DIMENSION OF STABLE SETS AND SCRAMBLED SETS IN POSITIVE FINITE ENTROPY SYSTEMS

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ABSTRACT. We study dimension of stable sets and scrambled sets of a dynamical system with positive finite entropy. We show that there is a measure-theoretically "large" set containing points whose sets of "hyperbolic points" (i.e., points lying in the intersections of the closures of the stable and unstable sets) admit positive Bowen dimension entropies, which, under a continuum hypothesis, also contains a scrambled set with positive Bowen dimension entropy. For several kinds of specific invertible dynamical systems, lower bounds of Hausdorff dimension of these sets are estimated. In particular, for a diffeomorphism on a smooth Riemannian manifold with positive entropy, such a lower bound is given in terms of the metric entropy and Lyapunov exponent.

1. INTRODUCTION

Throughout of the paper, by a *topologically dynamical system* (X,T) (TDS for short) we mean a compact metric space (X,d) with a continuous map T from X into itself, where d denotes the metric on X. If T is a homeomorphism, then we say that the TDS (X,T) is *invertible*. For a TDS (X,T), the *stable set* of a point $x \in X$ is defined as

$$W^s(x,T) = \{y \in X : \lim_{n\to+\infty} d(T^n x, T^n y) = 0\}.$$

If (X,T) is invertible, then we can also define the *unstable set* of a point $x \in X$ as

$$W^u(x,T) = \{y \in X : \lim_{n\to+\infty} d(T^{-n} x, T^{-n} y) = 0\}.$$

A pair of points $x, y \in X$ is said to be a *Li-Yorke pair with modulus* δ if

$$\limsup_{n\to+\infty} d(T^n x, T^n y) = \delta > 0 \text{ and } \liminf_{n\to+\infty} d(T^n x, T^n y) = 0.$$

A subset $S \subset X$ is called *scrambled* if any pair of distinct points $x, y \in S$ forms a Li-Yorke pair. If X contains an uncountable scrambled set, then the TDS (X,T) is said to be *chaotic in the sense of Li-Yorke* [20].

For a general TDS with positive entropy, it was shown by Blanchard et al [2] that any TDS with positive entropy always contains an uncountable scrambled set, hence it is chaotic in the sense of Li-Yorke. In fact, Blanchard and Huang [3] further showed that such a TDS even admits a certain amount of weak mixing dynamics – a stronger chaos than that defined by Li and Yorke.

For an Anosov diffeomorphism T on a compact manifold, it is well-known that points belonging to the stable set are asymptotic under T and tend to diverge under T^{-1}, while pairs belonging to the unstable set behave in the opposite way. A natural question concerning

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a general TDS with finite positive entropy is then whether it can retains a faint flavor of such a dynamical behavior. Recently, Blanchard, Host and Ruette [1] showed for any invertible TDS (X,T) with positive entropy that the stable sets for T are not stable for T^{-1}, i.e., if a T-invariant ergodic measure μ has positive entropy, then there is $\delta > 0$ such that for μ-a.e. $x \in X$, one can find an uncountable subset F_x of $W^s(x,T)$ such that for any $y \in F_x \setminus \{x\}$, $\{x,y\}$ forms a Li-Yorke pair for T^{-1} with modulus δ. For a C^2 diffeomorphism f on a closed smooth manifold M, it was further shown by Sumi [29] that if the metric entropy with respect to a f-invariant ergodic probability measure μ is positive, then, for μ-a.e. $x \in M$, both the stable set $\overline{W^s(x,f)}$ and the unstable set $\overline{W^u(x,f)}$ contain uncountable scrambled sets. Generalizing this to a TDS, it was shown by the second author in [14] that in any (invertible) TDS with positive entropy there is a measure-theoretically "rather big" set such that the closure of the stable (or unstable) sets of points in the set contains a weak mixing set.

With these known descriptions and characterizations, it is of fundamental importance to know how "big" these stable sets or scrambled sets can be in a positive entropy system. Usually one refers a set E in a TDS (X,T) as "big" if one of the following properties holds:

a) E is of a positive measure with respect to some T-invariant measure on X;

b) E has a non-empty interior or is a dense G_δ set in X;

c) $h^B_{top}(T|E)$, the Bowen dimension entropy of E, is positive;

d) $H_d(E)$, the Hausdorff dimension of E with respect to d, is positive.

It is not hard to see that, the properties a) and b) do not hold in general for a scrambled set or the closure of a stable set. This leave the properties c) and d) as possible criteria to describe the "bigness".

The purpose of this paper is to investigate the Bowen dimension entropy and Hausdorff dimension of the stable sets and scrambled sets in a TDS with positive finite entropy. Our main results state as follows.

Theorem 1. *Let (X,T) be a TDS with metric d and $h_{top}(T) < \infty$. If μ is a T-invariant ergodic measure on X with $h_\mu(T) > 0$, then the following holds.*

1) (Bowen dimension) $h^B_{top}(T|\overline{W^s(x,T)}) \geq h_\mu(T)$ *for μ-a.e $x \in X$, and moreover, under the continuum hypothesis, for μ-a.e $x \in X$ there exists a scrambled set $S_x \subseteq \overline{W^s(x,T)}$ for T satisfying $h^B_{top}(T|S_x) \geq h_\mu(T)$.*
2) (Hausdorff dimension) *If T is a Lipschitz continuous self-map with Lipschitz constant $L > 1$, then $H_d(\overline{W^s(x,T)}) \geq \frac{h_\mu(T)}{\log L}$ for μ-a.e $x \in X$, and moreover, under the continuum hypothesis, for μ-a.e $x \in X$ there exists a scrambled set $S_x \subseteq \overline{W^s(x,T)}$ for T satisfying $H_d(S_x) \geq \frac{h_\mu(T)}{\log L}$.*

Theorem 2. *Let (X,T) be an invertible TDS with metric d and $h_{top}(T) < \infty$. If μ is a T-invariant ergodic measure on X with $h_\mu(T) > 0$, then the following holds.*

1) (Bowen dimension) *For μ-a.e $x \in X$,*

$$h^B_{top}(T|\overline{W^s(x,T)} \cap \overline{W^u(x,T)}) \geq h_\mu(T) \text{ and } h^B_{top}(T^{-1}|\overline{W^s(x,T)} \cap \overline{W^u(x,T)}) \geq h_\mu(T).$$

Moreover, under the continuum hypothesis, for μ-a.e $x \in X$ there exists $S_x \subseteq \overline{W^s(x,T)} \cap \overline{W^u(x,T)}$ which is a scrambled set for both T and T^{-1} such that $h^B_{top}(T|S_x) \geq h_\mu(T)$ and $h^B_{top}(T^{-1}|S_x) \geq h_\mu(T)$.

2) (Hausdorff dimension) *If T is Lipschitz continuous with Lipschitz constant $L > 1$, then $H_d(\overline{W^s(x,T)} \cap \overline{W^u(x,T)}) \geq \frac{h_\mu(T)}{\log L}$ for μ-a.e $x \in X$, and moreover, under the continuum hypothesis, for μ-a.e $x \in X$ there exists $S_x \subseteq \overline{W^s(x,T)} \cap \overline{W^u(x,T)}$ which is a scrambled set for both T and T^{-1} such that $H_d(S_x) \geq \frac{h_\mu(T)}{\log L}$.*

Using the variational principle of entropy, the following result is a direct consequence of Theorems 1, 2.

Corollary. *Let (X,T) be a TDS with metric d. We assume that $h_{top}(T) < \infty$ and T is a Lipschitz continuous self-map with Lipschitz constant $L > 1$. Then the following holds.*

a) $\sup_{x\in X} H_d(\overline{W^s(x,T)}) \geq \frac{h_{top}(T)}{\log L}$*; and under the continuum hypothesis,*

$$\sup\{H_d(S) : S \text{ is a scrambled set for } T\} \geq \frac{h_{top}(T)}{\log L}.$$

b) *If T is invertible, then* $\sup_{x\in X} H_d(\overline{W^s(x,T)} \cap \overline{W^u(x,T)}) \geq \frac{h_{top}(T)}{\log L}$*; and under the continuum hypothesis,*

$$\sup\{H_d(S) : S \text{ is a scrambled set for } T, T^{-1}\} \geq \frac{h_{top}(T)}{\log L}.$$

For a differentiable self-map on a smooth Riemannian manifold, a lower bound of the Hausdorff dimension of the stable set and the scrambled set with respect to an ergodic measure μ can be estimated in terms of the metric entropy and the top Lyapunov exponent χ^1_μ (see Section 6 for detail).

Theorem 3. *Let f be a C^1 self-map on a smooth Riemannian manifold M and ρ be the metric on M induced by the Riemannian structure. If μ is a f-invariant ergodic measure with a compact support Λ such that $h_\mu(f) > 0$, then the following holds.*

1) *$H_\rho(\overline{W^s(x,f)}) \geq \frac{h_\mu(f)}{\chi^1_\mu}$ for μ-a.e. $x \in \Lambda$, and under the continuum hypothesis, for μ-a.e. $x \in \Lambda$ there exists a scrambled set $S_x \subseteq \overline{W^s(x,f)}$ for f satisfying $H_\rho(S_x) \geq \frac{h_\mu(f)}{\chi^1_\mu}$.*

2) *If f is a diffeomorphsim, then $H_\rho(\overline{W^s(x,f)} \cap \overline{W^u(x,f)}) \geq \frac{h_\mu(f)}{\chi^1_\mu}$ for μ-a.e. $x \in \Lambda$, and under the continuum hypothesis, for μ-a.e. $x \in \Lambda$ there exists a scrambled set $S_x \subseteq \overline{W^s(x,f)} \cap \overline{W^u(x,f)}$ for both f and f^{-1} satisfying $H_\rho(S_x) \geq \frac{h_\mu(f)}{\chi^1_\mu}$.*

Remark. *We conjecture that Theorems 1, 2 still hold without assuming $h_{top}(T) < \infty$. We also conjecture that the continuum hypothesis contained in all Theorems 1-3 should be removed.*

The paper is organized as follows. Section 2 is a preliminary section in which we review some notions of ergodic theory and TDS. In Section 3, we discuss some basic properties of Bowen dimension entropy for non-compact sets which will be used in later sections. In Section 4, we prove parts 1) of Theorems 1,2 by estimating a lower bound of Bowen dimension entropy for stable sets as well as for scrambled sets in a positive finite entropy system. In Section 5, we prove parts 2) of Theorems 1, 2 by estimating a lower bound of Hausdorff dimension for stable sets as well as for scrambled sets in a Lipschitz continuous TDS with positive finite entropy. We prove Theorem 3 in Section 6.

