STEADY STATES OF FOKKER-PLANCK EQUATIONS:
I. EXISTENCE

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ABSTRACT. This is the first paper in a series concerning the study of steady states of a
Fokker-Planck equation in a general domain in $\mathbb{R}^n$ with $L_{loc}^{\bar{p}}$ drift term and $W_{loc}^{1,\bar{p}}$ diffusion
term for any $\bar{p} > n$. In this paper, by using the level set method especially the inte-
gral identity which we introduced in [24], we obtain several new existence results of steady
states, including stationary solutions and measures, of the Fokker-Planck equation with
non-degenerate diffusion under Lyapunov-like conditions. As applications of these results,
we give some examples on the noise stabilization of an unstable equilibrium and the existence
and uniqueness of steady states subject to boundary degeneracy of diffusion in a bounded
domain.

1. INTRODUCTION

We consider a system of ordinary differential equations (ODE)

$$(1.1) \qquad \dot{x} = V(x), \qquad x \in \mathcal{U} \subset \mathbb{R}^n$$

under white noise perturbation $G(x)\dot{W}$, where $\mathcal{U} \subset \mathbb{R}^n$ is a connected open set which can be
bounded, unbounded, or the entire space $\mathbb{R}^n$, $V = (V^i)$ is a vector field on $\mathcal{U}$, called the *drift
field*, $G = (g^{ij})$ is an $n \times m$ matrix-valued function on $\mathcal{U}$ for some positive integer m, called
the *noise matrix*, and W is the standard m-dimensional Brownian motion. This leads to the
following system of Itô stochastic differential equations

$$(1.2) \qquad \mathrm{d}x = V(x)\mathrm{d}t + G(x)\mathrm{d}W, \qquad x \in \mathcal{U} \subset \mathbb{R}^n.$$

A fundamental issue is the impact of noises on the basic dynamics of (1.1). On one hand, as
all differential equations are idealized models of physical motions which are actually subjected
to noise perturbations, one would like to know to what extent or under what conditions basic
dynamics of a differential equation are robust under small noise perturbations. On the other
hand, it has also been observed that large noises can destroy otherwise deterministically
robust or unstable dynamics in a system, leading to interesting dynamical phenomena such
as random destabilization or stabilization.

Noise impacts on dynamics of differential equations have been extensively studied using a
"trajectory based approach" within the framework of random dynamical systems which are

skew-product flows over measurable, ergodic base flows. Due to its similarity with the study of a deterministic skew-product flow, this approach has been proven to be very useful in studying random perturbations with respect to problems such as random attractors, random invariant manifolds, and random bifurcations etc. We refer the reader to [4] and references therein for many interesting studies on these problems. However, when a physical system either has sufficiently high complexity or contains intrinsic uncertainties, "trajectory based" model and study would not provide much information to its dynamical description. Instead, a "distribution based approach" using Fokker-Planck equations seems necessary to synthesize the typical patterns of dynamics. The later approach is significantly different from the former one not only because of distinct dynamical objects considered but also because completely different natures of dynamical systems they adopt. Indeed, there are very limited cases of Itô stochastic differential equations which can be converted into random dynamical systems, unless white noises of Stratonovich type are considered in (1.2) instead.

In a series of works, we will study this fundamental issue by linking steady states of the Fokker-Planck equation associated with (1.2) in distribution sense to stochastic stability and robustness of invariant measures of (1.1). While Fokker-Planck equations are of importance on their own rights in various problems arising in chemical reactions, statistical physics, and fluid mechanics, our primary goal in this study is to develop a theoretical framework, through the study of stationary Fokker-Planck equations, for one to analyze the impact of white noises on compact invariant sets and invariant measures of an ODE system like (1.1).

In the case that both V and G are locally Lipschitz in $\mathcal{U}$, the stochastic differential equation (1.2) generates a Markov (diffusion) process in $\mathcal{U}$ whose *transition probability density function* $p(t, \xi, x)$, if exists, is actually a fundamental solution of the Fokker-Planck equation associated with (1.2). More precisely, denote $A = (a^{ij}) = \frac{GG^\top}{2}$, called the *diffusion matrix*. Then for any bounded, measurable function $f(x) \geq 0$ with $\int_{\mathcal{U}} f(x)\mathrm{d}x = 1$,

$$u(x, t) = \int_{\mathcal{U}} p(t, \xi, x) f(\xi) \mathrm{d}\xi$$

(formally) satisfies the *Fokker-Planck equation* (also called *Kolmogorov forward equation*):

(1.3)
$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = Lu(x,t), & x \in \mathcal{U}, \ t > 0, \\ u(x,t) \geq 0, & \int_{\mathcal{U}} u(x,t)\mathrm{d}x = 1 \end{cases}$$

with $u(x,0) = f(x)$, where L is the Fokker-Planck operator defined as

$$Lg(x) = \partial_{ij}^2 (a^{ij}(x)g(x)) - \partial_i(V^i(x)g(x)), \qquad g \in C^2(\mathcal{U}).$$

Among solutions of the Fokker-Planck equations (1.3), of particular interests are the *stationary solutions*, i.e., solutions $u(x)$ of the *stationary Fokker-Planck equation*

(1.4)
$$\begin{cases} Lu(x) = 0, & x \in \mathcal{U} \\ u(x) \geq 0, & \int_{\mathcal{U}} u(x)\mathrm{d}x = 1. \end{cases}$$

We note that, since $A = (a^{ij})$ is everywhere positive semi-definite, L is an elliptic operator (can be degenerate though).

In the above and also through the rest of the paper, we use short notations $\partial_i = \frac{\partial}{\partial x_i}$, $\partial_{ij}^2 = \frac{\partial^2}{\partial x_i \partial x_j}$, and we also adopt the usual summation convention on $i, j = 1, 2, \cdots, n$ whenever applicable.

For the generality of our theory, we will assume less regularity conditions on both the drift field and the diffusion matrix. Following [12]-[17] and others, we make the following standard hypothesis:

A) $a^{ij} \in W_{\text{loc}}^{1,\bar{p}}(\mathcal{U})$, $V^i \in L_{\text{loc}}^{\bar{p}}(\mathcal{U})$ for all $i, j = 1, \cdots, n$, where $\bar{p} > n$ is fixed.

There are many situations arising in applications in which the drift field is only continuous, or piece-wise continuous, or even measurable. Also, a white noise perturbed differential inclusion can often lead to a stochastic selection of form (1.2) in which V is of the class L_{loc}^p for some $p \geq 1$ in general (see e.g., [18]). Even when a Lipschitz drift field is considered, there is no physical reason for a multiplicative (i.e., spatially non-homogenous) white noise perturbation to have sufficiently smooth coefficients.

Due to the weak regularity condition A), it is necessary to consider *weak stationary solutions* of the Fokker-Planck equation (1.3) or *weak solutions* of the stationary Fokker-Planck equation (1.4), i.e., continuous functions satisfying the following weak form of the stationary Fokker-Planck equation:

$$(1.5) \qquad \begin{cases} \int_{\mathcal{U}} L^* f(x) u(x) \mathrm{d}x = 0, & \text{for all } f \in C_0^\infty(\mathcal{U}) \\ u(x) \geq 0, \quad \int_{\mathcal{U}} u(x) \mathrm{d}x = 1, \end{cases}$$

where

$$L^* = a^{ij} \partial_{ij}^2 + V^i \partial_i$$

is the adjoint Fokker-Planck operator and $C_0^\infty(\mathcal{U})$ denotes the space of C^∞ functions on $\mathcal{U}$ with compact supports.

In the absence of weak solutions satisfying (1.5), one further considers Borel probability measure-valued solutions μ satisfying

$$(1.6) \qquad V^i \in L_{loc}^1(\mathcal{U}, \mu), \quad i = 1, 2, \cdots, n, \text{ and,}$$

$$(1.7) \qquad \int_{\mathcal{U}} L^* f(x) \mathrm{d}\mu(x) = 0, \qquad \text{for all } f \in C_0^\infty(\mathcal{U}).$$

We refer to a Borel probability measure μ satisfying (1.6), (1.7) as a *stationary measure* of the Fokker-Planck equation (1.3) or a *measure solution* of the stationary Fokker-Planck equation (1.4). We call a stationary measure μ of (1.3) *regular* if μ admits a continuous density function u with respect to the Lebesgue measure, i.e., $\mathrm{d}\mu(x) = u(x)\mathrm{d}x$. It is clear that such a density u is necessarily a weak stationary solution of the Fokker-Planck equation (1.3). In fact, under the condition A) and the condition that (a^{ij}) is everywhere positive definite in $\mathcal{U}$, it follows from a regularity theorem due to Bogachev-Krylov-Röckner ([10], also see Theorem 2.2 in this paper) that all stationary measures of (1.3) are regular with density functions $u \in W_{loc}^{1,\bar{p}}(\mathcal{U})$. If $a^{ij} \in C_{loc}^{2,\alpha}(\mathcal{U})$, $V^i \in C_{loc}^{1,\alpha}(\mathcal{U})$, $i, j = 1, \cdots, n$, for some $\alpha \in (0,1)$, then it follows from the standard Schauder theory that the density functions become classical solutions of (1.4).

Suppose that $\mathcal{U} = \mathbb{R}^n$, V, G are locally Lipschitz, and (1.2) generates a global in time diffusion process in $\mathbb{R}^n$. It is well-known that any invariant measure of the diffusion process is necessarily a stationary measure of the Fokker-Planck equation (1.3), and vice versa under some suitable conditions (e.g., those in Theorem 2.12 and Proposition 2.9 of [16], see also [11, 17] for more interesting discussions in this regard in particular with respect to the uniqueness of stationary measures and their invariance). In cases that either (1.2) fails to generate a diffusion process due to low regularity of V, G etc or none stationary measure of (1.3) is invariant to the diffusion process, under certain mild conditions one can still define a semigroup generated by L^* so that there is a generalized diffusion process associated to it and stationary measures of (1.3) become invariant measures of this semigroup (see [17, 30]). We refer the reader to [20, 21] for more information on Fokker-Planck equations and Kolmogorov backward equations.

It is well-known that smooth stationary solutions of a Fokker-Planck equation defined on a compact manifold without boundary always exist and are unique when (a^{ij}) is a smooth, everywhere positive definite matrix on the manifold and the drift field is everywhere smooth (see e.g. [36] for an argument using Perron-Frobenius method). In a bounded domain $\Omega \subset \mathbb{R}^n$, with $(a^{ij}) \in W^{1,\bar{p}}(\Omega)$ being uniformly elliptic (see (2.4) for the definition) and $(V^i) \in L^{\bar{p}}(\Omega)$, the existence of weak stationary solutions of the associated Fokker-Planck equation in Ω follows from classical theory of linear elliptic equations (see Bogachev-Röckner [12] or Lemma 2.1 of this paper). These solutions depend on boundary conditions imposed and thus need not be unique.

