $\mathcal{B}^*(A)$ with respect to any $n \times n$ matrix-valued function A which is bounded under the sup-norm.*

2.2. Lyapunov-like functions. We recall the following definition from [15].

Definition 2.2. Let U be a C^2 compact function in \mathcal{U} with essential upper bound ρ_M .

- 1) U is called a *Lyapunov function* in \mathcal{U} with respect to (1.1) or L^* , if there is a $\rho_m \in (0, \rho_M)$, called *essential lower bound of U* , and a constant $\gamma > 0$, called *Lyapunov constant of U* , such that

$$(2.4) \quad L^*U(x) \leq -\gamma, \quad x \in \tilde{\mathcal{U}} =: \mathcal{U} \setminus \bar{\Omega}_{\rho_m},$$

where $\tilde{\mathcal{U}}$ is called the *essential domain of U* .

- 2) U is called a *strong Lyapunov function* in \mathcal{U} with respect to (1.1) or L^* , if

$$\lim_{x \rightarrow \partial\mathcal{U}} L^*U(x) = -\infty.$$

Now consider a family of adjoint Fokker-Planck operators:

$$L_{V,A}^* = a^{ij} \partial_{ij}^2 + V^i \partial_i, \quad V = (V^i) \in \mathcal{V}, \quad A = (a^{ij}) \in \mathcal{A},$$

where \mathcal{V} is a subset of vector-valued functions in $C(\mathcal{U}, \mathbb{R}^n)$ under the compact-open topology and \mathcal{A} is a subset of $n \times n$ matrix-valued, \mathcal{U} -everywhere positive definite functions of the class $W_{loc}^{1,\bar{p}}$ under the topology of $W^{1,\bar{p}}$ -convergence on any compact subsets of \mathcal{U} . By Sobolev embedding $W^{1,\bar{p}}(\Omega) \hookrightarrow C(\bar{\Omega})$ for any pre-compact set $\Omega \subset \mathcal{U}$, the topology on \mathcal{A} implies the compact-open topology in the space of $n \times n$ matrix-valued, continuous functions on \mathcal{U} .

Definition 2.3. Let \mathcal{V} , \mathcal{A} be the families defined in the above and U be a C^2 compact function on \mathcal{U} .

- 1) U is a *uniform Lyapunov function* with respect to the family $\{L_{V,A}^* : V \in \mathcal{V}, A \in \mathcal{A}\}$ if it is a Lyapunov function in \mathcal{U} with respect to each $L_{V,A}^*$, $V \in \mathcal{V}$, $A \in \mathcal{A}$, and the essential lower bound ρ_m and Lyapunov constant γ , of U , are independent of $V \in \mathcal{V}$ and $A \in \mathcal{A}$.
- 2) U is a *strong uniform Lyapunov function* with respect to the family $\{L_{V,A}^* : V \in \mathcal{V}, A \in \mathcal{A}\}$ if $L_{V,A}^* U \rightarrow -\infty$ uniformly in $V \in \mathcal{V}$ and $A \in \mathcal{A}$ as $x \rightarrow \partial\mathcal{U}$.
- 3) U is said to be *of the class $\mathcal{B}^*(\mathcal{A})$* with respect to the family \mathcal{A} if it satisfies Definition 2.1 for each $A \in \mathcal{A}$ with some $\rho_m \in (0, \rho_M)$ and function H independent of $A \in \mathcal{A}$.

2.3. Relative sequential compactness and tightness. For a Borel set $\Omega \subset \mathbb{R}^n$, we denote by $M(\Omega)$ the set of Borel probability measures on Ω furnished with the *weak*-topology*, i.e., $\mu_k \rightarrow \mu$ iff

$$\int_{\Omega} f(x) d\mu_k(x) \rightarrow \int_{\Omega} f(x) d\mu(x),$$

for every $f \in C_b(\Omega)$ - the space of bounded, continuous functions in Ω . It is well-known that $M(\Omega)$ with the weak*-topology is metrizable.

Definition 2.4. A subset $\mathcal{M} \subset M(\Omega)$ is said to be *tight* if for any $\epsilon > 0$ there exists a compact subset $K_\epsilon \subset \Omega$ such that $\mu(\Omega \setminus K_\epsilon) < \epsilon$ for all $\mu \in \mathcal{M}$.

Theorem 2.1. (Prokhorov's Theorem, [12, III-59]) *If a subset $\mathcal{M} \subset M(\Omega)$ is tight, then it is relatively sequentially compact in $M(\Omega)$.*

We note that if Ω is compact, then any subset of $M(\Omega)$ is tight.

2.4. Level Set Method. The level set method introduced in [14] contains two main ingredients: an integral identity and a derivative formula, both will play important roles to the measure estimates in our study of existence of stationary measures in the case of degenerate diffusion.