2. Preliminary

Given a TDS (X,T), we denote by $\mathcal{B}_X$ the σ-algebra of Borel subsets of X. A *cover* of X is a family of Borel subsets of X whose union is X. An *open cover* is one that consists of open sets. A *partition* of X is a cover of X consisting of pairwise disjoint sets. Given a partition α of X and $x \in X$, we denote by $\alpha(x)$ the atom of α containing x.

We denote the set of finite partitions, finite covers and finite open covers, of X, respectively, by $\mathcal{P}_X$, $\mathcal{C}_X$ and $\mathcal{C}_X^o$, respectively. Given two covers $\mathcal{U}, \mathcal{V}$ of X, $\mathcal{U}$ is said to be finer than $\mathcal{V}$ (denote by $\mathcal{U} \succeq \mathcal{V}$) if each element of $\mathcal{U}$ is contained in some element of $\mathcal{V}$. Let $\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$. It is clear that $\mathcal{U} \vee \mathcal{V} \succeq \mathcal{U}$ and $\mathcal{U} \vee \mathcal{V} \succeq \mathcal{V}$. Given integers M, N with $M \leq N$ and $\mathcal{U} \in \mathcal{C}_X$, we use $\mathcal{U}_M^N$ to denote $\bigvee_{n=M}^N T^{-n}\mathcal{U}$.

For $\mathcal{U} \in \mathcal{C}_X$, we define $N(\mathcal{U})$ as the minimum among the cardinalities of the subcovers of $\mathcal{U}$. Then the *topological entropy* of $\mathcal{U}$ with respect to T is defined by

$$h_{top}(T,\mathcal{U}) = \lim_{N \to +\infty} \frac{1}{N} \log N(\mathcal{U}_0^{N-1}) = \inf_{N \in \mathbf{N}} \frac{1}{N} \log N(\mathcal{U}_0^{N-1}).$$

The *topological entropy* of (X,T) is defined by

$$h_{top}(T) = \sup_{\mathcal{U} \in \mathcal{C}_X^o} h_{top}(T,\mathcal{U}).$$

We sometimes write $h_{top}(T,X)$ to emphasis the dependence of the entropy on the space X.

Let K be a non-empty closed subset of X. For $\epsilon > 0$, a subset F of X is called (n,ϵ)-*spanning set* of K, if for any $x \in K$, there exists $y \in F$ with $d_n(x,y) \leq \epsilon$, where $d_n(x,y) = \max_{0 \leq i \leq n-1} d(T^i x, T^i y)$; a subset E of K is called (n,ϵ)-*separated set* of K if $x, y \in E, x \neq y$ implies $d_n(x,y) > \epsilon$. Let $r_n(d,T,\epsilon,K)$ denote the smallest cardinality among all (n,ϵ)-spanning sets of K and $s_n(d,T,\epsilon,K)$ denote the largest cardinality among all (n,ϵ)-separated subsets of K. We define

$$r(d,T,\epsilon,K) = \limsup_{n \to +\infty} \tfrac{1}{n} \log r_n(d,T,\epsilon,K),$$
$$s(d,T,\epsilon,K) = \limsup_{n \to +\infty} \tfrac{1}{n} \log s_n(d,T,\epsilon,K).$$

Obviously, $r(d,T,\epsilon,K)$ and $s(d,T,\epsilon,K)$ are monotonically increasing when $\epsilon \searrow 0$. Let

$$h_*(d,T,K) = \lim_{\epsilon \to 0+} r(d,T,\epsilon,K) \text{ and } h^*(d,T,K) = \lim_{\epsilon \to 0+} s(T,d,\epsilon,K).$$

It is well-known that $h_*(d,T,K) = h^*(d,T,K)$ which is independent of the choice of a compatible metric d on X, so we simply denote it by $h(T,K)$. When $K = X$, $h(T,X) = h_{top}(T)$.

Given $\mathcal{U} \in \mathcal{C}_X$, define

$$N(\mathcal{U}|K) = \min\{\text{the cardinality of } \mathcal{F} | \mathcal{F} \subset \mathcal{U}, \bigcup_{F \in \mathcal{F}} F \supset K\},$$

and

$$h(T,\mathcal{U}|K) = \limsup_{n \to +\infty} \frac{1}{n} \log N(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}|K).$$

It is easy to see that $h(T,K) = \sup_{\mathcal{U} \in \mathcal{C}_X^o} h(T,\mathcal{U}|K)$.

Let $\mathcal{M}(X)$, $\mathcal{M}(X,T)$, and $\mathcal{M}^e(X,T)$, respectively, be the set of all Borel probability measures, T-invariant Borel probability measures, and T-invariant ergodic Borel probability measures on X, respectively. Then $\mathcal{M}(X)$ and $\mathcal{M}(X,T)$ are all convex, compact metric spaces when endowed with the weak*-topology.

For any given $\alpha \in \mathcal{P}_X$, $\mu \in \mathcal{M}(X)$ and any sub-σ-algebra $\mathcal{C} \subseteq \mathcal{B}_\mu$, where $\mathcal{B}_\mu$ is the completion of $\mathcal{B}_X$ under μ, the *conditional informational function* of α relevant to $\mathcal{C}$ is defined by

$$I_\mu(\alpha|\mathcal{C})(x) := \sum_{A\in\alpha} -1_A(x)\log \mathrm{E}(1_A|\mathcal{C})(x),$$

where $\mathrm{E}(1_A|\mathcal{C})$ is the conditional expectation of 1_A with respect to $\mathcal{C}$. Let

$$H_\mu(\alpha|\mathcal{C}) = \int_X I_\mu(\alpha|\mathcal{C})(x)\,d\mu(x) = \sum_{A\in\alpha}\int_X -\mathrm{E}(1_A|\mathcal{C})\log \mathrm{E}(1_A|\mathcal{C})d\mu.$$

Then $H_\mu(\alpha|\mathcal{C})$ increases with respect to α and decreases with respect to $\mathcal{C}$.

When $\mu \in \mathcal{M}(X,T)$ and $\mathcal{C}$ is T-invariant (i.e. $T^{-1}\mathcal{C} = \mathcal{C}$), it is not hard to see that $H_\mu(\alpha_0^{n-1}|\mathcal{C})$ is a non-negative and sub-additive sequence for a given $\alpha \in \mathcal{P}_X$. Thus

$$h_\mu(T,\alpha|\mathcal{C}) = \lim_{n\to+\infty}\frac{1}{n}H_\mu(\alpha_0^{n-1}|\mathcal{C}) = \inf_{n\geq 1}\frac{1}{n}H_\mu(\alpha_0^{n-1}|\mathcal{C})$$

is well defined. It is well-known that $h_\mu(T,\alpha|\mathcal{C}) = H_\mu(\alpha|\vee_{n=1}^\infty T^{-n}\alpha \vee \mathcal{C})$. If $\mathcal{C} = \{\emptyset, X\}(\bmod\ \mu)$, we write $H_\mu(\alpha|\mathcal{C})$ and $h_\mu(T,\alpha|\mathcal{C})$ by $H_\mu(\alpha)$ and $h_\mu(T,\alpha)$ respectively. The *measure-theoretic entropy* of μ is defined by

$$h_\mu(T) = \sup_{\alpha\in\mathcal{P}_X} h_\mu(T,\alpha).$$

If, in addition, T is invertible, then

$$h_\mu(T,\alpha|\mathcal{C}) = h_\mu(T^{-1},\alpha|\mathcal{C}) \text{ for any } \alpha\in\mathcal{P}_X \text{ and } h_\mu(T) = h_\mu(T^{-1}).$$

It is also well-known that for $\alpha \in \mathcal{P}_X$, $h_\mu(T,\alpha) = h_\mu(T,\alpha|P_\mu(T)) \leq H_\mu(\alpha|P_\mu(T))$, where $P_\mu(T)$ is the Pinsker σ-algebra of $(X,\mathcal{B}_\mu,\mu,T)$.

The following result is a conditional version of Shanon-McMillan-Breiman Theorem. Its proof is completely similar to the proof of Shanon-McMillan-Breiman Theorem (see e.g. [13]).

Theorem 2.1. *Let (X,T) be an invertible TDS, $\mu \in \mathcal{M}(X,T)$, $\alpha \in \mathcal{P}_X$ and $\mathcal{C} \subseteq \mathcal{B}_\mu$ a T-invariant sub-σ-algebra. Then there exists a T-invariant function $f \in L^1(\mu)$ such that*

$$\int_X f(x)d\mu(x) = h_\mu(T,\alpha|\mathcal{C}) \text{ and } \lim_{n\to+\infty}\frac{I_\mu(\alpha_0^{n-1}|\mathcal{C})(x)}{n} = f(x) \text{ for } \mu\text{-a.e. } x\in X$$

as well as in the sense of $L^1(\mu)$. Moreover, if μ is ergodic then $f(x) = h_\mu(T,\alpha|\mathcal{C})$ for μ-a.e. $x \in X$.