The existence of stationary solutions of Fokker-Planck equations (1.3) in an unbounded domain necessarily requires certain "dissipation" conditions, for otherwise the Laplacian equation in $\mathbb{R}^n$ provides a simple counter example. Some "dissipation" conditions are also needed for existence of stationary solutions of Fokker-Planck equations (1.3) even in a bounded domain when (a^{ij}) becomes degenerate on the boundary. For instance, taking $a(x) = x^2$, $\mathcal{U} = (0, 1)$, and $V \equiv 0$, the corresponding stationary Fokker-Planck equation $(x^2 u(x))'' = 0$, $x \in (0, 1)$, admits no solution that is non-negative and integrable.

In this paper, we will consider a general domain $\mathcal{U} \subset \mathbb{R}^n$, which can be bounded, unbounded or being $\mathbb{R}^n$, and focus on the existence of stationary measures of (1.3) in $\mathcal{U}$ that under the condition A) and that (a^{ij}) is everywhere positive definite in $\mathcal{U}$, by considering "dissipation" conditions which are compatible to those in deterministic ODE systems. One of such "dissipation" conditions is the existence of a *Lyapunov function* $U \in C^2(\mathcal{U})$ with respect to (1.4) which is a so-called *compact function* in $\mathcal{U}$ (see Section 2.1) satisfying the "dissipation" property that

$$(1.8) \qquad \limsup_{x \to \partial \mathcal{U}} L^* U(x) = \limsup_{x \to \partial \mathcal{U}} (a^{ij}(x) \partial_{ij}^2 U(x) + V^i(x) \partial_i U(x)) \leq -\gamma$$

for some constant $\gamma > 0$, called a *Lyapunov constant*. We note that when $\mathcal{U}$ is unbounded, the notion of $\partial \mathcal{U}$ and the limit $x \to \partial \mathcal{U}$ in the above should be understood under the topology which is defined through a fixed homeomorphism between the extended Euclidean space $\mathbb{E}^n = \mathbb{R}^n \cup \partial \mathbb{R}^n$ and the closed unit ball $\bar{\mathbb{B}}^n = \mathbb{B}^n \cup \partial \mathbb{B}^n$ in $\mathbb{R}^n$ which identifies $\mathbb{R}^n$ with B^n and $\partial \mathbb{R}^n$ with $\mathbb{S}^{n-1}$, and in particular, identifies each $x_* \in \mathbb{S}^{n-1}$ with the infinity element $x_*^\infty \in \partial \mathbb{R}^n$ of the ray through x_* (see Section 2 for detail). Consequently, if $\mathcal{U} = \mathbb{R}^n$, then

$x \to \partial \mathbb{R}^n$ under this topology simply means $x \to \infty$ and (1.8) becomes

$$(1.9) \qquad \limsup_{x \to \infty} L^* U(x) = \limsup_{x \to \infty} (a^{ij}(x)\partial_{ij}^2 U(x) + V^i(x)\partial_i U(x)) \leq -\gamma.$$

In the case $\mathcal{U} = \mathbb{R}^n$, following pioneer works of Has'minskiĭ [23, 22] for locally Lipschitz drift field and noise matrix, the existence of stationary measures of (1.3) for less regular coefficients has been extensively studied under various "dissipation" conditions (see e.g., [1, 5, 8], [10]-[17], [29], [33]-[35]). In particular, Veretennikov ([34]) showed the existence of a stationary measure of (1.3) when (a^{ij}) is continuous and bounded under sup-norm, and V is measurable, locally bounded and satisfies

$$(1.10) \qquad V(x) \cdot x \leq -\gamma, \qquad |x| \gg 1$$

for some positive constant γ depending on (a^{ij}). Later in [12], Bogachev-Röckner showed the existence of a regular stationary measure of (1.3) with density function lying in the space $W^{1,\bar{p}}_{loc}(\mathbb{R}^n)$ when the condition A) holds and there exists a non-negative function $U \in C^2(\mathbb{R}^n)$ with

$$(1.11) \qquad \lim_{x \to \infty} U(x) = +\infty$$

and

$$(1.12) \qquad \lim_{x \to \infty} L^* U(x) = -\infty.$$

Recently, the existence of stationary measures of (1.3) is proved for some cases when (1.9) is satisfied by some positive C^2 function U in $\mathbb{R}^n$, see [3, Theorem 2.6.10] for the case that $(g^{ij}), (V^i)$ are locally Lipschitz and do not grow faster than linearly at ∞ and U is so-called inf-compact function, and also see [14] for the case that A) holds and U satisfies (1.11).

In fact, if V is continuous on $\mathbb{R}^n$, then it is also shown in [12] that stationary measures of (1.3) in $\mathbb{R}^n$ still exist even when (a^{ij}) is degenerate in $\mathbb{R}^n$. We will leave more discussions on the degenerate case of (a^{ij}) to part III of the series ([26]).

One of our main results of this paper is as follows.

Theorem A. *Assume that* A) *holds in $\mathcal{U}$ and (a^{ij}) is everywhere positive definite in $\mathcal{U}$. If there exists a Lyapunov function with respect to (1.4) in $\mathcal{U}$, then (1.3) admits a stationary measure in $\mathcal{U}$ which is regular with positive density lying in the space $W^{1,\bar{p}}_{loc}(\mathcal{U})$. If, in addition, the Lyapunov function is unbounded, then stationary measures are unique in $\mathcal{U}$.*

A Lyapunv function with respect to (1.4) may be regarded as a stochastic counterpart of a Lyapunov function defined for a dissipative dynamical system. Recall that a smooth Lyapunov function U for an ODE system (1.1) is a compact function such that

$$(1.13) \qquad \limsup_{x \to \partial \mathcal{U}} V(x) \cdot \nabla U(x) = \limsup_{x \to \partial \mathcal{U}} V^i(x)\partial_i U(x) \leq -\gamma$$

for some constant $\gamma > 0$. On one hand, it is well-known in dissipative dynamical system theory that the existence of such a smooth Lyapunov function for the deterministic ODE system (1.1) with locally Lipschitz V implies the existence of global attractor of (1.1) in $\mathcal{U}$ (see the Appendix in part II of the series [25]). On the other hand, in the case that both V and G are locally Lipschitz in $\mathcal{U}$, it can be shown using the asymptotic theory of Markov transition probabilities ([19, 22]) that if there exists an unbounded Lyapunov function with respect to

(1.4), then the stationary measure μ of (1.3) (which must be unique in this case) attracts all orbits of the semi-flow defined on the space of probability measures on $\mathcal{U}$ which is generated by the Markov process. In this sense, one may conclude that a stationary measure of (1.3) obtained under the condition (1.8) resembles the global attractor in the deterministic case, though for the existence of a stationary measure the corresponding deterministic counterpart need not be always dissipative (see example (4.9)).

When the Lyapunov constant $\gamma = 0$ in (1.8), in particular $L^*U \leq 0$ near $\partial\mathcal{U}$, the compact function U is referred to as a *weak Lyapunov function*. Such a weak Lyapunov function is insufficient to yield a "dissipation" condition, just like the case of the ODE system when $\gamma = 0$ in (1.13). It turns out that a proper "dissipation" condition requires that U is of the *class of $\mathcal{B}_*(A)$* - a condition controlling the decay rates of $a^{ij}\partial_i U\partial_j U$ near $\partial\mathcal{U}$ (see Section 2.1 for detail), which is purely stochastic and unable to be satisfied by an ODE system. Under this additional condition, we are able to obtain the following result.

Theorem B. *Assume that* A) *holds and* (a^{ij}) *is everywhere positive definite in* $\mathcal{U}$. *If there exists a weak Lyapunov function with respect to* (1.4) *in* $\mathcal{U}$ *which is of the class* $\mathcal{B}_*(A)$, *then* (1.3) *admits a stationary measure in* $\mathcal{U}$ *which is regular with positive density lying in the space* $W^{1,\bar{p}}_{loc}(\mathcal{U})$.

By assuming weaker Lyapunov-like conditions, Theorems A, B allow a much broader class of applications. First of all, in many situations the commonly adopted or physical Lyapunov-like functions are often compact functions, which are not necessarily unbounded and/or associated with a finite (positive or zero) Lyapunov constant. As to be demonstrated in a separate work ([27]), such finiteness of Lyapunov constants are important in studying stochastic bifurcation problems in a general domain because it allows to define a function from the parameter space to Lyapunov constants. Secondly, still as to be seen in [27], when a family of positive definite diffusion matrices are considered, problems of concentration and limit behaviors of stationary measures crucially depend on the existence of Lyapunov functions associated with the family that can be merely of a finite Lyapunov constant. Thirdly but not lastly, Theorems A, B, when combined with the non-existence results in part II of the series ([25]), provide a very useful tool for one to study stochastic bifurcations from the existence to the non-existence of stationary measures with respect to parameters lying in either the drift term or the diffusion term. In such stochastic bifurcations problems, Theorem A is typically used at the non-critical parameter values and Theorem B should be useful at the critical parameter values.

Generality of the domain considered in the above results does allow a wide range of applications. For instance, in a population model describing the time evolution of n-species, the biologically meaningful domain is always taken to be $\mathcal{U} = \mathbb{R}^n_+ =: \{x = (x_i) \in \mathbb{R}^n : x_i > 0, \, i = 1, 2, \cdots, n\}$. Even for a system defined in the entire space $\mathbb{R}^n$, one can also apply the results to a general domain $\mathcal{U} \subset \mathbb{R}^n$ which is either the entire space or a bounded open set, to obtain a global or a local measure, resembling a global or a local attractor of the deterministic system, respectively. Though stationary measures need not exist in general when (a^{ij}) becomes degenerate on the usual boundary of $\mathcal{U}$, Theorem A says that they exist if a Lyapunov function exists in $\mathcal{U}$, and moreover, stationary measures are also unique when

the Lyapunov function is unbounded. Such uniqueness due to degeneracy of noise on the boundary is particularly interesting when applying to a bounded domain because if (a^{ij}) is uniformly elliptic in the domain, then stationary measures are always non-unique.

We remark that if the system (1.1) is defined on $\mathcal{U} \times M$, where $\mathcal{U} \subset \mathbb{R}^n$ is a connected open set and M is a smooth, compact manifold without boundary, then one can modify the definitions of Lyapunov, strong or weak Lyapunov functions in Section 2 in an obvious way by replacing the domain $\mathcal{U} \subset \mathbb{R}^n$ with $\mathcal{U} \times M$. Then the proofs in later sections can be modified accordingly so that Theorems A, B still hold with respect to such a generalized domain.