Recall that a bounded open set Ω in \mathbb{R}^n is called a *generalized Lipschitz domain* if

- i) Ω is a disjoint union of finitely many Lipschitz sub-domains;
- ii) intersections of boundaries among these Lipschitz sub-domains only occur at finitely many points.

Theorem 2.2. (Integral identity, [14, Theorem 2.1]) *Assume that A) holds in a domain $\Omega \subset \mathbb{R}^n$ and let $u \in W_{loc}^{1,\bar{p}}(\Omega)$ be a weak stationary solution of the Fokker-Planck equation corresponding to L in Ω . Then for any generalized Lipschitz domain $\Omega' \subset\subset \Omega$ and any function $F \in C^2(\bar{\Omega}')$ with $F|_{\partial\Omega'} = \text{constant}$,*

$$(2.5) \quad \int_{\Omega'} (L^* F)u \, dx = \int_{\partial\Omega'} (a^{ij} \partial_i F \nu_j)u \, ds,$$

where for a.e. $x \in \partial\Omega'$, $(\nu_j(x))$ denotes the unit outward normal vector of $\partial\Omega'$ at x .

Theorem 2.3. (Derivative formula, [14, Theorem 2.2]) *For a given compact function $U \in C^1(\mathcal{U})$ and a function $u \in C(\mathcal{U})$, consider the function*

$$y(\rho) := \int_{\Omega_\rho} u \, dx, \quad \rho \in (0, \rho_M)$$

and the open set

$$(2.6) \quad \mathcal{I} =: \{\rho \in (0, \rho_M) : \nabla U(x) \neq 0, x \in U^{-1}(\rho)\},$$

where ρ_M is the essential upper bound of U and Ω_ρ is the ρ -sublevel set of U for each $\rho \in (0, \rho_M)$. Then y is of class C^1 on \mathcal{I} with derivatives

$$(2.7) \quad y'(\rho) = \int_{\partial\Omega_\rho} \frac{u}{|\nabla U|} \, ds, \quad \rho \in \mathcal{I}.$$

2.5. Existence and regularity of stationary measures. The following existence result is obtained in [15].

Theorem 2.4. ([15, Theorem A]) *Assume that A) holds in \mathcal{U} and (a^{ij}) is everywhere positive definite in \mathcal{U} . If there exists a Lyapunov function in \mathcal{U} with respect to the stationary Fokker-Planck equation (1.1), then the Fokker-Planck equation corresponding to L admits a regular stationary measure in \mathcal{U} with positive density function lying in the space $W_{loc}^{1,\bar{p}}(\mathcal{U})$. If, in addition, the Lyapunov function is unbounded, then the stationary measure is unique in \mathcal{U} .*

The following regularity result for stationary measures of Fokker-Planck equations is proved in [5].

Theorem 2.5. (Bogachev-Krylov-Röckner [5]) *Assume that A) holds and (a^{ij}) is everywhere positive definite in \mathcal{U} . Then any stationary measure μ of the Fokker-Planck equation corresponding to L on \mathcal{U} admits a positive density function $u \in W_{loc}^{1,\bar{p}}(\mathcal{U})$.*

3. UPPER-SEMI-CONTINUITY OF STATIONARY MEASURES IN DRIFT AND DIFFUSION

3.1. Measure estimates via level set method. Applying the level set method, we have the following measure estimates for a stationary measure when a Lyapunov function exists for (1.1).

Lemma 3.1. *Assume that A) holds and (1.1) has a Lyapunov function U in \mathcal{U} with Lyapunov constant γ . Denote Ω_ρ as the ρ -sublevel set of U for each $\rho \in [\rho_m, \rho_M)$, where ρ_m , respectively ρ_M , is the essential lower, respectively upper, bound of U . Then the following hold for any weak solution $u \in W_{loc}^{1,\bar{p}}(\mathcal{U})$ of (1.1):*

- i) $\int_{\mathcal{U}} |L^*U(x)|u(x)dx \leq 2 \int_{\Omega_{\rho_m}} |L^*U(x)|u(x)dx.$
- ii) ([14, Theorem A b)]) *If U satisfies (2.1) and (2.2) for some non-negative measurable function H defined on $[\rho_m, \rho_M)$, then*

$$\mu(\mathcal{U} \setminus \Omega_\rho) \leq e^{-\gamma \int_{\rho_m}^\rho \frac{1}{H(t)} dt}, \quad \rho \in [\rho_m, \rho_M),$$

where μ is the probability measure with density u .