3. Bowen dimension entropy of non-compact set

The notion Bowen dimension entropy for non-compact sets in a TDS was introduced by Bowen in [7] and Pesin and Pitskel in [27]. Let (X,T) be a TDS and $\mathcal{U} \in \mathcal{C}_X$. For $E \subseteq X$ we write $E \prec \mathcal{U}$ if E is contained in some element of $\mathcal{U}$. Let $n_{T,\mathcal{U}}(E)$ be the biggest non-negative integer such that $T^kE \prec \mathcal{U}$ for every $k \in \{0,1,\cdots,n_{T,\mathcal{U}}(E)-1\}$; $n_{T,\mathcal{U}}(E) = 0$ if $E \not\prec \mathcal{U}$ and $n_{T,\mathcal{U}}(E) = +\infty$ if $T^kE \prec \mathcal{U}$ for any $k \in \mathbb{Z}_+$.

Let $Y \subseteq X$. For each $s \geq 0$ and $k \in \mathbb{N}$, denote

$$m_k(T, s, \mathcal{U}|Y) = \inf\{\sum_{i=1}^{\infty} e^{-s n_{T,\mathcal{U}}(U_i)} : \cup_{i \in \mathbf{N}} U_i \supset Y \text{ and } n_{T,\mathcal{U}}(U_i) \geq k \text{ for each } i \in \mathbb{N} \}.$$

Since $m_k(T, s, \mathcal{U}|Y)$ is increasing with respect to $k \in \mathbb{N}$,

$$m(T, s, \mathcal{U}|Y) =: \lim_{k \to +\infty} m_k(T, s, \mathcal{U}|Y)$$

is well defined. It is clear that $m(T, s, \mathcal{U}|Y) \leq m(T, s', \mathcal{U}|Y)$ if $s \geq s' \geq 0$ and $m(T, s, \mathcal{U}|Y) \notin \{0, +\infty\}$ for at most one point $s \geq 0$. We define *the Bowen dimension entropy* of Y relative to $\mathcal{U}$ by

$$\begin{aligned} h^B_{top}(T, \mathcal{U}|Y) &= \inf\{s \geq 0 : m(T, s, \mathcal{U}|Y) = 0\} \\ &= \sup\{s \geq 0 : m(T, s, \mathcal{U}|Y) = +\infty\}, \end{aligned}$$

and define *the Bowen dimension entropy of* Y by

$$h^B_{top}(T|Y) = \sup_{\mathcal{U} \in \mathcal{C}^o_X} h^B_{top}(T, \mathcal{U}|Y).$$

When $Y = X$, we omit the restriction on X and simply denote $h^B_{top}(T, \mathcal{U}|X)$, $h^B_{top}(T|X)$ by $h^B_{top}(T, \mathcal{U})$, $h^B_{top}(T)$, respectively. It is well-known that $h^B_{top}(T) = h_{top}(T)$ (see [7]).

We now give an equivalent definition of Bowen dimension entropy of non-compact set. For $k \in \mathbb{N}$, $x \in X$ and $r > 0$, let

$$B_k(x, r, T) = \{x' \in X : d_k(x, x') < r\}$$

be the *Bowen ball.* We define $B_\infty(x, r, T) = \{x\}$ when $x \in X$ and $r > 0$. Let $E \subseteq X$ and $\epsilon > 0$. For any $n \in \mathbf{N}$ and $s \geq 0$, denote

$$M_n(T, s, \epsilon|E) = \inf\{\sum_{i=1}^{\infty} e^{-n_i s} : \cup_{i=1}^{\infty} B_{n_i}(x_i, \epsilon, T) \supseteq E \text{ and } n_i \geq n \text{ for } i \in \mathbb{N}\}.$$

Since $M_n(T, s, \epsilon|Y)$ is increasing with respect to $n \in \mathbb{N}$,

$$M(T, s, \epsilon|E) =: \lim_{n \to +\infty} M_n(T, s, \epsilon|E)$$

is well defined. It is clear that $M(T, s, \epsilon|E) \leq M(T, s', \epsilon|E)$ if $s \geq s' \geq 0$ and $M(T, s, \epsilon|E) \notin \{0, +\infty\}$ for at most one point $s \geq 0$. Let

$$\begin{aligned} h^B_{top}(T, \epsilon|E) &= \inf\{s \geq 0 : M(T, s, \epsilon|E) = 0\} \\ &= \sup\{s \geq 0 : M(T, s, \epsilon|E) = +\infty\}. \end{aligned}$$

The following result is well-known (See e.g. [26], Remark (1), pp.74).

Proposition 3.1. *Let* (X, T) *be a TDS and* $E \subseteq X$. *Then*

$$h^B_{top}(T|E) = \lim_{\epsilon \to 0} h^B_{top}(T, \epsilon|E).$$

The Bowen dimension entropy is a monotonic function of sets, i.e., if $E \subseteq F$ then $h^B_{top}(T|E) \leq h^B_{top}(T|F)$. Moreover if $\{E_n\}_{n \geq 1}$ is a countable family of subsets of X then

$$h^B_{top}(T|\bigcup_{n=1}^{\infty} E_n) = \sup_{n \geq 1} h^B_{top}(T|E_n).$$

Hence if $E \subset X$ is countable, then $h^B_{top}(T|E) = 0$.

Let (X,T) and (Y,S) be two TDSs. A continuous map $\pi:(X,T)\to(Y,S)$ called a *homomorphism* or a *factor map* between (X,T) and (Y,S) if it is onto and $\pi T=S\pi$. In this case we say (X,T) is an *extension* of (Y,S) or (Y,S) is a *factor* of (X,T). The following results are elementary (See e.g., [7, Proposition 2]).

Proposition 3.2. *Let (X,T) and (Y,S) be two TDSs and $\pi:(X,T)\to(Y,S)$ be a factor map. Then for any $E\subset X$,*

(1) $h^B_{top}(T|E)=h^B_{top}(T|T(E))$;
(2) $h^B_{top}(T|E)\geq h^B_{top}(S|\pi(E))$;
(3) $h^B_{top}(T^k|E)=kh^B_{top}(T|E)$, $k\in\mathbf{N}$.

Theorem 3.3. *Let $\pi:(X,T)\to(Y,S)$ be a factor map between two TDSs. Then for any $E\subseteq X$,*

$$h^B_{top}(T|E)\leq h^B_{top}(S|\pi(E))+\sup_{y\in Y}h(T,\pi^{-1}(y)). \tag{3.1}$$

Proof. We follow the argument in the proof of Theorem 17 in [5]. Let d be a compatible metric on X and ρ a compatible metric on Y. If $\sup_{y\in Y}h(T,\pi^{-1}(y))=\infty$, (3.1) is clear. In the following we assume that $a:=\sup_{y\in Y}h(T,\pi^{-1}(y))<\infty$.

Fix $\epsilon>0$ and $\tau>0$. For each $y\in Y$, choose $m(y)\in\mathbf{N}$ such that

$$\frac{1}{m(y)}\log r_{m(y)}(d,T,\epsilon,\pi^{-1}(y))\leq h(T,\pi^{-1}(y))+\tau\leq a+\tau.$$

Let E_y be a $(m(y),\epsilon)$-spanning set of $\pi^{-1}(y)$ with $|E_y|=r_{m(y)}(d,T,\epsilon,\pi^{-1}(y))$. Denote

$$U_y=\{u\in X:\text{there exists } z\in E_y \text{ such that } d_{m(y)}(u,z)<2\epsilon\}.$$

Then U_y is an open neighborhood of $\pi^{-1}(y)$ and $(X\setminus U_y)\bigcap\bigcap_{\gamma>0}\pi^{-1}(\overline{B_\gamma(y)})=\emptyset$, where $B_\gamma(y)=\{y'\in Y:\rho(y',y)<\gamma\}$. By the finite intersection property of compact sets, there is a $W_y=B_{\gamma_y}(y)$ for some $\gamma_y>0$ such that $U_y\supset\pi^{-1}(W_y)$. Since Y is compact, there exist $y_1,\cdots,y_r$ such that $W_{y_1},\cdots,W_{y_r}$ cover Y. Let $\delta_1>0$ be a Lebesgue number of open cover $\{W_{y_1},\cdots,W_{y_r}\}$ with respect to ρ, and denote $\delta=\frac{\delta_1}{2}$, $M=\max\{m(y_1),...,m(y_r)\}$.

Let $y\in Y$ and $m\in\mathbb{N}$. We claim that *there exist $\ell(y)>0$ and $v_1(y),v_2(y),\cdots,v_{\ell(y)}(y)\in X$ such that $\ell(y)\leq e^{(a+\tau)(m+M)}$ and $\bigcup_{i=1}^{\ell(y)}B_m(v_i(y),4\epsilon,T)\supseteq\pi^{-1}(B_m(y,\delta,S))$, where $B_m(y,\delta,S)=\{y'\in Y:\rho_m(y,y')<\delta\}$ and $B_m(v_i(y),4\epsilon,T)=\{x'\in X:d_m(v_i(y),x')<4\epsilon\}$.*

For each $0\leq j<m$, we choose $\Delta_y(j)\in\{y_1,...,y_r\}$ such that $\overline{B_\delta(S^j(y))}\subset W_{\Delta_y(j)}$. Define the sequence $t_0,\cdots,t_q$ depending on y recursively such that $t_0(y)=0$ and $t_{s+1}(y)=t_s(y)+m(\Delta_y(t_s(y)))$ until one gets a $t_{q+1}(y)\geq m$. For $z_0\in E_{\Delta_y(t_0(y))},z_1\in E_{\Delta_y(t_1(y))},\cdots,z_{q(y)}\in E_{\Delta_y(t_{q(y)}(y))}$, we let

$$\begin{aligned}V(y;z_0,\ldots,z_{q(y)})=\{u\in X:d(T^{t+t_s(y)}(u),T^t(z_s))<2\varepsilon\\ \text{for all } 0\leq t<m(\Delta_y(t_s(y))) \text{ and } 0\leq s\leq q(y)\}.\end{aligned}$$