The proof of our results uses a level set method which crucially relies on the integral identity we derived in [24] (see also Theorem 3.1 below). Such a method, by overcoming limitations of the usual Lyapunov function method, the traditional large deviation theory, and the classical PDE estimates, has the advantage of allowing more delicate measure estimates than those made using classical methods (see [28] for some discussions in this regard), and moreover, unlike existing methods, it works the same in any domain: bounded, unbounded, or the entire space $\mathbb{R}^n$. Indeed, the integral identity reveals fundamental natures of stationary Fokker-Planck equations and plays a similar role as the Pohozaev Identity does to semi-linear elliptic equations. As in [24] and the present series, it enables one to obtain useful measure estimates for a stationary measure in a sub-domain by making use of information of noise distributions on the boundary of the domain. Besides their usefulness in the study of existence and non-existence of stationary measures, these estimates will also play important roles in studying problems like the concentration and limit behaviors of stationary measures when diffusion vanishes, as what we will explore in separate works (see e.g., [27]).

This paper is organized as follows. Section 2 is a preliminary section in which we introduce the notions of boundary, compact and Lyapunov-like functions, for a general domain $\mathcal{U} \subset \mathbb{R}^n$. We will also review an existence result of stationary solutions of (1.3) in a bounded domain from [12], a Harnack inequality from [31], and a regularity theorem from [10]. In Section 3, we recall two identities from [24] - the integral identity and the derivative formula, which are of fundamental importance to the level set method to be adopted in this paper and other parts of the series. In Sections 4, we recall from [24] a measure estimate for stationary measures using a Lyapunov function. Theorem A will be proved using these results and the level set method. Some examples and discussions on the noise stabilization of an unstable equilibrium and the existence and uniqueness of staitonary measure subject to boundary degeneracy of diffusion in a bounded domain are given. Theorem B will be proved in Section 5 using the level set method and a measure estimate for stationary measures using a weak Lyapunov function contained in [24]. Section 6 is an Appendix in which we give a characterization of Lyapunov functions in one dimension.

Through the rest of the paper, for simplicity, we will use the same symbol $|\cdot|$ to denote absolute value of a number, cardinality or Lebesgue measure of a set, and norm of a vector or a matrix.

2. Preliminary

In this section, we will introduce various notions to be used in the rest of the paper including boundary of a general domain in $\mathbb{R}^n$, compact functions, Lyapunov and weak Lyapunov functions, and generalized Lipschitz domains. We will also review a Harnack inequality on linear elliptic equations from [31] and a regularity theorem on stationary Fokker-Planck equations from [10].

Through the rest of the paper, we let $\mathcal{U} \subset \mathbb{R}^n$ be a connected open set which can be bounded, unbounded, or the entire space $\mathbb{R}^n$.

2.1. **Compact functions.** To unify both cases when $\mathcal{U}$ is bounded and unbounded, we consider the extended Euclidean space $\mathbb{E}^n = \mathbb{R}^n \cup \partial\mathbb{R}^n$ which is identified with the closed unit ball $\bar{\mathbb{B}}^n = \mathbb{B}^n \cup \partial\mathbb{B}^n$, through the homeomorphism $p : \mathbb{E}^n \to \bar{\mathbb{B}}^n$ to be defined as follows. Let $\partial\mathbb{R}^n = \{x_*^\infty : x_* \in \mathbb{S}^{n-1}\}$, where for each $x_* \in \mathbb{S}^{n-1}$, x_*^∞ denotes the infinity element of the ray through x_*. Define $p : \mathbb{E}^n \to \bar{\mathbb{B}}^n$:

$$p(x) = \begin{cases} \frac{x}{1+|x|}, & x \in \mathbb{R}^n; \\ x_*, & x = x_*^\infty \in \partial\mathbb{R}^n. \end{cases}$$

Then p clearly identifies $\mathbb{R}^n$ with $\mathbb{B}^n$ and $\partial\mathbb{R}^n$ with $\mathbb{S}^{n-1}$, and it becomes a homeomorphism when the topology of $\partial\mathbb{R}^n$ is defined as that inherited from this identification.

Definition 2.1. We call $\Gamma =: \partial\mathcal{U} \subset \mathbb{E}^n$ the *boundary* of $\mathcal{U}$ if $p(\Gamma)$ is the boundary of $p(\mathcal{U})$ in $\bar{\mathbb{B}}^n$.

For instance, when $\mathcal{U} = \mathbb{R}^n$, $\mathcal{U}$ has only one boundary component $\partial\mathbb{R}^n$, and when $\mathcal{U} = \mathbb{R}^n_+$ - the first octant of $\mathbb{R}^n$ for $n \geq 2$, $\mathcal{U}$ also has only one boundary component Γ which is the union of all non-negative, coordinate hyperplanes and the portion of $\partial\mathbb{R}^n$ lying in between these hyperplanes.

Definition 2.2. Let $U \in C(\mathcal{U})$ be a non-negative function and denote $\rho_M = \sup_{x \in \mathcal{U}} U(x)$, called the *essential upper bound of* U. U is said to be a *compact function in* $\mathcal{U}$ if

   i) $U(x) < \rho_M$, $x \in \mathcal{U}$; and
   ii) $\lim_{x \to \partial\mathcal{U}} U(x) = \rho_M$.

According to the topology we described earlier, $x \to \partial\mathcal{U}$ in the above means that the Hausdorff semi-distance $\text{dist}(p(x), p(\partial\mathcal{U})) \to 0$. Therefore, if Γ^* is a bounded boundary component of $\partial\mathcal{U}$, $x \to \Gamma^*$ under this topology is equivalent to $\text{dist}(x, \Gamma^*) \to 0$. Moreover, if $\mathcal{U} = \mathbb{R}^n$, then $x \to \partial\mathcal{U}$ under this topology simply means $x \to \infty$ in the usual sense.

From the definition, we immediately have the following result.

Proposition 2.1. *An unbounded, non-negative function $U \in C(\mathcal{U})$ is a compact function in $\mathcal{U}$ iff*

$$\lim_{x \to \partial\mathcal{U}} U(x) = +\infty.$$

Consequently, an unbounded, non-negative function $U \in C(\mathbb{R}^n)$ is a compact function in $\mathbb{R}^n$ iff it satisfies (1.11).

For a non-negative function $U \in C(\mathcal{U})$ and each $\rho \in [0, \rho_M)$, where ρ_M is the essential upper bound of U, we denote $\Omega_\rho = \{x \in \mathcal{U} : U(x) < \rho\}$ as the *ρ-sublevel set of U*.

Proposition 2.2. *A non-negative function U is a compact function in $\mathcal{U}$ iff the following holds:*

    a) $\Omega_\rho \subset\subset \mathcal{U}$ *for any* $0 \leq \rho < \rho_M$;
    b) $p(\partial\Omega_\rho) \to p(\partial\mathcal{U})$ *in Hausdorff metric as* $\rho \to \rho_M$.

Proof. Let i) ii) be as in Definition 2.2. It is clear that ii) implies a). We note that b) is equivalent to the following:

    (∗) For any neighborhood $\mathcal{N}$ of $\partial\mathcal{U}$, there exists a $\rho_\mathcal{N} \in (0, \rho_M)$ such that $\partial\Omega_\rho \subset \mathcal{N}$ for all $\rho \in [\rho_\mathcal{N}, \rho_M)$;
    (∗∗) For any $x \in \partial\mathcal{U}$ and any sequence $\rho_k \to \rho_M$, there is a sequence of points $x_k \in \partial\Omega_{\rho_k}$ such that $x_k \to x$.

Now, i) implies (∗) and ii) implies (∗∗) by continuity of U, i.e., i) and ii) together imply b).

Conversely, it is easy to see that a) and b) together imply i), and a) implies ii). $\qquad\square$

Definition 2.3. Let $A = (a^{ij})$ be a everywhere positive semi-definite, $n \times n$ matrix-valued function on $\mathcal{U}$. A compact function $U \in C^1(\mathcal{U})$ with essential upper bound ρ_M is said to be *of the class $\mathcal{B}_*(A)$* if there exist a $\rho_m \in (0, \rho_M)$ and a positive measurable function H on $[\rho_m, \rho_M)$ such that

$$(2.1) \qquad H(\rho) \leq a^{ij}(x)\partial_i U(x)\partial_j U(x), \qquad x \in U^{-1}(\rho), \ \rho \in [\rho_m, \rho_M), \text{ and}$$

$$(2.2) \qquad \int_{\rho_m}^{\rho_M} \frac{1}{H(\rho)}\mathrm{d}\rho < +\infty.$$

2.2. Lyapunov-like functions.
Below, we introduce two type of Lyapunov-like functions with respect to the stationary Fokker-Planck equation (1.4) or the adjoint Fokker-Planck operator L^*, each will play an important role to the existence of stationary measures.

Definition 2.4. Let U be a C^2 compact function in $\mathcal{U}$ with essential upper bound ρ_M.

    1. U is called a *Lyapunov function* in $\mathcal{U}$ with respect to (1.4) or L^*, if there is a $\rho_m \in (0, \rho_M)$, called *essential lower bound of U*, and a constant $\gamma > 0$, called *Lyapunov constant* of U, such that

$$(2.3) \qquad\qquad L^*U(x) \leq -\gamma, \qquad x \in \tilde{\mathcal{U}} =: \mathcal{U} \setminus \bar{\Omega}_{\rho_m},$$

        where $\tilde{\mathcal{U}}$ is called the *essential domain of U*.
    2. U is called a *weak Lyapunov function* in $\mathcal{U}$ with respect to (1.4) or L^*, if it satisfies (2.3) in an essential domain $\tilde{\mathcal{U}} = \mathcal{U} \setminus \bar{\Omega}_{\rho_m}$ with $\gamma = 0$. We still refer to such ρ_m as an *essential lower bound of U*.

2.3. **Stationary solutions in a bounded domain.** Let Ω be a bounded domain in $\mathbb{R}^n$ and consider the stationary Fokker-Planck equation (1.4) in Ω. Assume that the regularity conditions in A) hold globally in Ω and $A = (a^{ij})$ is *uniformly elliptic* in Ω, i.e., there are positive constants λ, Λ such that

$$(2.4) \qquad \lambda|\xi|^2 \le a^{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2, \qquad \xi \in \mathbb{R}^n,\ x \in \Omega.$$

As shown in [12, Theorem 1.2], by imposing Dirichlet boundary condition $u|_{\partial\Omega} = 1$, positive weak solutions of (1.4) in Ω can be obtained from classical works of Trudinger on the existence of weak solutions ([31, Theorem 3.2]) and weak maximal principle ([32, Theorem 7]) of linear elliptic equations with measurable coefficients. More precisely, the following holds.