Proof. i) Let $\rho \in (\rho_0, \rho_M)$ and $\rho^* \in (\rho, \rho_M)$. An application of Theorem 2.2 with $F = U$, $\Omega = \Omega_{\rho^*}$, and $\Omega' = \Omega_\rho$ yields that

$$\int_{\Omega_\rho} (L^*U)u \, dx = \int_{\partial\Omega_\rho} u a^{ij} \partial_i \phi(U) \nu_j \, ds,$$

where (ν_j) denote the unit outward normal vectors of $\partial\Omega_\rho$. Since

$$a^{ij}(x) \partial_i U(x) \nu_j(x) \geq 0, \quad x \in \partial\Omega_\rho,$$

we have

$$\int_{\Omega_\rho} (L^*U)u \, dx \geq 0.$$

Letting $\rho \rightarrow \rho_M$ in the above, we obtain

$$\int_{\mathcal{U}} (L^*U)u \, dx \geq 0,$$

or equivalently,

$$(3.1) \quad - \int_{\mathcal{U} \setminus \Omega_{\rho_m}} L^*U d\mu(x) \leq \int_{\Omega_{\rho_m}} L^*U d\mu(x) \leq \int_{\Omega_{\rho_m}} |L^*U| d\mu(x).$$

Since $L^*U < 0$ on $\mathcal{U} \setminus \Omega_{\rho_m}$, (3.1) becomes

$$\int_{\mathcal{U} \setminus \Omega_{\rho_m}} |L^*U| d\mu(x) = - \int_{\mathcal{U} \setminus \Omega_{\rho_m}} L^*U d\mu(x) \leq \int_{\Omega_{\rho_m}} |L^*U| d\mu(x).$$

Hence

$$\int_{\mathcal{U}} |L^*U| d\mu(x) = \int_{\mathcal{U} \setminus \Omega_{\rho_m}} |L^*U| d\mu(x) + \int_{\Omega_{\rho_m}} |L^*U| d\mu(x) \leq 2 \int_{\Omega_{\rho_m}} |L^*U| d\mu(x).$$

ii) To show the applications of the derivative formula in such a measure estimate, we give the proof below in the special case that $\nabla U \neq 0$ everywhere in the essential domain $\mathcal{U} \setminus \bar{\Omega}_{\rho_m}$ of U .

In this case, we note that for each $\eta \in (\rho_m, \rho_M)$, Ω_η is a C^2 domain, whose boundary $\partial\Omega_\eta$ coincides with $U^{-1}(\eta)$, and the unit outward normal vector $\nu(x)$ of $\partial\Omega_\eta$ at each x is well-defined and equals $\frac{\nabla U(x)}{|\nabla U(x)|}$.

Let $\eta^* \in (\rho_m, \rho_M)$. For any $\eta \in (\rho_m, \eta^*)$, applications of Theorem 2.2 with $F = U$ on $\Omega' = \Omega_{\eta^*}, \Omega_\eta$, respectively, yield that

$$\int_{\partial\Omega_\eta} u a^{ij} \frac{\partial_i U \partial_j U}{|\nabla U|} ds + \int_{\Omega_{\eta^*} \setminus \Omega_\eta} (a^{ij} \partial_{ij}^2 U + V^i \partial_i U) u dx = \int_{\partial\Omega_{\eta^*}} u a^{ij} \frac{\partial_i U \partial_j U}{|\nabla U|} ds.$$

Since the right hand side of the above is non-negative, applications of (2.2) with $H + \epsilon$, $0 < \epsilon \ll 1$, in place of H to the first term of the left hand side of above and the definition of Lyapunov function to the second term of the left hand side of above yield that

$$(3.2) \quad \gamma \int_{\Omega_{\eta^*} \setminus \Omega_\eta} u dx \leq (H(\eta) + \epsilon) \int_{\partial\Omega_\eta} \frac{u}{|\nabla U|} ds, \quad \eta \in [\rho_m, \eta^*].$$

Consider the function

$$y(\eta) = \mu(\Omega_{\eta^*} \setminus \Omega_\eta) = \int_{\Omega_{\eta^*} \setminus \Omega_\eta} u dx, \quad \eta \in (\rho_m, \eta^*).$$

By Theorem 2.3, $y(\eta)$ is of the class C^1 on (ρ_m, η^*) and

$$y'(\eta) = - \int_{\partial\Omega_\eta} \frac{u}{|\nabla U|} ds, \quad \eta \in (\rho_m, \eta^*).$$

Hence by (3.2),

$$(3.3) \quad y'(\eta) + \frac{\gamma}{H(\eta) + \epsilon} y(\eta) \leq 0, \quad \eta \in (\rho_m, \eta^*).$$

For any $\rho_0 \in (\rho_m, \eta^*), \rho \in (\rho_0, \eta^*)$, a direct integration of (3.3) in the interval $[\rho_0, \rho]$ yields that

$$\mu(\Omega_{\eta^*} \setminus \Omega_\rho) \leq \mu(\Omega_{\eta^*} \setminus \Omega_{\rho_0}) e^{-\gamma \int_{\rho_0}^\rho \frac{1}{H(t) + \epsilon} dt} < e^{-\gamma \int_{\rho_0}^\rho \frac{1}{H(t) + \epsilon} dt}.$$

The proof is complete by letting $\eta^* \rightarrow \rho_M, \rho_0 \rightarrow \rho_m$, and $\epsilon \rightarrow 0$ in the above. \square

Remark 3.1. As to be seen in a separate work ([17]), part ii) of the above lemma will also be useful in characterizing local concentration of a family of stationary measures associated with a so-called null family of diffusion matrices, as noises tend to zero.