It is not hard to see that

$$\bigcup_{z_0\in E_{\Delta_y(t_0(y))},\cdots,z_{q(y)}\in E_{\Delta_y(t_{q(y)}(y))}}V(y;z_0,\ldots,z_{q(y)})\supseteq\pi^{-1}(B_m(y,\delta,S)). \tag{3.2}$$

For $z_0 \in E_{\Delta_y(t_0(y))}, \dots, z_{q(y)} \in E_{\Delta_y(t_{q(y)}(y))}$, we pick any $v(z_0, \dots, z_{q(y)})$ from $V(y; z_0, ..., z_{q(y)})$. It is clear that $B_m(v(z_0, ..., z_{q(y)}), 4\epsilon, T) \supseteq V(y; z_0, ..., z_q(y))$. Moreover, by (3.2), we have

$$\bigcup_{z_0 \in E_{\Delta_y(t_0(y))}, \cdots, z_{q(y)} \in E_{\Delta_y(t_{q(y)}(y))}} B_m(v(z_0, ..., z_{q(y)}), 4\varepsilon, T) \supset \pi^{-1}(B_m(y, \delta, S)). \tag{3.3}$$

Let $\ell(y) = \prod_{s=0}^{q(y)} |E_{\Delta_y(t_s(y))}| = \prod_{s=0}^{q(y)} r_{m(\Delta_y(t_s(y)))}(d, T, \epsilon, \pi^{-1}(\Delta_y(t_s(y))))$. Clearly,

$$\begin{aligned} \ell(y) &= e^{\sum_{s=0}^{q(y)} \log r_{m(\Delta_y(t_s(y)))}(d,T,\epsilon,\pi^{-1}(\Delta_y(t_s(y))))} \\ &\le e^{(a+\tau)\sum_{s=0}^{q(y)} m(\Delta_y(t_s(y)))} = e^{(a+\tau)t_{q(y)+1}(y)} \le e^{(a+\tau)(m+M)}. \end{aligned}$$

Since the number of permissible $(z_0, ..., z_{q(y)})$ is $\ell(y)$, we may let $v_1(y), v_2(y), \cdots, v_{\ell(y)}(y)$ be an enumeration of $\{v(z_0, \dots, z_{q(y)}) : z_0 \in E_{\Delta_y(t_0(y))}, \dots, z_{q(y)} \in E_{\Delta_y(t_{q(y)}(y))}\}$. Then by (3.3), $\bigcup_{i=1}^{\ell(y)} B_m(v_i(y), 4\epsilon, T) \supseteq \pi^{-1}(B_m(y, \delta, S))$. This proves the claim.

For any $n \in \mathbb{N}$ and $s \ge a + \tau$, we are to show that

$$M_n(T, s, 4\epsilon|E) \le M_n(S, s - (a + \tau), \delta|\pi(E))e^{(a+\tau)M}.$$

Let $\{B_{n_j}(w_j, \delta, S)\}_{j=1}^{\infty}$ be a cover of $\pi(E)$ satisfying $n_j \ge n$ for each $j \in \mathbb{N}$. By the above claim, for each $B_{n_j}(w_j, \delta, S)$ there exist $\ell(w_j) > 0$ and $v_1(w_j), v_2(w_j), \cdots, v_{\ell(w_j)}(w_j) \in X$ such that $\ell(w_j) \le e^{(a+\tau)(n_j+M)}$ and $\bigcup_{i=1}^{\ell(w_j)} B_{n_j}(v_i(w_j), 4\epsilon, T) \supseteq \pi^{-1}(B_{n_j}(w_j, \delta, S))$.

Now $\bigcup_{j=1}^{\infty} \bigcup_{i=1}^{\ell(w_j)} B_{n_j}(v_i(w_j), 4\varepsilon, T) \supseteq \bigcup_{j=1}^{\infty} \pi^{-1}(B_{n_j}(w_j, \delta, S)) \supseteq \pi^{-1}(\pi(E)) \supseteq E$. Then

$$M_n(T, s, 4\epsilon|E) \le \sum_{j=1}^{\infty} \sum_{i=1}^{\ell(w_j)} e^{-sn_j} \le \sum_{j=1}^{\infty} e^{-sn_j} \ell(w_j) \le e^{(a+\tau)M} \sum_{j=1}^{\infty} e^{-(s-(a+\tau))n_j}.$$

Since the above inequality is true for any $\{B_{n_j}(w_j, \delta, S)\}_{j=1}^{\infty}$, we have

$$M_n(T, s, 4\epsilon|E) \le M_n(S, s - (a + \tau), \delta|\pi(E))e^{(a+\tau)M}.$$

Let $n \to +\infty$, we have $M(T, s, 4\epsilon|E) \le M(S, s - (a + \tau), \delta|\pi(E))e^{(a+\tau)M}$. This implies that $h_{top}(T, 4\epsilon|E) \le h_{top}(S, \delta|\pi(E)) + a + \tau \le h_{top}(S|\pi(E)) + a + \tau$ (see Proposition 3.1). Finally, let $\epsilon \searrow 0$ and $\tau \searrow 0$, we have $h_{top}(T|E) \le h_{top}(S|\pi(E)) + a$. This completes the proof. □

For a TDS (X, T) with metric d and surjective map T, we define a *natural extension* $(\widetilde{X}, \widetilde{T}) \to (X, T)$, where $\widetilde{X} = \{(x_1, x_2, \cdots) : T(x_{i+1}) = x_i, x_i \in X, i \in \mathbb{N}\}$ is a subspace of the product space $X^{\mathbb{N}} = \Pi_{i=1}^{\infty} X$ endowed with the compatible metric d_T:

$$d_T((x_1, x_2, \cdots), (y_1, y_2, \cdots)) = \sum_{i=1}^{\infty} \frac{d(x_i, y_i)}{2^i},$$

$\widetilde{T} : \widetilde{X} \to \widetilde{X}$ is the shift homeomorphism defined by $\widetilde{T}(x_1, x_2, \cdots) = (T(x_1), x_1, x_2, \cdots)$. For each $i \in \mathbb{N}$, let $\pi_i : \widetilde{X} \to X$ be the projection onto the i-th coordinate. Clearly, each $\pi_i : (\widetilde{X}, \widetilde{T}) \to (X, T)$ is a factor map.

Lemma 3.4. *Let (X, T) be a TDS with a metric d and a surjective map T, $(\widetilde{X}, \widetilde{T})$ be the natural extension of (X, T) and $\pi_1 : \widetilde{X} \to X$ be the projection onto the first coordinate. Then $h^B_{top}(\widetilde{T}|K) = h^B_{top}(T|\pi_1(K))$ for any subset K of $\widetilde{X}$.*

Proof. By Theorem 3.3, we only need to prove that $h(\widetilde{T},\pi_1^{-1}(x))=0$ for any $x\in X$. Fix a $x\in X$. For any $\epsilon>0$, take an $N\in\mathbb{N}$ large enough such that $\sum_{i=N}^{\infty}\frac{\text{diam(X)}}{2^i}<\epsilon$.

Let $E_N\subseteq\pi_1^{-1}(x)$ be a finite (N,ϵ)-spanning set of $\pi_1^{-1}(x)$. We want to show that E_N is also an (n,ϵ)-spanning set of $\pi_1^{-1}(x)$ for $n>N$.

Let $n\in\mathbb{N}$ with $n>N$. For any $\widetilde{y}\in\pi_1^{-1}(x)$, since E_N is an (N,ϵ)-spanning set of $\pi_1^{-1}(x)$ there exists a $\widetilde{x}\in E_N$ such that $d_T(\widetilde{T}^i\widetilde{x},\widetilde{T}^i\widetilde{y})<\epsilon$ for all $i=0,1,\cdots,N-1$. Since for any $k\in\{N,N+1,\cdots,n-1\}$, $\pi_j(\widetilde{T}^k\widetilde{x})=\pi_j(\widetilde{T}^k\widetilde{y}))=T^{k-j+1}(x)$ for all $j=1,\cdots,k,k+1$, we have

$$\begin{aligned}
d_T(\widetilde{T}^k\widetilde{x},\widetilde{T}^k\widetilde{y}) &= \sum_{j=1}^{\infty}\frac{d(\pi_j(\widetilde{T}^k\widetilde{x}),\pi_j(\widetilde{T}^k\widetilde{x}))}{2^j}=\sum_{j=k+2}^{\infty}\frac{d(\pi_j(\widetilde{T}^k\widetilde{x}),\pi_j(\widetilde{T}^k\widetilde{x}))}{2^j}\\
&\leq \sum_{j=k+2}^{\infty}\frac{\text{diam(X)}}{2^j}\leq\sum_{j=N}^{\infty}\frac{\text{diam(X)}}{2^j}<\epsilon.
\end{aligned}$$

It follows that $(d_T)_n(\widetilde{x},\widetilde{y})<\epsilon$. Hence E_N is also an (n,ϵ)-spanning set of $\pi_1^{-1}(x)$ for $n>N$. Therefore,

$$r(d_T,\widetilde{T},\epsilon,\pi_1^{-1}(x))=\limsup_{n\to\infty}\frac{1}{n}\log r_n(d_T,\widetilde{T},\epsilon,\pi_1^{-1}(x))\leq\limsup_{n\to\infty}\frac{1}{n}\log|E_N|=0.$$

By taking $\epsilon\to 0$, we have $h(\widetilde{T},\pi_1^{-1}(x))=0$. This completes the proof. □

4. Bowen dimension entropy of stable sets and scrambled sets

Let (X,T) be an invertible TDS, $\mu\in\mathcal{M}(X,T)$, and $\mathcal{B}_\mu$ be the completion of $\mathcal{B}_X$ with respect to μ. Then $(X,\mathcal{B}_\mu,\mu,T)$ turns out to be a *Lebesgue system*. If $\{\alpha_i\}_{i\in I}$ is a countable family of finite partitions of X, then the partition $\alpha=\bigvee_{i\in I}\alpha_i$ is called a *measurable partition*. The sets $A\in\mathcal{B}_\mu$, which are unions of atoms of α, form a sub-σ-algebra of $\mathcal{B}_\mu$, denoted by $\widehat{\alpha}$ or simply by α if there is no confusion. Every sub-σ-algebra of $\mathcal{B}_\mu$ coincides with a σ-algebra constructed in this way (mod μ). For a given measurable partition α, we define $\alpha^-=\bigvee_{n=1}^{\infty}T^{-n}\alpha$ and $\alpha^T=\bigvee_{n=-\infty}^{+\infty}T^{-n}\alpha$. In the same way we can define $\mathcal{F}^-$ and $\mathcal{F}^T$ where $\mathcal{F}$ is a sub-σ-algebra of $\mathcal{B}_\mu$. It is clear that for a measurable partition α of X, $\widehat{\alpha^-}=(\widehat{\alpha})^-$ and $\widehat{\alpha^T}=(\widehat{\alpha})^T(\text{mod }\mu)$.