Lemma 2.1. ([12]) *Assume that $A = (a^{ij})$ is uniformly elliptic in Ω and $a^{ij} \in W^{1,\bar{p}}(\Omega)$, $V^i \in L^{\bar{p}}(\Omega)$, $i, j = 1, \cdots, n$, for some $\bar{p} > n$. Then (1.4) with Ω in place of $\mathcal{U}$ admits a positive weak solution $u \in W^{1,\bar{p}}(\Omega)$.*

Remark 2.1. We note that if Ω is a C^2 domain and $\{a^{ij}\} \subset C^2(\bar{\Omega})$, $\{V^i\} \subset C(\bar{\Omega})$ in the operator L, then the above lemma can be alternatively proved by imposing a homogeneous Robin boundary condition and a result of Amann ([2]) concerning the principle eigenvalues of general elliptic operators. In fact, consider the equation

$$Lu = \partial_i(a^{ij}(x)\partial_j u + b^i(x)u) = 0, \qquad x \in \Omega,$$

with the boundary condition

$$(2.5) \qquad \mathcal{B}u =: \partial_j(a^{ij}u)\nu_i - (V^i\nu_i)u = 0, \qquad x \in \partial\Omega,$$

where $b^i = \partial_j a^{ij} - V^i$, $i = 1, \cdots, n$, and (ν_i) denotes the field of unit outward normal vectors on $\partial\Omega$. Since, for each $x \in \partial\Omega$, $(a^{ij}(x)\nu_i(x))$ is transversal to the tangent vector of $\partial\Omega$ at x, the boundary condition (2.5) is of Robin type. Let

$$\mathcal{L}: \mathrm{Dom}(\mathcal{L}) \hookrightarrow L^{\bar{p}}(\Omega) \longrightarrow L^{\bar{p}}(\Omega): \quad \mathcal{L}u = Lu,$$

be the linear operator with domain

$$\mathrm{Dom}(\mathcal{L}) = \{u \in W^{2,\bar{p}}(\Omega) : \mathcal{B}u|_{\partial\Omega} = 0\}.$$

By [2, Theorem 12.1], $\mathcal{L}$ has a principle eigenvalue λ_0, i.e, λ_0 is a simple eigenvalue with each eigenfunction being positive in $\bar{\Omega}$. Let $u_0 \in \mathrm{Dom}(\mathcal{L})$ be such an eigenfunction. Then

$$\lambda_0 \int_\Omega u_0(x)\, \mathrm{d}x = \int_\Omega Lu_0(x)\, \mathrm{d}x = \int_{\partial\Omega} \mathcal{B}u_0\, \mathrm{d}s = 0.$$

Hence $\lambda_0 = 0$, i.e., u_0 is a positive solution of the equation $Lu = 0$. Thus, (1.4) with Ω in place of $\mathcal{U}$ admits a solution $u =: u_0(\int_\Omega u_0 \mathrm{d}x)^{-1} \in W^{2,\bar{p}}(\Omega)$.

2.4. **Harnack inequality.** Consider the divergence operators of the form

$$Lu := \partial_i(a^{ij}(x)\partial_j u + b^i(x)u) + c^i(x)\partial_i u + d(x)u,$$

where a^{ij}, b^i, c^i, d, $i, j = 1, \ldots, n$, are measurable functions on a bounded domain $\Omega \subset \mathbb{R}^n$, $A = (a^{ij})$ is almost everywhere positive definite in Ω and satisfies

$$\lambda(x)|\xi|^2 \le a^{ij}(x)\xi_i\xi_j \le \Lambda(x)|\xi|^2, \qquad \xi \in \mathbb{R}^n, \ x \in \Omega,$$

for some positive, integrable functions λ^{-1}, Λ on Ω.

Let

$$g(x) := b_{ij}(x)(b^i(x)b^j(x) + c^i(x)c^j(x)) + |d(x)|,$$

where $(b_{ij}) = (a^{ij})^{-1}$ and assume that g is integrable on Ω. Denote $\mathcal{H}^1(A, \Omega)$ as the completion of $C_0^\infty(\Omega)$ under the inner product

$$\langle u, v \rangle = \int_\Omega a^{ij}u_{x_i}v_{x_j}\mathrm{d}x, \qquad u, v \in C_0^\infty(\Omega).$$

We note by [31, Proposition 1.3] that if $\lambda^{-1}, \frac{\Lambda}{\lambda} \in L^\infty(\Omega)$, then $\mathcal{H}^1(A, \Omega) = W^{1,2}(A, \Omega) = H^1(\Omega)$, where $W^{1,2}(A, \Omega)$ denotes the class of strongly differentiable functions u in Ω with $\langle u, u \rangle < \infty$.

Theorem 2.1. (Harnack Inequality, Trudinger [31, Corollary 5.3]) *Let $u \in \mathcal{H}^1(A, \Omega)$ be a nonnegative solution of $Lu = 0$ in Ω. Assume that*

$$\lambda^{-2} \in L^p(\Omega), \quad \Lambda, g \in L^q(\Omega), \quad \frac{1}{p} + \frac{1}{q} < \frac{2}{n}.$$

Then for any $\Omega' \subset\subset \Omega$,

$$\sup_{x \in \Omega'} u(x) \le C \inf_{x \in \Omega'} u(x),$$

where $C > 0$ is a constant depending only on n, p, q, $|\lambda^{-1}|_{L^p(\Omega)}$, $|\Lambda|_{L^q(\Omega)}$, $|g|_{L^q(\Omega)}$ and $\mathrm{dist}(\Omega', \partial\Omega)$.

2.5. **Regularity of stationary measures.** The following regularity result on stationary measures of Fokker-Planck equations is proved in [10].

Theorem 2.2. (Bogachev-Krylov-Röckner [10]) *Assume that* A) *holds and (a^{ij}) is everywhere positive definite in $\mathcal{U}$. Then any stationary measure μ of* (1.3) *admits a positive density function $u \in W^{1,\bar{p}}_{loc}(\mathcal{U})$, i.e., $\mathrm{d}\mu(x) = u(x)\mathrm{d}x$.*

3. Integral identity and derivative formula

In this section, we will recall from [24] two fundamental identities to be used in our level set method for studying stationary measures of (1.3). One is the integral identity which will play a crucial role in capturing information of a weak stationary solution in each sublevel set of a Lyapunov-like function from its boundary. The other one is a derivative formula which will be useful in conducting measure estimates of a stationary measure on sublevel sets of a Lyapunov-like function using differential inequalities.

Theorem 3.1. (Integral Identity, [24, Theorem 2.1]) *Assume that* A) *holds in a domain* $\mathcal{U} =: \Omega \subset \mathbb{R}^n$ *and let* $u \in W_{loc}^{1,\bar{p}}(\Omega)$ *be a weak solution of* (1.4). *Then for any generalized Lipschitz domain* $\Omega' \subset\subset \Omega$ *and any function* $F \in C^2(\bar{\Omega}')$ *with* $F|_{\partial\Omega'} =$ *constant,*

$$(3.1) \qquad \int_{\Omega'} (L^*F)u \; \mathrm{d}x = \int_{\partial\Omega'} (a^{ij}\partial_i F\nu_j)u \; \mathrm{d}s,$$

where for a.e. $x \in \partial\Omega'$, $(\nu_j(x))$ *denotes the unit outward normal vector of* $\partial\Omega'$ *at* x.

Proof. For the reader's convenience, we sketch the proof.

Let $F|_{\partial\Omega'} = c$ and Ω_* be a smooth domain such that $\Omega' \subset\subset \Omega_* \subset\subset \Omega$. Consider the function

$$\tilde{F}(x) = \begin{cases} F(x) - c, & x \in \Omega', \\ 0, & x \in \Omega \setminus \Omega' \end{cases}$$

and its regularizations $\tilde{F}_h$, $0 < h < 1$. Then $\tilde{F}_h \in C_0^\infty(\Omega)$, $\mathrm{supp}(\tilde{F}_h) \subset \bar{\Omega}_*$ as $0 < h \ll 1$, and $\tilde{F}_h \to \tilde{F}$ in $W^{1,q}(\Omega_*)$, as $h \to 0$, for any $0 < q < \infty$. Since u is a weak solution of (1.4) in Ω,

$$\int_\Omega (a^{ij}\partial_{ij}^2 \tilde{F}_h + V^i\partial_i\tilde{F}_h)u \; \mathrm{d}x = 0, \qquad \text{as } 0 < h \ll 1.$$

Since A) holds and $u \in W_{loc}^{1,\bar{p}}(\Omega)$, one can show that, as $h \to 0$,

$$\int_\Omega (a^{ij}\partial_{ij}^2 \tilde{F}_h)u \; \mathrm{d}x \to \int_{\Omega'} ua^{ij}\partial_{ij}^2 F \; \mathrm{d}x - \int_{\partial\Omega'} ua^{ij}\partial_i F\nu_j \; \mathrm{d}s,$$

$$\int_\Omega (V^i\partial_i\tilde{F}_h)u \; \mathrm{d}x \to \int_{\Omega'} uV^i\partial_i F \; \mathrm{d}x.$$

Hence (3.1) holds. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 3.1. 1. We note that the theorem does not require (a^{ij}) to be even positive semi-definite. It also holds for less regular (a^{ij}), (V^i), and u, as long as $a^{ij}u \in W_{loc}^{1,\alpha}(\Omega)$ and $V^iu \in L_{loc}^\alpha(\Omega)$, $\forall i,j, = 1,2,\cdots,n$, for some $\alpha > 1$.

2. We recall from [24] that a bounded open set Ω in $\mathbb{R}^n$ is called a *generalized Lipschitz domain* if i) Ω is a disjoint union of finitely many Lipschitz sub-domains; and ii) intersections of boundaries among these Lipschitz sub-domains only occur at finitely many points. A generalized Lipschitz domain need not be a Lipschitz domain in the usual sense because the boundary need not be orientable. As a simple example, consider the function $U : \mathbb{R}^2 \to \mathbb{R}^1$:

$$(3.2) \qquad U(u,z) = \begin{cases} -u^2 + z^2 + 1, & |u| \leq 1, \\ (u-1)^2 + z^2, & u > 1, \\ (u+1)^2 + z^2, & u < -1. \end{cases}$$

Let $\Omega = \Omega_1$ - the sublevel set of U corresponding to $\rho = 1$. Then Ω has only one boundary component $U^{-1}(1)$ which is the union of two closed Lipschitz curves intersecting at 0 (i.e., the boundary self-intersects at 0). We note that $\partial\Omega = U^{-1}(1)$ encloses two disjoint connected open sets, and hence Ω itself is not a single Lipschitz domain but rather a generalized Lipschitz domain.