3.2. Upper-semi-continuity of a family of stationary measures. Consider a family of adjoint Fokker-Planck operators:

$$L_{V,A}^* = a^{ij} \partial_{ij}^2 + V^i \partial_i, \quad V = (V^i) \in \mathcal{V}, \quad A = (a^{ij}) \in \mathcal{A},$$

where \mathcal{V} is a subset of vector-valued functions in $C(\mathcal{U}, \mathbb{R}^n)$ under the compact-open topology and \mathcal{A} is a subset of $n \times n$ matrix-valued, \mathcal{U} -everywhere positive definite functions of the class $W_{loc}^{1,\bar{p}}$ under the topology of $W^{1,\bar{p}}$ -convergence on any compact subsets of \mathcal{U} . For each $V \in \bar{\mathcal{V}}$ and $A \in \bar{\mathcal{A}}$, we denote $\mathcal{M}_{V,A}$ as the set of stationary measures of the Fokker-Planck equation corresponding to $L = L_{V,A}$, i.e.,

$$(3.4) \quad \mathcal{M}_{V,A} = \{\mu \in M(\mathcal{U}) : \mu \text{ satisfies (1.4) with } L^* = L_{V,A}^*\},$$

where $M(\mathcal{U})$ denotes the space of the Borel probability measures on \mathcal{U} furnished with the weak*- topology.

Assume that there exists a uniform Lyapunov function in \mathcal{U} with respect to the family $\{L_{V,A}^* : V \in \mathcal{V}, A \in \mathcal{A}\}$. Then it follows from Theorem 2.4 that $\mathcal{M}_{V,A}$ is non-empty for

each $V \in \mathcal{V}$ and $A \in \mathcal{A}$. The following theorem, from which Theorem B in Section 1 follows, says that under additional conditions on the uniform Lyapunov function on \mathcal{U} the map $(V, A) \in \mathcal{V} \times \mathcal{A} \mapsto \mathcal{M}_{V,A}$ is upper semi-continuous and can be extended to $\bar{\mathcal{V}} \times \bar{\mathcal{A}}$ in an upper semi-continuous way. This amounts to showing the relative sequential compactness of $\mathcal{M} = \bigcup_{V \in \mathcal{V}, A \in \mathcal{A}} \mathcal{M}_{V,A}$ in $M(\mathcal{U})$.

The following theorem is just Theorem B in Section 1.

Theorem 3.1. *Assume that there is a uniform Lyapunov function U in \mathcal{U} with respect to the family $\{L_{V,A}^* : V \in \mathcal{V}, A \in \mathcal{A}\}$. Denote ρ_m , respectively ρ_M , as the essential lower, respectively upper, bound of U , and Ω_ρ as the ρ -sublevel set of U in \mathcal{U} for each $\rho \in [\rho_m, \rho_M]$. Further assume that*

i) *either U is a uniform strong Lyapunov function such that*

$$(3.5) \quad \sup_{V \in \mathcal{V}, A \in \mathcal{A}} (|V|_{C(\bar{\Omega}_{\rho_m})} + |A|_{C(\bar{\Omega}_{\rho_m})}) < \infty;$$

or

ii) *U is of the class $\mathcal{B}^*(\mathcal{A})$.*

Then the following holds:

- a) *The set $\mathcal{M} = \bigcup_{V \in \mathcal{V}, A \in \mathcal{A}} \mathcal{M}_{V,A}$ is relatively sequentially compact in $M(\mathcal{U})$;*
- b) *For any $(V_0, A_0) \in \bar{\mathcal{V}} \times \bar{\mathcal{A}}$ and any sequence $\{(V_k, A_k)\} \subset \mathcal{V} \times \mathcal{A}$ with $(V_k, A_k) \rightarrow (V_0, A_0)$, there is a subsequence (V_{k_i}, A_{k_i}) such that $\mu_{V_{k_i}, A_{k_i}}$ converges to a stationary measure $\mu_0 \in \mathcal{M}_{V_0, A_0}$. In particular, $\mathcal{M}_{V_0, A_0} \neq \emptyset$.*

Proof. By Theorem 2.5, every stationary measure $\mu_{V,A}$, for $V \in \mathcal{V}$, $A \in \mathcal{A}$, admits a positive density function in $W_{loc}^{1,\bar{p}}(\mathcal{U})$ which is necessarily a weak solution of (1.1).