The *Pinsker σ-algebra* $P_\mu(T)$ of $(X,\mathcal{B}_\mu,\mu,T)$ is defined as the smallest sub-σ-algebra of $\mathcal{B}_\mu$ containing the collection $\{\xi\in\mathcal{P}_X:h_\mu(T,\xi)=0\}$. It is well-known that $P_\mu(T)=P_\mu(T^{-1})$ and $P_\mu(T)$ is T-invariant, i.e. $T^{-1}P_\mu(T)=P_\mu(T)$. Let γ be a measurable partition of X with $\widehat{\gamma}=P_\mu(T)$ (mod μ). Then μ can be disintegrated over $P_\mu(T)$ as $\mu=\int_X\mu_x d\mu(x)$ where $\mu_x\in\mathcal{M}(X)$ and $\mu_x(\gamma(x))=1$ for μ-a.e. $x\in X$. The disintegration is characterized by properties (4.1) and (4.2) below:

$$\text{for every } f\in L^1(X,\mathcal{B}_X,\mu), f\in L^1(X,\mathcal{B}_X,\mu_x) \text{ for } \mu\text{-a.e. } x\in X, \tag{4.1}$$
$$\text{and the map } x\mapsto\textstyle\int_X f(y)\,d\mu_x(y) \text{ is in } L^1(X,P_\mu(T),\mu);$$

$$\text{for every } f\in L^1(X,\mathcal{B}_X,\mu), \mathrm{E}_\mu(f|P_\mu(T))(x)=\int_X f\,d\mu_x \;\text{ for } \mu\text{-a.e. } x\in X. \tag{4.2}$$

For any $f \in L^1(X, \mathcal{B}_X, \mu)$, we also have

$$\int_X \left(\int_X f \, d\mu_x \right) \, d\mu(x) = \int_X f \, d\mu.$$

Define for μ-a.e. $x \in X$ the set $\Gamma_x = \{y \in X : \mu_x = \mu_y\}$. Then $\mu_x(\Gamma_x) = 1$ for μ-a.e. $x \in X$.

Lemma 4.1. *Let (X, T) be an invertible TDS and $\mu \in \mathcal{M}(X, T)$. Then there exist a sequence of partitions $W_i \in \mathcal{P}_X$ and a sequence of integers $0 = k_1 < k_2 < \cdots$ satisfying:*

(1) $\lim_{i \to +\infty} diam(W_i) = 0$,

(2) $\lim_{k \to +\infty} H_\mu(P_k|\mathcal{P}^-) = h_\mu(T)$, *where* $P_k = \bigvee_{i=1}^{k} T^{-k_i} W_i$ *and* $\mathcal{P} = \bigvee_{k=1}^{\infty} P_k$,

(3) $\bigcap_{n=0}^{\infty} \widehat{T^{-n}\mathcal{P}^-} = P_\mu(T)$.

Proof. The lemma follows directly from the proof of Lemma 4 in [1]. For the sake of completeness, we outline the construction of $\{W_i\}_{i=1}^\infty \subset \mathcal{P}_X$ and $0 = k_1 < k_2 < \cdots$ below.

Let $\{W_i\}_{i=1}^\infty$ be an increasing sequence of finite partitions such that $\lim_{i \to +\infty} \text{diam}(W_i) = 0$. Take $k_1 = 0$ and inductively define $k_1, k_2, \cdots$ such that

$$H_\mu(P_k|P_{q-1}^-) - H_\mu(P_k|P_q^-) < (\frac{1}{k})\frac{1}{2^{q-k}}, \ \ k = 1, 2, \cdots, q-1,$$

for each $q \geq 2$, where $P_j = \bigvee_{i=1}^{j} T^{-k_i} W_i$. It is not hard to check that (1)-(3) are satisfied (see e.g., the proof in [13] or [25]). □

Remark 4.2. *Since* $\lim_{i \to +\infty} diam(W_i) = 0$*, it is easy to see that* $(T^{-n}\mathcal{P}^-)(x) \subseteq W^s(x, T)$ *for each* $n \in \mathbb{N} \cup \{0\}$ *and* $x \in X$*, where* $(T^{-n}\mathcal{P}^-)(x)$ *is the atom of* $T^{-n}\mathcal{P}^-$ *containing* x.

Let X be a compact metric space and $\mu \in \mathcal{M}(X)$. For $K \subseteq X$, the *outer measure* of K for μ is defined by

$$\mu^*(K) = \inf\{\sum_{n \in \mathbf{N}} \mu(A_n) : A_n \in \mathcal{B}_X, K \subseteq \bigcup_{n \in \mathbf{N}} A_n\}.$$

Lemma 4.3. *Let X be a compact metric spaces and $\mu \in \mathcal{M}(X)$. If $G \subseteq X \times X$ is a Borel set with $\mu \times \mu(G) = 1$, then under the continuum hypothesis there exists a set $K \subseteq X$ with $\mu^*(K) = 1$ and $K \times K \setminus \Delta_X \subseteq G$, where $\Delta_X = \{(x, x) : x \in X\}$.*

Proof. This result is proved in [30] (see also [9] or [4, Lemma 52]). For the sake of completeness, we give a proof below.

Let $G' = \{(x, y) : (y, x) \in G\}$ and set $G^* = G \cap G'$. Since $\mu \times \mu(G^*) = 1$, we may assume without loss of generality that $G = G'$. By Fubini's Theorem, there exists a μ-measurable set $E \subseteq X$ such that $\mu(E) = 1$ and $\mu(G_y) = 1$ for any $y \in E$, where $G_y = \{x \in X : (x, y) \in G\}$.

Using the Continuum Hypothesis, we let ω be the first uncountable ordinal and let

$$B_0, B_1, \cdots, B_\alpha, \cdots \ \ (\alpha < \omega)$$

be the collection of all closed subsets B of X with $\mu(B) > 0$.

Since $\mu(E) = 1$, $\mu(B_0 \cap E) = \mu(B_0) > 0$. Thus we can choose a $x_0 \in B_0 \cap E$. Since $\mu(B_1 \cap E \cap G_{x_0}) = \mu(B_1) > 0$, we can choose a $x_1 \in B_1 \cap E \cap G_{x_0}$. Similarly we can choose a $x_2 \in B_2 \cap E \cap G_{x_0} \cap G_{x_1}$. Suppose that for all $\beta < \alpha < \omega$, we have chosen $x_\beta \in B_\beta \cap E \cap \bigcap_{\gamma<\beta} G_{x_\gamma}$.

To find x_α, consider the set

$$S_\alpha = B_\alpha \cap E \cap \bigcap_{\gamma<\alpha} G_{x_\gamma}.$$

As a countable intersection of full measure sets, $\mu(E \cap \bigcap_{\gamma<\alpha} G_{x_\gamma}) = 1$. Hence $\mu(S_\alpha) = \mu(B_\alpha) > 0$ and we can choose a $x_\alpha \in S_\alpha$. Let $K = \{x_\alpha : \alpha < \omega\}$. The fact that K intersects every closed subset B of X with $\mu(B) > 0$ implies that $\mu^*(K) = 1$. Indeed, suppose for otherwise $\mu^*(K) < 1$. Then there exists a μ-measurable subset F of X such that $K \subseteq F$ and $\mu(X \setminus F) > 0$. Since the measure μ is regular, there exists an $\alpha_0 < \omega$ such that $B_{\alpha_0} \subseteq X \setminus F$. Then $K \cap B_{\alpha_0} \subseteq K \cap (X \setminus F) = \emptyset$, which is a contradiction to the fact that $x_{\alpha_0} \in K \cap B_{\alpha_0}$.

Now for any $(x, y) \in K \times K \setminus \Delta_X$, we have $x = x_\alpha$ and $y = x_\beta$ for some $\alpha \neq \beta$. If $\alpha < \beta$, then $x_\beta \in G_{x_\alpha}$ and $(x_\alpha, x_\beta) \in G$. If $\alpha > \beta$, then we similarly have $(x_\beta, x_\alpha) \in G$ and $(x_\alpha, x_\beta) \in G$ by the symmetry of G. In any case, we have $(x, y) \in G$. □

Remark 4.4. *We conjecture that Lemma 4.3 holds without the continuum hypothesis. If this conjecture is true, then conclusions (2) in our main Theorems 1-3 will be true without the continuum hypothesis.*

Let (X, T) be a TDS. A pair $\{x, y\} \in X \times X$ is said to be a *strong Li-Yorke pair* for T if it is a Li-Yorke pair and recurrent (meaning that (x, y) lies in the closure of $\{(T^n x, T^n y) \in X \times X : n \geq 1\}$); a subset S of X is called a *strong scrambled set* for T if for any $x, y \in S$ with $x \neq y$, $\{x, y\}$ is a strong Li-Yorke pair for T. It is clear that a strong scrambled set for T is a scrambled set for T. Recall that a TDS (X, T) is *transitive* if for each pair of non-empty open subsets U and V of X, there exists $n \geq 0$ such that $U \cap T^{-n}V \neq \emptyset$. A point $x \in X$ is said to be *a transitive point* if $\mathrm{orb}(x, T) = \{Tx, T^2x, ...\}$ is dense in X. If (X, T) is transitive then it is well-known that the set of transitive points forms a dense G_δ set of X (denoted by $X_{trans}(T)$).