Theorem 3.2. (Derivative Formula, [24, Theorem 2.2]) *Let μ be a measure with density function $u \in C(\mathcal{U})$. For a compact function $U \in C^1(\mathcal{U})$, consider the measure function*

$$y(\rho) := \mu(\Omega_\rho) = \int_{\Omega_\rho} u \; \mathrm{d}x, \qquad \rho \in (0, \rho_M),$$

and the open set

(3.3) $$\mathcal{I} =: \{\rho \in (0, \rho_M) : \nabla U(x) \neq 0, \; x \in U^{-1}(\rho)\},$$

where ρ_M is the essential upper bound of U and Ω_ρ is the ρ-sublevel set of U for each $\rho \in (0, \rho_M)$. Then y is of the class C^1 on $\mathcal{I}$ with derivatives

(3.4) $$y'(\rho) = \int_{\partial\Omega_\rho} \frac{u}{|\nabla U|} \; \mathrm{d}s, \qquad \rho \in \mathcal{I}.$$

Proof. As the proof is short, we include it below for the reader's convenience.

Since U is a compact function on $\mathcal{U}$, it follows from Proposition 2.2 a) that $\partial\Omega_\rho \subset U^{-1}(\rho)$ for all $\rho \in (0, \rho_M)$. Let $\rho \in \mathcal{I}$. We note that $\partial\Omega_\rho$ is in fact a C^1 hyper-surface which coincides with $U^{-1}(\rho)$.

Let $\mathcal{T} = \{(x, e_j) : j = 1, 2, \cdots, n\}$ be an orientation preserving, orthonormal, moving frame defined over $\partial\Omega_\rho$ such that for each $x \in \partial\Omega_\rho$, $e_j = e_j(x)$, $j = 1, 2, \cdots, n-1$, are tangent vectors, and $e_n = e_n(x)$ is the outward unit normal vector, of $\partial\Omega_\rho$ at x. We denote $\{(x, \omega^j) : j = 1, 2, \cdots, n\}$ as the dual frame of $\mathcal{T}$ defined over $\partial\Omega_\rho$, i.e., $\omega^i(e_j) = \delta^i_j$, $i, j = 1, 2, \cdots, n$. Since, for each $x \in \partial\Omega_\rho$, $e_n = \frac{\nabla U}{|\nabla U|}$, we have $\omega^n = \frac{\mathrm{d}U}{|\nabla U|}$. Therefore

$$\mathrm{d}x = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n = \mathrm{d}s \wedge \omega^n = \frac{1}{|\nabla U|} \mathrm{d}s\mathrm{d}U,$$

where $\mathrm{d}s = \omega^1 \wedge \cdots \wedge \omega^{n-1}$ is a volume form defined on $\partial\Omega_\rho$, from which (3.4) easily follows.

Continuity of $y'(\rho)$ on $\mathcal{I}$ follows from (3.4). □

Remark 3.2. The derivative formula (3.4) is known in the case that $|\nabla U(x)| \geq c > 0$ a.e. in $\mathbb{R}^n$ (see [7, Proposition 5.8.34]), and it is also used in [9, Proposition 2] for level set estimates concerning functions that satisfy (1.11).

4. STATIONARY MEASURES UNDER LYAPUNOV CONDITION

We will show Theorem A in this section. Our approach differs from existing works on existence of stationary measures of (1.3) in order to apply general Lyapunov functions as defined in Definition 2.4. In particular, the integral identity (3.1) contained in Theorem 3.1 will play an important role in our study.

4.1. **Measure estimates via level set method.** We first recall from [24] the following measure estimates with respect to a regular stationary measure of (1.3) in the essential domain of a Lyapunov function. As to be seen in a separate work ([27]), such estimates will also be useful in showing the relative sequential compactness of a family of stationary measures associated with a so-called null family of diffusion matrices, as well as in characterizing the concentration of these measures as noises tend to zero.

Lemma 4.1. ([24, Theorem A a)]) *Assume that* A) *holds and let* U *be a Lyapunov function in* $\mathcal{U}$ *with Lyapunov constant* γ *and essential lower bound* ρ_m *and upper bound* ρ_M. *Then for any weak solution* $u \in W^{1,\bar{p}}_{loc}(\mathcal{U})$ *of* (1.4) *and any* $\rho_0 \in (\rho_m, \rho_M)$,

$$(4.1) \qquad \mu(\mathcal{U} \setminus \Omega_{\rho_0}) \le \gamma^{-1} C_{\rho_m, \rho_0} |A|_{C(\Omega_{\rho_0} \setminus \Omega_{\rho_m})} |\nabla U|^2_{C(\Omega_{\rho_0} \setminus \Omega_{\rho_m})} \mu(\Omega_{\rho_0} \setminus \Omega_{\rho_m}),$$

where μ *is the measure with density function* u, *i.e.,* $\mathrm{d}\mu(x) = u(x)\mathrm{d}x$, *the constant* $C_{\rho_m, \rho_0} > 0$ *depends only on* ρ_m, ρ_0, *and* Ω_ρ *denotes the* ρ-*sublevel set of* U *for each* $\rho \in [\rho_m, \rho_M)$.

Proof. The lemma follows immediately from [24, Theorem A a)] by taking

$$H(\rho) = \sup_{x \in U^{-1}(\rho)} (a^{ij}(x)\partial_i U(x)\partial_j U(x)).$$

The original proof of [24, Theorem A a)] uses the approximation of U by a sequence of Morse functions because its sublevel sets need not be generalized Lipschitz domains. In order to highlight the main ideas of using the integral identity (3.1), we give the proof below for the special case that $\nabla U \ne 0$ everywhere in the essential domain $\mathcal{U} \setminus \bar{\Omega}_{\rho_m}$ of U.

For given $\rho_0 \in (\rho_m, \rho_M)$, consider a fixed monotonically increasing function $\phi \in C^2(\mathbb{R}_+)$ satisfying

$$\phi(t) = \begin{cases} 0, & \text{if } t \in [0, \rho_m]; \\ t, & \text{if } t \in [\rho_0, +\infty). \end{cases}$$

Then $\phi'(t) = 0$, $t \in [0, \rho_m]$ and $\phi''(t) = 0$, $t \in [0, \rho_m] \cup [\rho_0, +\infty)$.

Let $\rho \in (\rho_0, \rho_M)$ and $\rho^* \in (\rho, \rho_M)$. An application of Theorem 3.1 with $F = \phi \circ U$, $\Omega = \Omega_{\rho^*}$, and $\Omega' = \Omega_\rho$ yields that

$$\int_{\Omega_\rho} (a^{ij}\partial^2_{ij}\phi(U) + V^i\partial_i\phi(U))u \; \mathrm{d}x = \int_{\partial\Omega_\rho} ua^{ij}\partial_i\phi(U)\nu_j \; \mathrm{d}s,$$

i.e.,

$$\int_{\Omega_\rho} \phi'(U)(L^*U)u \; \mathrm{d}x \;\; + \;\; \int_{\Omega_\rho} \phi''(U)(a^{ij}\partial_i U\partial_j U)u \; \mathrm{d}x$$

$$(4.2) \qquad\qquad = \int_{\partial\Omega_\rho} \phi'(U)ua^{ij}\partial_i U\nu_j \; \mathrm{d}s = \int_{\partial\Omega_\rho} ua^{ij}\partial_i U\nu_j \; \mathrm{d}s,$$

where $(\nu_j) = \frac{\nabla U}{|\nabla U|}$ denote the unit outward normal vectors of $\partial\Omega_\rho$. Since

$$a^{ij}(x)\partial_i U(x)\nu_j(x) \ge 0, \qquad x \in \partial\Omega_\rho,$$

(4.2) yields that

$$\int_{\Omega_\rho} \phi'(U)(L^*U)u \; \mathrm{d}x + \int_{\Omega_\rho} \phi''(U)(a^{ij}\partial_i U\partial_j U)u \; \mathrm{d}x \ge 0.$$

By the definition of ϕ, it follows that

$$\int_{\Omega_\rho \setminus \Omega_{\rho_m}} \phi'(U)(L^*U)u \; \mathrm{d}x \ge - \int_{\Omega_\rho \setminus \Omega_{\rho_m}} \phi''(U)(a^{ij}\partial_i U\partial_j U)u \; \mathrm{d}x.$$

Letting $\rho \to \rho_M$ in the above, we obtain

$$(4.3) \qquad \int_{\mathcal{U} \setminus \Omega_{\rho_m}} \phi'(U)(L^*U)u \; \mathrm{d}x \ge - \int_{\mathcal{U} \setminus \Omega_{\rho_m}} \phi''(U)(a^{ij}\partial_i U\partial_j U)u \; \mathrm{d}x.$$

Using the facts $\phi'(t) \geq 0$, $\phi'(t) = 1$ as $t \geq \rho_0$, and that U is a Lyapunov function, we have

$$(4.4) \qquad \int_{\mathcal{U} \setminus \Omega_{\rho_m}} \phi'(U)(L^*U)u \ \mathrm{d}x \leq -\gamma \int_{\mathcal{U} \setminus \Omega_{\rho_m}} \phi'(U)u \ \mathrm{d}x \leq -\gamma\mu(\mathcal{U} \setminus \Omega_{\rho_0}).$$

Denote $C_{\rho_m, \rho_0} = \max_{\rho_m \leq \rho \leq \rho_0} |\phi''(\rho)|$. We also have

$$\int_{\mathcal{U} \setminus \Omega_{\rho_m}} |\phi''(U)|(a^{ij}\partial_i U \partial_j U)u \ \mathrm{d}x = \int_{\Omega_{\rho_0} \setminus \Omega_{\rho_m}} |\phi''(U)|(a^{ij}\partial_i U \partial_j U)u \ \mathrm{d}x$$

$$(4.5) \qquad \leq \quad C_{\rho_m, \rho_0} |A|_{C(\Omega_{\rho_0} \setminus \Omega_{\rho_m})} |\nabla U|^2_{C(\Omega_{\rho_0} \setminus \Omega_{\rho_m})} \mu(\Omega_{\rho_0} \setminus \Omega_{\rho_m}).$$

The lemma now follows from (4.3)-(4.5). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4.2. **Existence of regular stationary measures.** The following theorem is just Theorem A stated in Section 1.