In the case i), we have

$$\gamma_{V,A}(\rho) =: \inf_{x \in \mathcal{U} \setminus \Omega_\rho} |L_{V,A}^* U(x)| \rightarrow +\infty,$$

as $\rho \rightarrow \rho_M$, uniformly with respect to $V \in \mathcal{V}, A \in \mathcal{A}$. By (3.5),

$$C =: \sup_{V \in \mathcal{V}, A \in \mathcal{A}} |L_{V,A}^* U|_{C(\bar{\Omega}_{\rho_m})} < \infty.$$

It then follows from Lemma 3.1 i) that

$$(3.6) \quad \begin{aligned} \mu_{V,A}(\mathcal{U} \setminus \bar{\Omega}_\rho) \leq \mu_{V,A}(\mathcal{U} \setminus \Omega_\rho) &\leq \gamma_{V,A}^{-1}(\rho) \int_{\mathcal{U} \setminus \Omega_\rho} |L_{V,A}^* U(x)| d\mu(x) \\ &\leq \gamma_{V,A}^{-1}(\rho) \int_{\mathcal{U}} |L_{V,A}^* U(x)| d\mu(x) \leq 2C \gamma_{V,A}^{-1}(\rho) \rightarrow 0, \end{aligned}$$

as $\rho \rightarrow \rho_M$, uniformly with respect to $V \in \mathcal{V}, A \in \mathcal{A}$.

In the case ii), we let γ be a Lyapunov constant of U and H be a function on $[\rho_m, \rho_M]$ satisfying Definition 2.3 3) with respect to the family \mathcal{A} . It follows from Lemma 3.1 ii) and the condition (2.3) that

$$(3.7) \quad \mu_{V,A}(\mathcal{U} \setminus \bar{\Omega}_\rho) \leq \mu_{V,A}(\mathcal{U} \setminus \Omega_\rho) \leq e^{-\gamma \int_{\rho_m}^\rho \frac{1}{H(t)} dt} \rightarrow 0,$$

as $\rho \rightarrow \rho_M$, uniformly with respect to $V \in \mathcal{V}, A \in \mathcal{A}$.

It now follows from (3.6), (3.7) that \mathcal{M} is tight, hence by Theorem 2.1 it is relatively sequentially compact. This proves a).

To prove b), we let $\{(V_k, A_k)\} \subset \mathcal{V} \times \mathcal{A}$ be a sequence such that $(V_k, A_k) \rightarrow (V_0, A_0) \in \bar{\mathcal{V}} \times \bar{\mathcal{A}}$. By Theorem 2.4, $\mathcal{M}_{V_k, A_k} \neq \emptyset$ for each k . By a), we may assume without loss of generality that $\mu_k =: \mu_{V_k, A_k}$ converges, under weak*-topology, to a measure $\mu_0 \in M(\mathcal{U})$. Since

$$\int_{\mathcal{U}} L_{V_k, A_k}^* f(x) d\mu_k(x) = 0, \quad f \in C_0^\infty(\mathcal{U}), \quad k = 1, 2, \dots,$$

passing to the limit $k \rightarrow \infty$ yields that

$$\int_{\mathcal{U}} L_{V_0, A_0}^* f(x) d\mu_0(x) = 0, \quad f \in C_0^\infty(\mathcal{U}),$$

i.e., $\mu_0 \in \mathcal{M}_{V_0, A_0}$. Hence $\mathcal{M}_{V_0, A_0} \neq \emptyset$. \square

Remark 3.2. The upper semi-continuity of the set-valued map $\bar{\mathcal{V}} \times \bar{\mathcal{A}} \rightarrow \bar{\mathcal{M}} : (V, A) \mapsto \mathcal{M}_{V, A} \cap \bar{\mathcal{M}}$, shown in above theorem, resembles the deterministic case for which it is well-known that global attractors associated with a continuous family of dissipative systems vary upper semi-continuously.

4. STATIONARY MEASURES CORRESPONDING TO DEGENERATE DIFFUSION

In this section, we will consider the problem of the existence of stationary measures of the Fokker-Planck equation corresponding to L when the diffusion matrix $A = (a^{ij})$ is only positive semi-definite in \mathcal{U} . As in [7], we will tackle this problem by perturbing A with a family of positive definite matrixes then pass weak*-limit of the corresponding stationary measures as the perturbations tend to zero.

The following theorem is just Theorem A in Section 1.

Theorem 4.1. *Assume that there exists a Lyapunov function U with respect to the operator L_{V_0, A_0}^* , where $A_0 = (a_0^{ij})$ with $a_0^{ij} \in W_{loc}^{1, \bar{p}}(\mathcal{U})$, $i, j = 1, \dots, n$, and $V_0 \in C(\mathcal{U}, \mathbb{R}^n)$. Also assume that*

- i) *either U is a strong Lyapunov function in \mathcal{U} with respect to L_{V_0, A_0}^* ; or*
- ii) *U is of the class $\mathcal{B}^*(A_0)$.*

Then the Fokker-Planck equation corresponding to $L = L_{V_0, A_0}$ admits a stationary measure in \mathcal{U} .