For $\nu \in \mathcal{M}(X, T)$, the set of *generic points* of ν with respect T is defined by

$$G_\nu = \{x \in X : \lim_{n\to+\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(T^i x) = \int_X \phi \, d\nu \text{ holds for any } \phi \in C(X; \mathbb{R})\}. \tag{4.3}$$

We note that if ν is ergodic, then $(\mathrm{supp}(\nu), T)$ is transitive and $\nu(G_\nu) = 1$ by Birkhoff Pointwise Ergodic Theorem.

Proposition 4.5. *Let (X, T) be a zero-dimensional invertible TDS. If $\mu \in \mathcal{M}^e(X, T)$ with $h_\mu(T) > 0$ and $E \in \mathcal{B}_X$ with $\mu(E) = 1$, then the following holds.*

(1) *For μ-a.e $x \in X$,*

$$h^B_{top}(T|\overline{W^s(x,T)} \cap \overline{W^u(x,T)} \cap E) \geq h_\mu(T) \text{ and}$$
$$h^B_{top}(T^{-1}|\overline{W^s(x,T)} \cap \overline{W^u(x,T)} \cap E) \geq h_\mu(T).$$

(2) *Under the continuum hypothesis, for μ-a.e $x \in X$ there exists a set $S_x \subseteq \overline{W^s(x,T)} \cap \overline{W^u(x,T)} \cap E$ such that*
   (a) *S_x is a strongly scrambled set for T, T^{-1},*
   (b) *$h^B_{top}(T|S_x) \geq h_\mu(T)$ and $h^B_{top}(T^{-1}|S_x) \geq h_\mu(T)$.*

Proof. Let $\mathcal{B}_\mu$ be the completion of $\mathcal{B}_X$ with respect to μ, $P_\mu(T)$ be the Pinsker σ-algebra of $(X, \mathcal{B}_\mu, \mu, T)$, and γ be the measurable partition of X with $\widehat{\gamma} = P_\mu(T)$ (mod μ). Then μ can be disintegrated over $P_\mu(T)$ as $\mu = \int_X \mu_x d\mu(x)$ where $\mu_x \in M(X)$ and $\mu_x(\gamma(x)) = 1$ for μ-a.e. $x \in X$. Define for μ-a.e. $x \in X$ the set $\Gamma_x = \{y \in X : \mu_x = \mu_y\}$. Then for μ-a.e. $x \in X$, $\mu_x(\Gamma_x) = 1$.

Claim 1: $\text{supp}(\mu_x) \subseteq \overline{W^s(x,T)} \cap \overline{W^u(x,T)}$ for μ-a.e. $x \in X$.

This is already proved in [15] (Step 1 of the proof of Theorem 4.6). For completion, we include the proof below.

Since $P_\mu(T)$ is also the Pinsker σ-algebra of $(X, \mathcal{B}_\mu, \mu, T^{-1})$ and $W^s(x, T^{-1}) = W^u(x,T)$, by symmetry it remains to prove that for μ-a.e. $x \in X$, $\text{supp}(\mu_x) \subseteq \overline{W^s(x,T)}$. By Lemma 4.1, there exist $\{W_i\}_{i=1}^{\infty} \subset \mathcal{P}_X$ and $0 = k_1 < k_2 < \cdots$ satisfying

(1) $\lim_{i\to+\infty} \text{diam}(W_i) = 0$,
(2) $\lim_{k\to+\infty} H_\mu(P_k|\mathcal{P}^-) = h_\mu(T)$, where $P_k = \bigvee_{i=1}^{k} T^{-k_i} W_i$ and $\mathcal{P} = \bigvee_{k=1}^{\infty} P_k$,
(3) $\bigcap_{n=0}^{\infty} \widehat{T^{-n}\mathcal{P}^-} = P_\mu(T)$.

It is clear that $\mathcal{P}^-(x) \subseteq W^s(x,T)$ for $x \in X$.

Let $\mu = \int_X \mu_{n,x} d\mu(x)$ be the disintegration of μ over $\widehat{T^{-n}\mathcal{P}^-}$ for $n \in \mathbb{N}$. Then for $n \in \mathbb{N}$, $\mu_{n,x}((T^{-n}\mathcal{P}^-)(x)) = 1$ for μ-a.e. $x \in X$. Moreover, since $(T^{-n}\mathcal{P}^-)(x) \subseteq W^s(x,T)$ for each $x \in X$, $\text{supp}(\mu_{n,x}) \subseteq \overline{W^s(x,T)}$ for μ-a.e. $x \in X$.

Let $\{f_i\}_{i=1}^{\infty}$ be a dense subset of $C(X;\mathbb{R})$ with respect to the supremum norm. For each $i \in \mathbb{N}$, by *Martingale Theorem* for μ-a.e. $x \in X$

$$\lim_{n\to+\infty} \int_X f_i(y) d\mu_{n,x}(y) = \lim_{n\to+\infty} \text{E}(f_i|\widehat{T^{-n}\mathcal{P}^-})(x) = \text{E}(f_i|P_\mu(T))(x) = \int_X f_i(y) d\mu_x(y).$$

Hence there exists a measurable subset $X_0 \subseteq X$ with $\mu(X_0) = 1$ such that for each $x \in X_0$ and $i \in \mathbb{N}$,

$$\lim_{n\to+\infty} \int_X f_i(y) d\mu_{n,x}(y) = \int_X f_i(y) d\mu_x(y).$$

By a simple approximation argument, we have for each $f \in C(X;\mathbb{R})$,

$$\lim_{n\to+\infty} \int_X f(y) d\mu_{n,x}(y) = \int_X f(y) d\mu_x(y) \text{ for each } x \in X_0,$$

i.e., $\lim_{n\to+\infty} \mu_{n,x} = \mu_x$ for $x \in X_0$ under the weak*-topology. For μ-a.e. $x \in X$, since $\text{supp}(\mu_{n,x}) \subseteq \overline{W^s(x,T)}$ for all $n \in \mathbb{N}$, we have $\text{supp}(\mu_x) \subseteq \overline{W^s(x,T)}$. This proves the Claim.

Since X is zero dimensional, there exists a sequence of finite clopen partitions $\{\alpha_j\}_{j=1}^{\infty}$ (i.e. each element in α_j is closed and open) of X such that $\lim_{j\to+\infty} \text{diam}(\alpha_j) = 0$.

By Theorem 2.1, for each $j \in \mathbb{N}$ we have

$$\lim_{n\to+\infty} \frac{1}{n} I_\mu(\bigvee_{i=0}^{n-1} T^{-i}\alpha_j|P_\mu(T))(x) = h_\mu(T, \alpha_j|P_\mu(T)) = h_\mu(T, \alpha_j) \text{ for } \mu\text{-a.e. } x \in X.$$

In the above, we have used the fact that $h_\mu(T, \alpha|P_\mu(T)) = h_\mu(T, \alpha)$ for any $\alpha \in \mathcal{P}_X$.

Now since for μ-a.e. $x \in X$

$$I_\mu(\bigvee_{i=0}^{n-1} T^{-i}\alpha_j | P_\mu(T))(x) = \sum_{A \in \bigvee_{i=0}^{n-1} T^{-i}\alpha_j} -1_A(x) \log \mathrm{E}(1_A | P_\mu(T))(x)$$
$$= \sum_{A \in \bigvee_{i=0}^{n-1} T^{-i}\alpha_j} -1_A(x) \log \mu_x(A) = -\log \mu_x((\bigvee_{i=0}^{n-1} T^{-i}\alpha_j)(x)),$$

we have

$$\lim_{n\to+\infty} \frac{-\log \mu_x((\bigvee_{i=0}^{n-1} T^{-i}\alpha_j)(x))}{n} = h_\mu(T,\alpha_j) \text{ for } \mu\text{-a.e. } x \in X.$$

Thus we easily find a Borel subset X_1 of X satisfying $\mu(X_1) = 1$ and for all $j \in \mathbf{N}$, $x \in X_1$

$$\lim_{n\to+\infty} \frac{-\log \mu_x((\bigvee_{i=0}^{n-1} T^{-i}\alpha_j)(x))}{n} = h_\mu(T,\alpha_j). \tag{4.4}$$

Since $\mu(X_1) = 1$, $\mu_x(X_1) = 1$ for μ-a.e. $x \in X$. Moreover $\mu_x(\Gamma_x \cap X_1) = 1$ for μ-a.e. $x \in X$. Thus there exists a Borel set $X_2 \subseteq X_1$ of X with $\mu(X_2) = 1$ satisfying $\mu_x(\Gamma_x \cap X_1) = 1$.