Theorem 4.1. *Assume that* A) *holds in $\mathcal{U}$ and (a^{ij}) is everywhere positive definite in $\mathcal{U}$. If there exists a Lyapunov function in $\mathcal{U}$ with respect to the stationary Fokker-Planck equation* (1.4), *then* (1.3) *admits a regular stationary measure in $\mathcal{U}$ with positive density function lying in the space $W^{1,\bar{p}}_{loc}(\mathcal{U})$. If, in addition, the Lyapunov function is unbounded, then the stationary measure is unique in $\mathcal{U}$.*

Proof. Let U be a Lyapunov function in $\mathcal{U}$ with Lyapunov constant γ, essential lower bound ρ_m and upper bound ρ_M. We also denote by Ω_ρ the ρ-sublevel set of U.

Let $\rho^k \in (\rho_m, \rho_M)$, $k = 1, 2, \cdots$, be an increasing sequence such that $\rho^k \to \rho_M$ as $k \to \infty$. We denote $\Omega_k = \Omega_{\rho^k}$ and $\Gamma_k = \partial\Omega_{\rho^k}$, $k = 1, 2, \cdots$. Since $\Gamma_k \to \partial\mathcal{U}$, for any domain $\Omega_* \subset\subset \mathcal{U}$ we have that $\Omega_* \subset \Omega_k$ as $k \gg 1$.

Since (a^{ij}) is continuous and everywhere positive definite in $\mathcal{U}$, (a^{ij}) is uniformly elliptic in Ω_k for each k. Hence by Lemma 2.1, for each k, the stationary Fokker-Planck equation (1.4) with Ω_k in place of $\mathcal{U}$ admits a positive solution $u_k \in W^{1,\bar{p}}(\Omega_k) \hookrightarrow C(\bar{\Omega}_k)$. For each k, we extend u_k to $\mathcal{U}$ by setting $u_k \equiv 0$ in $\mathcal{U} \setminus \Omega_k$, and still denote it by u_k.

For any fixed domain $\Omega_* \subset\subset \mathcal{U}$, we let λ, Λ, and η be positive constants such that

$$\lambda|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j, \qquad\qquad \xi \in \mathbb{R}^n, \ x \in \Omega_*,$$

$$|a^{ij}|_{W^{1,\bar{p}}(\Omega_*)} \leq \Lambda, \ |V^i|_{L^{\bar{p}}(\Omega_*)} \leq \eta, \ \ i,j = 1, \cdots, n.$$

Then for any two domains B_*, B^* with $B_* \subset\subset B^* \subset\subset \Omega_*$, an application of Harnack inequality (Theorem 2.1) to $\Omega = \Omega_*$, $\Omega' = B^*$ with $2q = \bar{p}$, $b^i = \partial_j a^{ij} - V^i$, $c_i = d \equiv 0$, $i = 1, \cdots, n$, yields that

$$\sup_{B^*} u_k \leq C_1 \inf_{B^*} u_k,$$

for some constant $C_1 > 0$ depending only on n, $\bar{p}$, λ, Λ, η, and $\mathrm{dist}(B^*, \partial\Omega_*)$, in particular, not on k. Since

$$|B^*| \inf_{B^*} u_k \leq \int_{B^*} u_k \ \mathrm{d}x \leq \int_{\Omega_k} u_k \ \mathrm{d}x \equiv 1,$$

where $|B^*|$ stands for the volume of B^*, we have

$$\inf_{B^*} u_k \leq |B^*|^{-1}.$$

This implies that

$$\sup_{B^*} u_k \le C_1 |B^*|^{-1}.$$

Using the above and Hölder estimate, we also have

$$|u_k|_{C^\alpha(B_*)} \le C_2 |u_k|_{L^2(B^*)} \le C_1 C_2 |B^*|^{-\frac{1}{2}},$$

where $\alpha > 0$ and $C_2 > 0$ are constants depending only on n, $\bar{p}$, λ, Λ, η, and $\text{dist}(B_*, \partial B^*)$.

Since B_*, B^*, Ω_* are arbitrary, the sequence $\{u_k\}$ is equi-continuous on any compact subset of $\mathcal{U}$. By taking a subsequence if necessary, we assume without loss of generality that $\{u_k\}$ converges, as $k \to \infty$, to some non-negative function $\tilde{u} \in C(\mathcal{U})$ under the compact-open topology of $C(\mathcal{U})$. For any given $f \in C_0^\infty(\mathcal{U})$, we let $k \gg 1$ such that $\text{supp}(f) \subset \Omega_k$, i.e., $f \in C_0^\infty(\Omega_k)$. Then

$$\int_{\mathcal{U}} L^* f(x) u_k(x) \, \mathrm{d}x = 0, \qquad \text{as } k \gg 1.$$

Since $u_k \to \tilde{u}$ uniformly on $\text{supp}(f)$ as $k \to \infty$, we have

$$\int_{\mathcal{U}} L^* f(x) \tilde{u}(x) \, \mathrm{d}x = 0.$$

Moreover, since $\int_{\mathcal{U}} u_k \mathrm{d}x = \int_{\Omega_k} u_k \, \mathrm{d}x \equiv 1$ and $u_k \ge 0$, it follows from Fatou's Lemma that

$$\int_{\mathcal{U}} \tilde{u} \mathrm{d}x = \int_{\mathcal{U}} \lim_{k \to \infty} u_k \mathrm{d}x \le \varliminf_{k \to \infty} \int_{\mathcal{U}} u_k \mathrm{d}x = 1.$$

Thus, if $\tilde{u}$ is not identically zero in $\mathcal{U}$, then μ with $\mathrm{d}\mu(x) = \frac{\tilde{u}(x)}{\int_{\mathcal{U}} \tilde{u} \mathrm{d}x} \mathrm{d}x$ is a stationary measure of (1.3), and we have by Theorem 2.2 that μ admits a positive density function $u \in W_{loc}^{1,\bar{p}}(\mathcal{U})$.

We now show that $\tilde{u} \not\equiv 0$ in $\mathcal{U}$. For $k \ge 1$, since the set Ω_k contains $\bar{\Omega}_{\rho_m}$, U is a Lyapunov function in Ω_k with respect to L^* having the essential domain $\Omega_k \setminus \bar{\Omega}_{\rho_m}$, Lyapunov constant γ, and essential lower bound ρ_m and upper bound ρ^k. For fixed $\rho_0 \in (\rho_m, \rho_1) \subset (\rho_m, \rho^k)$, an application of Lemma 4.1 to μ_k with $\mathrm{d}\mu_k(x) = u_k \mathrm{d}x$ on Ω_k for every $k \ge 1$ yields that

$$\mu_k(\Omega_k \setminus \Omega_{\rho_0}) \le \gamma^{-1} C_* \mu_k(\Omega_{\rho_0} \setminus \Omega_{\rho_m}) \le \gamma^{-1} C_* |\Omega_{\rho_0} \setminus \Omega_{\rho_m}| |u_k|_{C(\Omega_{\rho_0} \setminus \Omega_{\rho_m})},$$

where $C_* = C_{\rho_m, \rho_0} |A|_{C(\Omega_{\rho_0} \setminus \Omega_{\rho_m})} |\nabla U|^2_{C(\Omega_{\rho_0} \setminus \Omega_{\rho_m})}$ for a positive constant C_{ρ_m, ρ_0} depending only on ρ_m, ρ_0 as in Lemma 4.1. It follows that

$$
\begin{aligned}
1 \equiv \mu_k(\Omega_k) \;&=\; \mu_k(\Omega_k \setminus \Omega_{\rho_0}) + \mu_k(\Omega_{\rho_0}) \\
&\le\; \gamma^{-1} C_* |\Omega_{\rho_0} \setminus \Omega_{\rho_m}| |u_k|_{C(\Omega_{\rho_0} \setminus \Omega_{\rho_m})} + |\Omega_{\rho_0}| |u_k|_{C(\Omega_{\rho_0})}.
\end{aligned}
$$

(4.6)

If $\tilde{u} \equiv 0$ in $\mathcal{U}$, then $u_k \to 0$ as $k \to \infty$ uniformly on Ω_{ρ_0}, leading to a contradiction to (4.6).

The uniqueness follows from [17, Example 5.1], where it is shown that if there exists an unbounded function $U \in C^2(\mathcal{U})$ satisfying: i) each set $\Omega_\rho \cup U^{-1}(\rho)$, $\rho > 0$, is compact; ii) for a constant $\alpha > 0$ and a compact set K,

(4.7) $$L^* U(x) \le \alpha U(x), \qquad a.e. \ x \in \mathcal{U} \setminus K,$$

then (1.4) admits at most one stationary measure in $\mathcal{U}$. Thus, by taking U as the Lyapunov function when it is unbounded and also taking K as $\bar{\Omega}_{\rho_m}$, we obtain the uniqueness result. $\square$

We note that by Proposition 2.1, a non-negative function $U \in C^2(\mathbb{R}^n)$ is an unbounded Lyapunov function with respect to (1.4) in $\mathbb{R}^n$ iff it satisfies (1.11) and (1.9). Applying the above theorem to $\mathcal{U} = \mathbb{R}^n$, we immediately obtain the following result which is just the one contained in [14].

Corollary 4.1. *Assume that* A) *holds in* $\mathbb{R}^n$, (a^{ij}) *is everywhere positive definite in* $\mathbb{R}^n$, *and there is a function* $U \in C^2(\mathbb{R}^n)$ *satisfying* (1.11) *and* (1.9). *Then the Fokker-Planck equation* (1.3) *has a unique regular stationary measure in* $\mathbb{R}^n$ *with positive density function lying in the space* $W^{1,\bar{p}}_{loc}(\mathbb{R}^n)$.

Proof. We note that (1.11) implies that there exists a constant $c \geq 0$ such that $U + c \geq 0$. It then becomes a unbounded Lyapunov function with respect to (1.4). The corollary now follows from Theorem 4.1. $\qquad\square$

Remark 4.1. 1) In applying Theorem 4.1 or Corollary 4.1 to a particular stochastic system, one often need to construct a Lyapunov function. Just like the case for deterministic systems, such a Lyapunov function is usually constructed (for the sake of using undetermined coefficients technique) as a polynomial which often ends up with a finite Lyapunov constant. To give an example, consider

$$(4.8) \qquad \mathrm{d}x = -\frac{bx}{1+x^2}\mathrm{d}t + \sqrt{2}\,\mathrm{d}W, \qquad x \in \mathbb{R}^1,$$

where $b > 1$ is a constant.