Proof. Let

$$H^*(\rho) = \max_{x \in \partial\Omega_\rho} a_0^{ij}(x) \partial_i U(x) \partial_j U(x), \quad \rho \in [0, \rho_M),$$

where Ω_ρ denotes the ρ -sublevel set of U in \mathcal{U} . It is clear that the function H^* is upper semi-continuous, hence measurable.

Claim 1. If U is of the class $\mathcal{B}^*(A_0)$, then there is a positive measurable function H_1 on $[0, \rho_M)$ such that $\min_{t \in [0, \rho]} (H_1 - H^*)(t) =: c_\rho > 0$ for any $\rho \in [0, \rho_M)$ and $\int_0^{\rho_M} \frac{1}{H_1(t)} dt = +\infty$.

Since U is of the class $\mathcal{B}^*(A_0)$, there exist a constant $\rho_m \geq 0$ and a non-negative measurable function H on $[\rho_m, \rho_M)$ such that

$$a_0^{ij}(x) \partial_i U(x) \partial_j U(x) \leq H(\rho), \quad x \in \partial\Omega_\rho, \quad \rho \in [\rho_m, \rho_M), \text{ and}$$

$$\int_{\rho_m}^{\rho_M} \frac{1}{H(t)} dt = +\infty.$$

We let $\{s_i : i = 0, 1, \dots\}$ be a strictly increasing sequence in $[\rho_m, \rho_M)$ such that $s_0 = \rho_m$ and $\lim_{i \rightarrow \infty} s_i = \rho_M$. For each $i = 1, 2, \dots$, choose a continuous function $u_i : [s_{i-1}, s_i] \rightarrow [\frac{1}{2}, 1]$ such that $u_i(s_{i-1}) = u_i(s_i) = 1$ and $u_i(t) < 1$, $t \in (s_{i-1}, s_i)$. Since $\lim_{n \rightarrow \infty} \int_{s_{i-1}}^{s_i} \frac{1}{H+u_i^n} dt = \int_{s_{i-1}}^{s_i} \frac{1}{H} dt < \infty$, there exists an integer $n(i) \in \mathbb{N}$ such that $\int_{s_{i-1}}^{s_i} \frac{1}{H+u_i^{n(i)}} dt \geq \frac{1}{2} \int_{s_{i-1}}^{s_i} \frac{1}{H} dt$.

Let $H_1(t) = H^*(t) + u_i^{n(i)}(t)$, $t \in [s_{i-1}, s_i]$, $i = 1, 2, \dots$, and $H_1(t) = H^*(t) + 1$, $t \in [0, \rho_m]$. For a given $\rho \in [0, \rho_M)$, there exists i_0 sufficiently large such that $\rho \leq s_{i_0}$. It is clear that for $t \in [0, \rho]$, $(H_1 - H^*)(t) \geq \min_{i \leq i_0} u_i^{n(i)}(t) \geq \min_{i \leq i_0} 2^{-n(i)}$. Hence H_1 is measurable and $c_\rho > 0$. Moreover, since $H^* \leq H$, we have

$$\int_{\rho_m}^{\rho_M} \frac{1}{H_1} dt \geq \sum_{i=1}^{\infty} \int_{s_{i-1}}^{s_i} \frac{1}{H + u_i^{n(i)}} dt \geq \frac{1}{2} \sum_{i=1}^{\infty} \int_{s_{i-1}}^{s_i} \frac{1}{H} dt = +\infty.$$

This proves *Claim 1*.

Now we let V_0 be fixed and perturb $A_0 = (a_0^{ij})$ by a set \mathcal{A} of everywhere positive definite, matrix-valued functions in \mathcal{U} . Consider the function

$$\epsilon(x) = \frac{\min\{\epsilon_0, H_1(U(x)) - H^*(U(x))\}}{|D^2U(x)| + |\nabla U(x)|^2 + 1}, \quad x \in \mathcal{U},$$

where D^2U denotes the Hessian matrix of U and

$$\epsilon_0 = \begin{cases} 1, & \text{in the case i);} \\ \frac{\gamma}{2}, & \text{in the case ii)} \end{cases}$$

with γ being the Lyapunov constant of U in the case ii). By *Claim 1*, the function $\epsilon(x)$ has a positive minimum on any compact subset of \mathcal{U} .

Let \mathcal{A} be the family of everywhere positive definite, matrix-valued functions $A = (a^{ij})$ in \mathcal{U} of the class $W_{loc}^{1, \bar{p}}$ such that

$$\max_{1 \leq i, j \leq n} |a^{ij}(x) - a_0^{ij}(x)| \leq \epsilon(x), \quad x \in \mathcal{U}.$$

Then for each $A \in \mathcal{A}$, we clearly have

$$\begin{aligned} |(L_{V_0, A}^* - L_{V_0, A_0}^*)U(x)| &\leq \epsilon_0, \\ |(a^{ij}(x) - a_0^{ij}(x))\partial_i U(x)\partial_j U(x)| &\leq H_1(U(x)) - H^*(U(x)). \end{aligned}$$

It follows that, with respect to the family $\{L_{V_0, A}^* : A \in \mathcal{A}\}$, U is a uniform strong Lyapunov function in the case i) and a uniform Lyapunov function of the class $\mathcal{B}^*(\mathcal{A})$ for the function H_1 in the case ii) by *Claim 1*.