Claim 2. If $x \in X_2$ and $B_x \subseteq X$ with $\mu_x^*(B_x) > 0$, then

$$h_{top}^B(T|B_x) \geq h_\mu(T) \text{ and } h_{top}^B(T^{-1}|B_x) \geq h_\mu(T).$$

Let $x \in X_2$ and $B_x \subseteq X$ with $\mu_x^*(B_x) > 0$. Put $D_x = B_x \cap \Gamma_x \cap X_1$. Since $x \in X_2$, $\mu_x(\Gamma_x \cap X_1) = 1$ and so $\mu_x^*(B_x) = \mu_x^*(D_x) > 0$. By symmetry of T, T^{-1}, $h_\mu(T) = h_\mu(T^{-1})$ and $D_x \subseteq B_x$. It remains to prove that $h_{top}^B(T|D_x) \geq h_\mu(T)$. Since $h_{top}^B(T|D_x) = \lim_{j\to+\infty} h_{top}^B(T,\alpha_j|D_x)$ and $h_\mu(T) = \lim_{j\to+\infty} h_\mu(T,\alpha_j)$, it is sufficient to show that

$$h_{top}^B(T,\alpha_j|D_x) \geq h_\mu(T,\alpha_j)$$

for all $j \in \mathbb{N}$.

Fix $j \in \mathbb{N}$. Without loss of generality, we suppose $h_\mu(T,\alpha_j) > 0$. For any $\epsilon \in (0, h_\mu(T,\alpha_j))$ and $k \in \mathbb{N}$, it follows from the fact $\mu_x = \mu_y$ for all $y \in D_x \subseteq \Gamma_x$ that

$$D_x(k,\epsilon) =: \{y \in D_x : \mu_y((\bigvee_{i=0}^{n-1} T^{-i}\alpha_j)(y)) \leq e^{-n(h_\mu(T,\alpha_j)-\epsilon)} \text{ for all } n \geq k\}$$
$$= \{y \in D_x : \mu_x((\bigvee_{i=0}^{n-1} T^{-i}\alpha_j)(y)) \leq e^{-n(h_\mu(T,\alpha_j)-\epsilon)} \text{ for all } n \geq k\}.$$

Since $D_x \subseteq X_1$, we have by (4.4) that $\bigcup_{k=1}^\infty D_x(k,\epsilon) = D_x$ for any $\epsilon > 0$. The fact $\mu_x^*(D_x) = 1$ implies that there is an $N \in \mathbb{N}$ such that $\mu_x^*(D_x(N,\epsilon)) > 0$.

Let $n \in \mathbb{N}$ with $n \geq N$ and $\{U_i : i = 1, 2, \cdots\}$ be a countable cover of $D_x(N,\epsilon)$ with $n_{T,\alpha_j}(U_i) \geq n$ for any $i \in \mathbb{N}$. For each U_i there exists a $B_i \in \bigvee_{\ell=0}^{n_{T,\alpha_j}(U_i)-1} T^{-\ell}\alpha_j$ such that $U_i \subseteq B_i$. Hence if $U_i \cap D_x(N,\epsilon) \neq \emptyset$, then $B_i \cap D_x(N,\epsilon) \supseteq U_i \cap D_x(N,\epsilon) \neq \emptyset$. Taking $x_i \in B_i \cap D_x(N,\epsilon)$, then

$$\mu_x(B_i) = \mu_x((\bigvee_{\ell=0}^{n_{T,\alpha_j}(U_i)-1} T^{-\ell}\alpha_j)(x_i)) \leq e^{-n_{T,\alpha_j}(U_i)(h_\mu(T,\alpha_j)-\epsilon)}.$$

Using the fact

$$\bigcup_{i\in\mathbb{N}: U_i \cap D_x(N,\epsilon) \neq \emptyset} B_i \supseteq \bigcup_{i\in\mathbb{N}: U_i \cap D_x(N,\epsilon) \neq \emptyset} U_i \supseteq D_x(N,\epsilon),$$

we have

$$\sum_{i=1}^{\infty} e^{-n_{T,\alpha_j}(U_i)(h_\mu(T,\alpha_j)-\epsilon)} \geq \sum_{i\in\mathbb{N}:U_i\cap D_x(N,\epsilon)\neq\emptyset} e^{-n_{T,\alpha_j}(U_i)(h_\mu(T,\alpha_j)-\epsilon)} \geq \sum_{i\in\mathbb{N}:U_i\cap D_x(N,\epsilon)\neq\emptyset} \mu_x(B_i) \geq \mu_x^*(D_x(N,\epsilon)).$$

Since $\{U_i : i = 1, 2, \cdots\}$ is arbitrary,

$$m_n(T, h_\mu(T,\alpha_j) - \epsilon, \alpha_j|D_x(N,\epsilon)) \geq \mu_x^*(D_x(N,\epsilon) > 0$$

for all $n \geq N$, which, when passing to the limit $n \to +\infty$, yields

$$m(T, h_\mu(T,\alpha_j) - \epsilon, \alpha_j|D_x(N,\epsilon)) \geq \mu_x^*(D_x(N,\epsilon)) > 0.$$

This implies that $h^B_{top}(T,\alpha_j|D_x(N,\epsilon)) \geq h_\mu(T,\alpha_j) - \epsilon$ and therefore

$$h^B_{top}(T,\alpha_j|D_x) \geq h^B_{top}(T,\alpha_j|D_x(N,\epsilon)) \geq h_\mu(T,\alpha_j) - \epsilon.$$

Now, by taking $\epsilon \searrow 0$, we have $h^B_{top}(T,\alpha_j|D_x) \geq h_\mu(T,\alpha_j)$, and the Claim is proved.

To prove (i), we note that by Claim 1 and the fact that $\mu(E) = 1$ there exists a Borel subset X_3 of X with $\mu(X_3) = 1$ such that

$$\mu_x(\overline{W^s(x,T)} \cap \overline{W^u(x,T)} \cap E) = \mu_x(E) = 1$$

for all $x \in X_3$. For each $x \in X_2 \cap X_3$, since $\mu_x(\overline{W^s(x,T)} \cap \overline{W^u(x,T)} \cap E) = 1$, we have by Claim 2 that $h^B_{top}(T|\overline{W^s(x,T)} \cap \overline{W^u(x,T)} \cap E) \geq h_\mu(T)$ and $h^B_{top}(T^{-1}|\overline{W^s(x,T)} \cap \overline{W^u(x,T)} \cap E) \geq h_\mu(T)$. This proves (i) since $\mu(X_2 \cap X_3) = 1$.

To prove (ii), we note that $\mu(X_2) = 1$. By Claim 2 it is sufficient to show under the continuum hypothesis that for μ-a.e. $x \in X$ there exists a strong scrambled set $S_x \subseteq \overline{W^s(x,T)} \cap \overline{W^u(x,T)} \cap E$ for T, T^{-1} with $\mu_x^*(S_x) = 1$.

Define a measure $\lambda(\mu)$ on X^2 by

$$\lambda(\mu) = \int_X \mu_x \times \mu_x d\mu(x).$$

It is well-known (see e.g. [2, 15]) that μ_x is non-atomic for μ-a.e. $x \in X$ and $\lambda(\mu)$ is a $T \times T$-invariant ergodic measure on $X \times X$. Let $W = \mathrm{supp}(\lambda(\mu))$. Since $\lambda(\mu)$ is an ergodic measure for $T \times T$, both $(W, T \times T)$ and $(W, (T \times T)^{-1})$ are transitive.

Since $\mu(E) = 1$, $\mu_x(E) = 1$ for μ-a.e. $x \in X$. Hence

$$\lambda(\mu)(E \times E) = \int_X \mu_x \times \mu_x(E \times E) d\mu(x) = 1.$$

Let G^+ be the set of generic points of $\lambda(\mu)$ for $T \times T$ and G^- be the set of generic points of $\lambda(\mu)$ for $(T \times T)^{-1}$. Then $\lambda(\mu)(G^+ \cap G^- \cap (E \times E)) = 1$ and

$$G^+ \cap G^- \cap (E \times E) \subset W_{trans}(T \times T) \cap W_{trans}((T \times T)^{-1}).$$

Since

$$1 = \lambda(\mu)(G^+ \cap G^- \cap (E \times E)) = \int_X \mu_x \times \mu_x(G^+ \cap G^- \cap (E \times E)) d\mu(x)$$

and μ_x is non-atomic for μ-a.e. $x \in X$, there exists a subset $X_4 \subset X$ with $\mu(X_4) = 1$ such that $\mu_x \times \mu_x(G^+ \cap G^- \cap (E \times E)) = 1$, $\mathrm{supp}(\mu_x) \subseteq \overline{W^s(x,T)} \cap \overline{W^u(x,T)}$, $\mu_x(\Gamma_x) = 1$, and μ_x is non-atomic for $x \in X_4$.

For each $x \in X_4$, let $C_x = \overline{W^s(x,T)} \cap \overline{W^u(x,T)} \cap E$. Since $\mu_x \times \mu_x(G^+ \cap G^- \cap (E \times E)) = 1$ and $\text{supp}(\mu_x) \subseteq \overline{W^s(x,T)} \cap \overline{W^u(x,T)}$, we have

$$\mu_x(C_x) = 1 \text{ and } \mu_x \times \mu_x(G^+ \cap G^- \cap (C_x \times C_x)) = 1.$$

By Lemma 4.3 there exists $S_x \subseteq X$ such that $\mu_x^*(S_x) = 1$ and $S_x \times S_x \setminus \Delta_X \subseteq G^+ \cap G^- \cap (C_x \times C_x)$. This implies that $S_x \subseteq C_x$. Since μ_x is non-atomic, S_x must be uncountable.

Next we show that for each $x \in X_4$, S_x is a strong scrambled set for T, T^{-1}. Let $(x_1, x_2) \in W_{trans}(T \times T) \cap W_{trans}((T \times T)^{-1})$. On one hand, since $\{(z,z) : z \in \text{supp}(\mu)\} \subseteq W$, we have

$$\liminf_{n \to +\infty} d(T^n x_1, T^n x_2) = 0 \text{ and } \liminf_{n \to +\infty} d(T^{-n} x_1, T^{-n} x_2) = 0.$$

On the other hand, since μ_x is non-atomic for μ-a.e $x \in X$, we have $W \not\subseteq \Delta_X$. This implies that $x_1 \neq x_2$. Hence $\{x_1, x_2\}$ is a strong Li-Yorke pair for T, T^{-1}. Since $S_x \times S_x \setminus \Delta_X \subseteq W_{trans}(T \times T) \cap W_{trans}((T \times T)^{-1})$, S_x is a strong scrambled set for T, T^{-1}. □

Definition 4.6. As in [18], an extension $\pi : (Z,R) \to (X,T)$ between two TDSs is said to be a *principal extension* if $h_\nu(R) = h_{\pi\nu}(T)$ for every $\nu \in \mathcal{M}(Z,R)$.