We find that the functions $I_\pm(x)$ defined in the Appendix (Proposition 6.1) are constant multiples of $(1+x^2)^{-\frac{b}{2}}$ in the present situation. It is clear that (6.2) is satisfied and hence by Proposition 6.1 there exists an unbounded Lyapunov function U with respect to the stationary Fokker-Planck equation associated with (4.8). Therefore, by Theorem 4.1 or Corollary 4.1, the Fokker-Planck equation associated with (4.8) admits a unique regular stationary measure.

Now let $b \in (1,3)$. If one would prefer to have a polynomial Lyapunov function U, then the Lyapunov constant is always finite. To see this, we note that for U being a polynomial Lyapunov function, it has to be of even degree with a positive leading coefficient because it need to satisfy (1.11). Therefore, we may assume without loss of generality that $U = x^{2p} + P(x)$, $x \in \mathbb{R}^1$, where $p \geq 1$ is a natural number and P is a polynomial of degree $k < 2p$. A direct calculation yields that

$$L^*U = 2px^{2p-2}(2p - 1 - \frac{bx^2}{1+x^2}) + O(|x|^{k-2}), \qquad |x| \gg 1.$$

As $x \to \infty$, the coefficient of x^{2p-2} tends to $2p(2p - 1 - b)$, which must be non-positive for U being a Lyapunov function. Since $b \in (1,3)$, it cannot be zero, and it is negative only when $p = 1$, in which case we see that $\gamma =: b - 1$ is a Lyapunov constant.

2) It is already known that a sufficiently large noise can stabilize an unstable equilibrium of a deterministic system. By choosing an appropriate Lyapunov function, one can in fact quantify the optimal lower bound of amplitudes of diffusions needed for the stabilization process. For example, consider the stochastic differential equation

$$(4.9) \qquad \mathrm{d}x = bx\mathrm{d}t + \sqrt{2\sigma(x^2+1)}\,\mathrm{d}W, \qquad x \in \mathbb{R}^1,$$

where $b > 0$ is a fixed constant. We note that the unperturbed ODE has a unique equilibrium $\{0\}$ which is unstable and repelling.

When $\sigma > b$, we claim that the Fokker-Planck equation associated with (4.9) has a unique stationary measure. Hence in this parameter range the diffusion stabilizes the unperturbed ODE. To show the claim, we consider $U(x) = \log(x^2 + 1)$. Then

$$L^*U(x) = -2 \cdot \frac{(\sigma - b)x^2 - \sigma}{x^2 + 1}.$$

Hence $L^*U(x) < -(\sigma - b)$ as $|x| \gg 1$. Therefore, U is an unbounded Lyapunov function with respect to the stationary Fokker-Planck equation associated with (4.9). The claim now follows from Corollary 4.1.

In fact, it can be shown that (4.9) admits no stationary measure when $0 < \sigma \leq b$ (see [25]), i.e., b is the optimal lower bound of amplitudes of diffusions needed for this particular stabilization process.

3) When dealing with a stochastic system over a bounded domain, Theorem 4.1 is particularly useful in showing the existence of stationary measures when the diffusion matrix becomes degenerate on the boundary, for which Lemma 2.1 fails to apply. For example, consider

$$(4.10) \qquad \mathrm{d}x = bx\mathrm{d}t + \sqrt{2}(1 - x^2)\mathrm{d}W, \qquad x \in \mathcal{U} = (-1, 1),$$

where $b < 0$ is a constant. We note that the noise coefficient vanishes at the boundary of $\mathcal{U}$.

Consider $U(x) = -\log(1 - x^2)$, $x \in (-1, 1)$. It is clear that $U(x) \to +\infty$, as $|x| \to 1$, and

$$L^*U(x) = 2 + 2x^2 + \frac{2bx^2}{1 - x^2}, \qquad x \in (-1, 1).$$

Hence, $L^*U(x) \to -\infty$, as $|x| \to 1$, i.e., U is in fact an unbounded Lyapunov function with respect to the stationary Fokker-Planck equation associated with (4.10). It follows from Theorem 4.1 that the Fokker-Planck equation associated with (4.10) has a unique stationary measure in $\mathcal{U}$.

5. Stationary measures under weak Lyapunov condition

In this section, we will study the existence of regular stationary measures under a weak Lyapunov condition. Theorem B will be proved.

5.1. Measure estimates via level set method.
We recall the following measure estimate from [24].

Lemma 5.1. ([24, Theorem A c)]) *Let* A) *hold in* $\mathcal{U}$. *Assume that* (1.4) *admits a weak Lyapunov function* U *in* $\mathcal{U}$ *with essential lower bound* ρ_m *and upper bound* ρ_M *such that*

$$(5.1) \qquad H_1(\rho) \leq a^{ij}(x)\partial_i U(x)\partial_j U(x) \leq H_2(\rho), \qquad x \in U^{-1}(\rho), \ \rho \in [\rho_m, \rho_M)$$

for some positive measurable function H_1 *and continuous function* H_2 *on* $[\rho_m, \rho_M)$. *Then for any weak solution* $u \in W^{1,\bar{p}}_{loc}(\mathcal{U})$ *of* (1.4) *and any* $\rho_0 \in (\rho_m, \rho_M)$,

$$(5.2) \qquad \mu(\mathcal{U} \setminus \Omega_{\rho_m}) \leq \mu(\Omega_{\rho_0} \setminus \Omega_{\rho_m})\mathrm{e}^{\int_{\rho_0}^{\rho_M} \frac{1}{H(\rho)}\mathrm{d}\rho},$$

where μ is the probability measure with density function u, i.e., $\mathrm{d}\mu(x) = u(x)\mathrm{d}x$, Ω_ρ denotes the ρ-sublevel set of U for each $\rho \geq 0$, and $\tilde{H}(\rho) = H_1(\rho) \int_{\rho_m}^{\rho} \frac{1}{H_2(s)} \mathrm{d}s$, $\rho \in [\rho_m, \rho_M)$.

Proof. We sketch the proof in order to show the application of the derivative formula (3.4).

Let $\rho \in [\rho_m, \rho_M)$. Since by (5.1), $\nabla U(x) \neq 0$ for all $x \in U^{-1}(\rho)$, Ω_ρ is a C^2 domain with

$$(5.3) \qquad \partial\Omega_\rho = U^{-1}(\rho).$$

Consider

$$H(t) = \int_{\rho_m}^{t} \int_{\rho_m}^{\tau} \frac{1}{H_2(s)} \ \mathrm{d}s\mathrm{d}\tau, \qquad t \in [\rho_m, \rho_M).$$

Since H_2^{-1} is continuous, H is a C^2 function on $[\rho_m, \rho_M)$. We extend H to a C^2 function on $[0, \rho_M)$ and still denote it by H. Let $F = H \circ U$. For any $\rho \in (\rho_m, \rho_M)$, we apply Theorem 3.1 to F with Ω' being Ω_ρ, Ω_{ρ_m}, respectively, to obtain

$$(5.4) \qquad \int_{\Omega_\rho \backslash \Omega_{\rho_m}} (a^{ij}\partial_{ij}^2 F + V^i\partial_i F)u \ \mathrm{d}x = H'(\rho) \int_{\partial\Omega_\rho} ua^{ij}\frac{\partial_i U \partial_j U}{|\nabla U|} \ \mathrm{d}s.$$

For $x \in \Omega_\rho \backslash \Omega_{\rho_m}$, denote $\rho' := U(x)$. Using facts $H''(\rho') = H_2^{-1}(\rho')$, $H'(\rho') \geq 0$, and that U is a weak Lyapunov function, we have by (5.1) that

$$a^{ij}(x)\partial_{ij}^2 F(x) + V^i(x)\partial_i F(x) \leq 1, \qquad x \in \Omega_\rho \backslash \Omega_{\rho_m}.$$

Also by (5.1), the right hand side of (5.4) simply satisfies

$$H'(\rho) \int_{\partial\Omega_\rho} ua^{ij}\frac{\partial_i U \partial_j U}{|\nabla U|} \ \mathrm{d}s \geq \int_{\rho_m}^{\rho} \frac{1}{H_2(s)}\mathrm{d}s \ H_1(\rho) \int_{\partial\Omega_\rho} \frac{u}{|\nabla U|} \ \mathrm{d}s.$$

Hence by (5.4),

$$(5.5) \qquad \int_{\Omega_\rho \backslash \Omega_{\rho_m}} u(x) \ \mathrm{d}x \geq \tilde{H}(\rho) \int_{\partial\Omega_\rho} \frac{u}{|\nabla U|} \ \mathrm{d}s, \qquad \rho \in (\rho_m, \rho_M).$$

Consider the function

$$y(\rho) = \int_{\Omega_\rho \backslash \Omega_{\rho_m}} u \ \mathrm{d}x, \qquad \rho \in (\rho_m, \rho_M).$$

Then by Theorem 3.2 and (5.5),

$$y'(\rho) \leq \frac{1}{\tilde{H}(\rho)}y(\rho), \qquad \rho \in (\rho_m, \rho_M).$$

For any $\rho_0 \in (\rho_m, \rho_M)$, a direct integration of the above inequality yields that

$$y(\rho) \leq y(\rho_0)\mathrm{e}^{\int_{\rho_0}^{\rho} \frac{1}{\tilde{H}(r)}\mathrm{d}r}, \qquad \rho \in (\rho_0, \rho_M).$$

The proof is complete simply by taking limit $\rho \to \rho_M$ in the above. $\qquad \square$

5.2. **Existence of regular stationary measure.** The following theorem is just Theorem B stated in Section 1.

Theorem 5.1. *Assume that* A) *holds and* (a^{ij}) *is everywhere positive definite in* $\mathcal{U}$. *If there exists a weak Lyapunov function in* $\mathcal{U}$ *with respect to the stationary Fokker-Planck equation* (1.4) *which is of the class* $\mathcal{B}_*(A)$, *then* (1.3) *admits a regular stationary measure in* $\mathcal{U}$ *with positive density function lying in* $W^{1,\bar{p}}_{loc}(\mathcal{U})$.

Proof. Let U denote the weak Lyapunov function in $\mathcal{U}$ with respect to (1.4). Without loss of generality, we may assume that for U the essential lower bound ρ_m and the constant ρ_m appearing in Definition 2.3 are the same. Let $\{\rho_k\} \subset (\rho_m, \rho_M)$ be a sequence such that $\rho_k \to \rho_M$ as $k \to \infty$, where ρ_M denotes the essential upper bound of U. As in the proof of Theorem 4.1, we obtain a sequence of $\{u_k\}$ such that, for each k, $u_k \in W^{1,\bar{p}}_{loc}(\mathcal{U})$ is a weak solution of (1.4) in the domain $\Omega_k =: \Omega_{\rho_k}$, and, as $k \to \infty$, u_k converges under the compact-open topology of $C(\mathcal{U})$ to a non-negative function $\tilde{u} \in C(\mathcal{U})$ which satisfies

$$\int_{\mathcal{U}} L^* f(x) \tilde{u}(x) \, \mathrm{d}x = 0, \qquad \text{for all } f \in C_0^\infty(\mathcal{U}).$$

As argued in the proof of Theorem 4.1, we only need to show that $\tilde{u}$ is not identically zero in $\mathcal{U}$.