Claim 2. \mathcal{A} contains A_0 as a limit point.

Using partition of unity, there exists a sequence $\{f_i : i \in \mathcal{I}\}$, $\mathcal{I} \subset \mathbb{N}$, of non-negative, C^∞ functions on \mathcal{U} such that $\text{supp}(f_i) \subset \subset \mathcal{U}$, $i \in \mathcal{I}$, $\{\text{supp}(f_i) : i \in \mathcal{I}\}$ is a locally finite cover of \mathcal{U} , and

$$\sum_{i \in \mathcal{I}} f_i(x) = 1, \quad x \in \mathcal{U}.$$

For each $i \in \mathcal{I}$, denote $c_i = \min_{x \in \text{supp}(f_i)} \epsilon(x)$. Since each $\text{supp}(f_i)$ is a compact subset of \mathcal{U} , we have $c_i > 0$ for each $i \in \mathcal{I}$. Consider the function $G(x) = \sum_{i \in \mathcal{I}} c_i f_i(x)$, $x \in \mathcal{U}$. Then it is not hard to see that G is a well-defined C^∞ function, and $0 < G(x) \leq \epsilon(x)$, $x \in \mathcal{U}$.

Let $A_\epsilon(x) = A_0 + \epsilon G(x)I$, $\epsilon \in (0, 1]$, $x \in \mathcal{U}$, where I denotes the identity matrix. Then it is clear that $\{A_\epsilon\} \subset \mathcal{A}$ and $A_\epsilon \rightarrow A_0$ as $\epsilon \rightarrow 0$. This proves *Claim 2*.

Since $A_0 \in \bar{\mathcal{A}}$, an application of Theorem 3.1 yields the existence of a stationary measure of the Fokker-Planck equation in \mathcal{U} corresponding to L with $V = V_0, A = A_0$. \square

Remark 4.1. We note that when A is degenerate in \mathcal{U} the stationary measures need not admit density functions. For example, consider $\mathcal{U} = \mathbb{R}^1$, $A \equiv 0$, and $V(x) = x$, $x \in \mathbb{R}^1$. It is easy to see by a simple calculation that the corresponding Fokker-Planck equation admits no weak stationary solution at all. However, it is clear that the Dirac measure at the origin is a stationary measure of the corresponding Fokker-Planck equation.

When applying the above theorem to \mathbb{R}^n , we obtain a generalization to the result of Bogachev-Röckner in [7] in the case of degenerate diffusion. Indeed, in the following corollary, part a) is precisely the result of Bogachev-Röckner when A is degenerate in \mathbb{R}^n .

Corollary 4.1. *Let $\mathcal{U} = \mathbb{R}^n$ and consider $A = (a^{ij})$ with $a^{ij} \in W_{loc}^{1, \bar{p}}(\mathbb{R}^n)$, $i, j = 1, \dots, n$, and $V \in C(\mathbb{R}^n, \mathbb{R}^n)$. Further assume that there is a function $U \in C^2(\mathbb{R}^n)$ such that*

- a) *either U satisfies (1.5) and (1.6); or*
- b) *U satisfies (1.7), both A and D^2U are bounded under the sup-norm, and there is a constant $r_0 > 0$ such that D^2U is uniformly positive definite in $\{x \in \mathbb{R}^n : |x| \geq r_0\}$.*

Then the Fokker-Planck equation corresponding to $L = L_{V,A}$ admits a stationary measure in \mathbb{R}^n .

Proof. In the case a), we note that $U + c$ is a non-negative function for some constant $c \geq 0$, hence is a strong Lyapunov function in \mathbb{R}^n . The existence of stationary measures then follows from Theorem 4.1 i).

In the case b), we have by Proposition 2.2 that there is a constant $c \geq 0$ such that $U + c$ is an unbounded compact function in \mathbb{R}^n which is of the class $\mathcal{B}^*(A)$. Since $U + c$ also satisfies (1.7), it is a Lyapunov function in \mathbb{R}^n . Therefore, all conditions in Theorem 4.1 ii) are satisfied. \square

REFERENCES

- [1] S. Albeverio, V. Bogachev, and M. Röckner, On uniqueness of invariant measures for finite- and infinite-dimensional diffusions, *Comm. Pure and Appl. Math.* **52** (1999), 325–362.
- [2] A. Arapostathis, V. S. Borkar, and M. K. Ghosh, *Ergodic Control of Diffusion Processes*, Cambridge University Press, Cambridge, 2012.
- [3] A. Bensoussan, *Perturbation Methods in Optimal Control* (Translated from French by C. Tomson), Wiley/Gauthier-Villars Series in Modern Applied Mathematics, John Wiley & Sons, Ltd., Chichester; Gauthier-Villars, Montrouge, 1988.
- [4] V. I. Bogachev, G. Da Prato, and M. Röckner, On parabolic equations for measures, *Comm. Partial Diff. Equations* **33** (2008), 397–418.