For a given $\rho_0 \in (\rho_m, \rho_M)$, we let $\bar{k}$ be such that $\rho_k \in (\rho_0, \rho_M)$ as $k \geq \bar{k}$. For each $k \geq \bar{k}$, U remains a weak Lyapunov function with respect to (1.4) in the domain Ω_k with essential upper bound ρ_k. Since U is of the class $\mathcal{B}_*(A)$, there is a positive measurable function H which satisfies the properties (2.1) and (2.2) in Definition 2.3 on $[\rho_m, \rho_M)$. Clearly (5.1) is satisfied with

$$H_1(\rho) =: \min_{x \in U^{-1}(\rho)} \left(a^{ij}(x) \partial_i U(x) \partial_j U(x) \right),$$
$$H_2(\rho) =: \max_{x \in U^{-1}(\rho)} \left(a^{ij}(x) \partial_i U(x) \partial_j U(x) \right), \qquad \rho \in [\rho_m, \rho_M).$$

By (2.1), $H \leq H_1$. Then H_1 and H_2 are positive and continuous. For each $k \geq \bar{k}$, applying (5.2) in Lemma 5.1 with u_k, Ω_k, ρ_k in place of u, $\mathcal{U}$, ρ_M respectively, yields that

$$\int_{\Omega_k \setminus \Omega_{\rho_m}} u_k(x) \mathrm{d}x \leq e^{\int_{\rho_0}^{\rho_k} \frac{1}{\tilde{H}(\rho)} \mathrm{d}\rho} \int_{\Omega_{\rho_0} \setminus \Omega_{\rho_m}} u_k(x) \mathrm{d}x,$$

i.e.,

(5.6) $$1 = \int_{\Omega_k} u_k(x) \mathrm{d}x \leq \int_{\Omega_{\rho_m}} u_k(x) \mathrm{d}x + e^{\int_{\rho_0}^{\rho_k} \frac{1}{\tilde{H}(\rho)} \mathrm{d}\rho} \int_{\Omega_{\rho_0} \setminus \Omega_{\rho_m}} u_k(x) \mathrm{d}x,$$

where

$$\tilde{H}(\rho) = H_1(\rho) \int_{\rho_m}^{\rho} \frac{1}{H_2(s)} \mathrm{d}s, \qquad \rho \geq \rho_0.$$

Let $C =: \int_{\rho_m}^{\rho_0} \frac{1}{H_2(s)} \mathrm{d}s$. Then C is a positive constant and $CH_1(\rho) \leq \tilde{H}(\rho)$, $\rho \in [\rho_0, \rho_M)$. By (2.2) and $H_1 \geq H$, $\int_{\rho_m}^{\rho_M} \frac{1}{H_1(\rho)} \mathrm{d}\rho < +\infty$. It follows that

(5.7) $$\int_{\rho_0}^{\rho_M} \frac{1}{\tilde{H}(\rho)} \mathrm{d}\rho < +\infty.$$

Now, $\tilde{u} \not\equiv 0$, for otherwise, (5.7) would lead to a contradiction when taking $k \to \infty$ in (5.6). $\qquad\square$

Remark 5.1. 1) From above proof, we see that (5.7), which appears weaker than (2.2), is sufficient to conclude Theorem 5.1. But we remark that condition (2.2) is actually equivalent to (5.7). To show (5.7) implies (2.2), we note that $\tilde{H}(\rho) \leq H_1(\rho) \int_{\rho_m}^{\rho} \frac{1}{H_1(s)} \mathrm{d}s$, i.e.,

$$y'(\rho) \leq \frac{1}{\tilde{H}(\rho)} y(\rho), \qquad \rho \in (\rho_m, \rho_M),$$

where $y(\rho) = \int_{\rho_m}^{\rho} \frac{1}{H_1(s)} \mathrm{d}s$. It follows that, for any $\rho_0 \in (\rho_m, \rho_M)$,

$$\int_{\rho_0}^{\rho_M} \frac{1}{\tilde{H}(\rho)} \mathrm{d}\rho \geq \log \int_{\rho_m}^{\rho_M} \frac{1}{H_1(s)} \mathrm{d}s - \log \int_{\rho_m}^{\rho_0} \frac{1}{H_1(s)} \mathrm{d}s.$$

Since H_1 is positive and continuous on $[\rho_m, \rho_0]$, we have from (5.7) that (2.2) is satisfied with $H = H_1$.

2) As to be seen in part II of the series ([25]), the condition that U is of the class $\mathcal{B}_*(A)$ cannot be removed in the above theorem. In fact, an example of a stationary Fokker-Planck equation will be given there in such way that it even admits a strictly weak Lyapunov function U, i.e., $L^*U(x) < 0$ in the essential domain of U, but it does not have any measure solution.

6. Appendix

When $\mathcal{U} = \mathbb{R}^1$, Lyapunov functions can be explicitly classified in the case of non-degenerate and bounded diffusion. Consider

$$(6.1) \qquad\qquad \mathrm{d}x = V(x)\mathrm{d}t + \sqrt{2a(x)}\mathrm{d}W, \qquad x \in \mathbb{R}^1,$$

where V, a are continuous functions, and $0 < a(x) \leq \Lambda$ in $\mathbb{R}^1$ for some constant Λ. We denote the adjoint Fokker-Planck operator associated with (6.1) by $L_{V,a}^*$.

Proposition 6.1. $L_{V,a}^*$ *admits a Lyapunov function in* $\mathbb{R}^1$ *iff there is a continuous function* $\gamma(x)$ *such that for some constant* $K > 0$, $\inf_{|x| \geq K} \gamma(x) > 0$, *and*

$$(6.2) \qquad\qquad \int_{-\infty}^{-K} I_-(t)q(t)\mathrm{d}t + \int_K^{+\infty} I_+(t)q(t)\mathrm{d}t < \infty,$$

where

$$I_\pm(x) = \mathrm{e}^{\int_{\pm K}^{x} p(s)\mathrm{d}s}, \quad p(x) = \frac{V(x)}{a(x)}, \qquad q(x) = \frac{\gamma(x)}{a(x)}.$$

Moreover, if $L_{V,a}^*$ *admits a Lyapunov function in* $\mathbb{R}^1$, *then it also admits an unbounded Lyapunov function in* $\mathbb{R}^1$.

Proof. Suppose $U \in C^2(\mathbb{R}^1)$ is a Lyapunov function with respect to $L_{V,a}^*$. Denote

$$(6.3) \qquad \gamma(x) =: -L^*U(x) = -a(x)U''(x) - V(x)U'(x), \qquad x \in \mathbb{R}^1.$$

Then $\gamma(x)$ is continuous in $\mathbb{R}^1$ and there exists a $K > 0$ such that $\inf_{|x| \geq K} \gamma(x) > 0$. We note that (6.3) is equivalent to

$$(6.4) \qquad\qquad U'' + p(x)U' = -q(x).$$

A direct integration of (6.4) yields that

$$(6.5) \qquad U(x) = \begin{cases} U_-(x), & x \leq -K, \\ U_+(x), & x \geq K, \end{cases}$$

where

$$(6.6) \qquad U_\pm(x) = \int_{\pm K}^x U'_\pm(u)\mathrm{d}u + U_\pm(\pm K),$$

and

$$U'_\pm(x) = I_\pm(x)^{-1}\left(-\int_{\pm K}^x I_\pm(t)q(t)\mathrm{d}t + U'_\pm(\pm K)\right).$$

Since U, as a Lyapunov function, must satisfy $0 \leq U(x) < U(\infty) = U(-\infty)$, $x \in \mathbb{R}^1$, there are sequences $x_i^\pm \to \pm\infty$ such that $\pm U'_\pm(x_i^\pm) > 0$ for all i. It follows that

$$(6.7) \qquad \int_K^{+\infty} I_+(t)q(t)\mathrm{d}t \leq U'_+(K),$$

$$(6.8) \qquad \int_{-\infty}^{-K} I_-(t)q(t)\mathrm{d}t \leq -U'_-(-K),$$

i.e., (6.2) holds.

Conversely, suppose that there is a continuous γ satisfying (6.2) with $\inf_{|x|\geq K}\gamma(x) > 0$ for some constant $K > 0$. Let $U'_\pm(\pm K)$ be chosen such that both (6.7) and (6.8) are valid as strict inequalities. For arbitrarily fixed $U_\pm(\pm K) > 0$, we define positive functions $U_\pm$ through the formula in (6.6) and let U be a positive C^2 function in $\mathbb{R}^1$ which is defined by (6.5) when $|x| \geq K$. Then U satisfies the equation (6.3) when $|x| > K$, hence it is a Lyapunov function in $\mathbb{R}^1$ with respect to $L^*_{V,a}$ if it is also a compact function.

We now claim that $U(+\infty) = U(-\infty) = +\infty$, which implies that U is an unbounded Lyapunov function. Suppose for contradiction that U_+ is bounded. Since $U'_+(x) > 0$ as $x \geq K$, $U_+(+\infty)$ exists. It follows that

$$\begin{aligned} \infty &> \int_K^\infty I_+(x)^{-1}\left(-\int_K^x I_+(u)q(u)\mathrm{d}u + U'(K)\right)\mathrm{d}x \\ &\geq \int_K^\infty I_+(x)^{-1}\mathrm{d}x\left(U'(K) - \int_K^\infty I_+(u)q(u)\mathrm{d}u\right) =: C\int_K^\infty I_+(x)^{-1}\mathrm{d}x. \end{aligned}$$

Since $a(t)$ is bounded from above and $\gamma(t)$ is bounded away from 0, $q(t)$ is bounded away from 0. It follows from (6.2) that

$$\int_K^\infty I_+(x)\mathrm{d}x < \infty.$$

Then by Hölder's inequality,

$$\infty = \int_K^\infty I_+(x)^{-\frac{1}{2}}I_+(x)^{\frac{1}{2}}\mathrm{d}x \leq \left(\int_K^\infty I_+(x)^{-1}\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_K^\infty I_+(x)\mathrm{d}x\right)^{\frac{1}{2}} < \infty,$$

a contradiction. Similarly, $U_-(-\infty) = +\infty$. $\qquad\square$

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