- [5] V. I. Bogachev, N. V. Krylov, and M. Röckner, On regularity of transition probabilities and invariant measures of singular diffusions under minimal conditions, *Comm. Partial Diff. Equations* **26** (2001), 2037–2080.
- [6] V. I. Bogachev, N. V. Krylov, and M. Röckner, Elliptic and parabolic equations for measures, *Russian Math. Surveys* **64** (2009), 973–1078.
- [7] V. Bogachev and M. Röckner, A generalization of Khasminskii’s theorem on the existence of invariant measures for locally integrable drifts, *Theory Probab. Appl.* **45** (2001), 363–378.
- [8] V. I. Bogachev and M. Röckner, Invariant measures of diffusion processes: regularity, existence, and uniqueness problems. Stochastic partial differential equations and applications (Trento, 2002), 69–87, Lecture Notes in Pure and Appl. Math., 227, Dekker, New York, 2002.
- [9] V. I. Bogachev, M. Röckner, and S. V. Shaposhnikov, On positive and probability solutions of the stationary Fokker-Planck-Kolmogorov equation, *Dokl. Math.* **85** (2012), 350–354.
- [10] V. I. Bogachev, M. Röckner, and V. Stannat, Uniqueness of invariant measures and maximal dissipativity of diffusion operators on L^1 . Infinite dimensional stochastic analysis (Amsterdam, 1999), 39–54, Verh. Afd. Natuurkd. 1. Reeks. K. Ned. Akad. Wet., 52, R. Neth. Acad. Arts Sci., Amsterdam, 2000.
- [11] V. I. Bogachev, M. Röckner, and V. Stannat, Uniqueness of solutions of elliptic equations and uniqueness of invariant measures of diffusions. (Russian) *Mat. Sb.* **193** (2002), no. 7, 3–36; translation in *Sb. Math.* **193** (2002), no. 7–8, 945–976.
- [12] C. Dellacherie and P. A. Meyer, Probabilities and Potential, Vol. 29, North-Holland Mathematics Studies, North-Holland Publishing Co., Amsterdam, 1978.
- [13] V. Guillemin and A. Pollack, Differential Topology, Prentice-Hall, 1974, Reprinted by AMS Chelsea Publishing, Providence, 2010.
- [14] W. Huang, M. Ji, Z. Liu, and Y. Yi, Integral Identity and Measure Estimates for Stationary Fokker-Planck Equations, submitted.
- [15] W. Huang, M. Ji, Z. Liu, and Y. Yi, Steady states of Fokker-Planck equations: I. Existence, submitted.
- [16] W. Huang, M. Ji, Z. Liu, and Y. Yi, Steady states of Fokker-Planck equations: II. Non-existence, submitted.
- [17] W. Huang, M. Ji, Z. Liu, and Y. Yi, Concentration and limit behaviors of stationary measures, preprint, 2012.
- [18] A. V. Skorohod, Asymptotic Methods in the Theory of Stochastic Differential Equations (Translated from Russian by H. H. McFaden), Translations of Mathematical Monographs, 78. American Mathematical Society, Providence, RI, 1989.
- [19] A. Yu. Veretennikov, Bounds for the mixing rate in the theory of stochastic equations, *Theory Probab. Appl.* **32** (1987), 273–281.
- [20] A. Yu. Veretennikov, On polynomial mixing bounds for stochastic differential equations, *Stochastic Process. Appl.* **70** (1997), 115–127.
- [21] A. Yu. Veretennikov, On polynomial mixing and convergence rate for stochastic difference and differential equations, *Theory Probab. Appl.* **44** (1999), 361–374.

W. HUANG: WU WEN-TSUN KEY LABORATORY OF MATHEMATICS, USTC, CHINESE ACADEMY OF SCIENCES, HEFEI ANHUI 230026, PRC

E-mail address: wenh@mail.ustc.edu.cn

M. JI: (CORRESPONDING AUTHOR) ACADEMY OF MATHEMATICS AND SYSTEM SCIENCES, CHINESE ACADEMY OF SCIENCES, BEIJING 100080, PRC

E-mail address: jimin@math.ac.cn

Z. LIU: SCHOOL OF MATHEMATICS, JILIN UNIVERSITY, CHANGCHUN, 130012, PRC

E-mail address: zxliu@jlu.edu.cn

Y. YI: SCHOOL OF MATHEMATICS, GEORGIA INSTITUTE OF TECHNOLOGY, ATLANTA, GA 30332, USA, AND SCHOOL OF MATHEMATICS, JILIN UNIVERSITY, CHANGCHUN, 130012, PRC

E-mail address: yi@math.gatech.edu