Lemma 4.7. ([8]) *Every invertible TDS (X,T) with $h_{top}(T) < \infty$ has a zero dimensional principal extension (Z,R) with R being invertible.*

Proof. See Proposition 7.8 in [8]. □

Remark 4.8. *For an invertible TDS (X,T) Lindenstrauss and Weiss* [22] *introduced the mean dimension $mdim(X,T)$ (an idea suggested by Gromov). It is well-known that for an invertible TDS (X,T), if $h_{top}(T) < \infty$ or the topological dimension of X is finite, then $mdim(X,T) = 0$ (see Definition 2.6 and Theorem 4.2 in* [22]*).*

In general, one can show that for an invertible TDS (X,T), if $mdim(X,T) = 0$ then (X,T) has a zero dimensional principal extension (Z,R) with R being invertible. Indeed, let (Y,S) be an irrational rotation on the circle. Then $(X \times Y, T \times S)$ admits a nonperiodic minimal factor (Y,S) and $mdim(X \times Y, T \times S) = 0$. Hence $(X \times Y, T \times S)$ has the so-called small boundary property [21, Theorem 6.2]*, which implies the existence of a basis of the topology consisting of sets whose boundaries have measure zero for every invariant measure. With these results it is easy to construct a refining sequence of small-boundary partitions for $(X \times Y, T \times S)$, where the partitions have small boundaries if their boundaries have measure zero for all $\mu \in \mathcal{M}(X \times Y, T \times S)$. Then by a standard construction (see* [8]*), associated to this sequence there exists a zero dimensional principal extension (Z,R) of $(X \times Y, T \times S)$ with R being invertible. Finally, noting that $(X \times Y, T \times S)$ is a principal extension of (X,T), we know that (Z,R) is also a zero dimensional principal extension of (X,T) since the composition of two principal extensions is still a principal extension.*

The following theorem implies Theorem 2 1).

Theorem 4.9. *Let (X,T) be an invertible TDS with $h_{top}(T) < \infty$. If $\mu \in \mathcal{M}^e(X,T)$ with $h_\mu(T) > 0$ and $E \in \mathcal{B}_X$ with $\mu(E) = 1$, then the following holds.*

(1) *For μ-a.e $x \in X$,*

$$h_{top}^B(T | \overline{W^s(x,T)} \cap \overline{W^u(x,T)} \cap E) \geq h_\mu(T) \text{ and}$$
$$h_{top}^B(T^{-1} | \overline{W^s(x,T)} \cap \overline{W^u(x,T)} \cap E) \geq h_\mu(T).$$

(2) *Under the continuum hypothesis, for μ-a.e $x \in X$ there exists a set $S_x \subseteq \overline{W^s(x,T)} \cap \overline{W^u(x,T)} \cap E$ such that*

(a) *S_x is a strong scrambled set for T, T^{-1},*
(b) *$h^B_{top}(T|S_x) \geq h_\mu(T)$ and $h^B_{top}(T^{-1}|S_x) \geq h_\mu(T)$.*

Proof. We only show (2) as the proof of (1) is similar to that of (2).

By Lemma 4.7 there exists a principal extension $\pi : (Z, R) \to (X, T)$ where Z is zero-dimensional and R is invertible. Take a $\nu \in \mathcal{M}^e(Z, R)$ such that $\pi\nu = \mu$. Since $\pi^{-1}(E) \in \mathcal{B}_Z$ with $\nu(\pi^{-1}(E)) = 1$, Proposition 4.5 implies that there exists a Borel set $Z_0 \subseteq Z$ with $\nu(Z_0) = 1$ such that for each $z \in Z_0$ there exists a strong scrambled set $S_z \subseteq \overline{W^s(z, R)} \cap \overline{W^u(z, R)} \cap \pi^{-1}(E)$ for both R and R^{-1}, and $h^B_{top}(R|S_z) \geq h_\nu(R)$ and $h^B_{top}(R^{-1}|S_z) \geq h_\nu(R)$.

Let $X_0 = \pi(Z_0)$. Then $\mu(X_0) = 1$. For $x \in X_0$, we take $z \in Z_0$ with $\pi(z) = x$ and define $S_x = \pi(S_z)$. It is clear that $S_x \subseteq \overline{W^s(x, T)} \cap \overline{W^u(x, T)} \cap E$ and S_x is a strong scrambled set for T, T^{-1}. Since $h_{top}(T) < \infty$, by the variational principle of condition entropy (see [19, 11, 16]), we have

$$\sup_{x \in X} h(R, \pi^{-1}(x)) = \sup_{\theta \in \mathcal{M}(Z,R)} (h_\theta(R) - h_{\pi\theta}(T)) = 0, \text{ and}$$
$$\sup_{x \in X} h(R^{-1}, \pi^{-1}(x)) = \sup_{\theta \in \mathcal{M}(Z,R^{-1})} (h_\theta(R^{-1}) - h_{\pi\theta}(T^{-1})) = 0.$$

Now, we have by Theorem 3.3 that $h^B_{top}(T|S_x) = h^B_{top}(R|S_z) \geq h_\nu(R) = h_\mu(T)$. Similarly, $h^B_{top}(T^{-1}|S_x) \geq h_\mu(T)$. This proves (2). □

The following theorem implies Theorem 1 1).

Theorem 4.10. *Let (X, T) be a TDS with $h_{top}(T) < \infty$. If $\mu \in \mathcal{M}^e(X, T)$ with $h_\mu(T) > 0$ and $E \in \mathcal{B}_X$ with $\mu(E) = 1$, then the following holds.*

(1) *$h^B_{top}(T|\overline{W^s(x, T)} \cap E) \geq h_\mu(T)$ for μ-a.e $x \in X$.*
(2) *Under the continuum hypothesis, for μ-a.e $x \in X$ there exists a scrambled set $S_x \subseteq \overline{W^s(x, T)} \cap E$ for T satisfying $h^B_{top}(T|S_x) \geq h_\mu(T)$.*

Proof. We only prove (2) since the proof of (1) is similar to that of (2).

Without loss of generality, we assume that T is surjective, for otherwise we can replace X with supp(μ). Let $\pi_1 : (\widetilde{X}, \widetilde{T}) \to (X, T)$ be the natural extension. Then $\widetilde{T}$ is a homeomorphism and $h_{top}(\widetilde{T}) = h_{top}(T) < \infty$. Take $\nu \in \mathcal{M}^e(\widetilde{X}, \widetilde{T})$ with $\pi_1\nu = \mu$.

By Theorem 4.9, under the continuum hypothesis there exists a Borel subset $\widetilde{X}_0$ of $\widetilde{X}$ with $\mu(\widetilde{X}_0) = 1$ such that for $\widetilde{x} \in \widetilde{X}_0$ there exists a strong scrambled set $S_{\widetilde{x}} \subseteq \overline{W^s(\widetilde{x}, \widetilde{T})}$ for T satisfying $h^B_{top}(\widetilde{T}|S_{\widetilde{x}}) \geq h_\nu(\widetilde{T})$.

Let $X_0 = \pi_1(\widetilde{X}_0)$. Obviously $\mu(X_0) = 1$. For any $x \in X_0$ take $\widetilde{x} \in \widetilde{X}_0$ with $x = \pi_1(\widetilde{x})$ and let $S_x = \pi_1(S_{\widetilde{x}})$. Then $S_x \subseteq \overline{W^s(x, T)}$ and by Lemma 3.4 $h^B_{top}(T|S_x) = h^B_{top}(\widetilde{T}|S_{\widetilde{x}}) \geq h_\nu(\widetilde{T}) = h_\mu(T)$. Now, since $\{\widetilde{y}, \widetilde{z}\} \subset \widetilde{X}$ is a Li-Yorke pair for $\widetilde{T}$ iff $\{\pi_1(\widetilde{y}), \pi_1(\widetilde{z})\}$ is a Li-Yorke pair for T, S_x is a scrambled set for T. This proves (2). □

Let (X, T) be a TDS with a compatible metric d. Given $\epsilon > 0$, the *ϵ-stable set* of x under T is the set of points whose forward orbit ϵ-shadows that of x:

$$W^s_\epsilon(x, T) = \{y \in X : d(T^n x, T^n y) \leq \epsilon \text{ for all } n = 0, 1, \cdots\}.$$

Definition 4.11. As in Bowen [6], a TDS (X,T) is called *h-expansive* if there exists an $\epsilon > 0$ such that

$$\sup_{x\in X} h(T, W^s_\epsilon(x,T)) = 0,$$

while as in Misiurewicz [23], (X,T) is called *asymptotically h-expansive* if

$$\lim_{\epsilon\to 0}\sup_{x\in X} h(T, W^s_\epsilon(x,T)) = 0.$$

It was shown by Bowen [6] that all expansive systems, expansive homeomorphisms, endomorphisms of a compact Lie group, and Axiom A diffeomorphisms are h-expansive, by Misiurewicz [23] that every continuous endomorphism of a compact metric group is asymptotically h-expansive if its entropy is finite, and by Buzzi [10] that each C^∞ diffeomorphism on a compact manifold is asymptotically h-expansive. In [23], Misiurewicz showed that for an asymptotically h-expansive system (X,T), the entropy map $\nu \in \mathcal{M}(X,T) \mapsto h_\nu(T) \in \mathbf{R}_+$ is upper semi-continuous. Hence for an asymptotically h-expansive system (X,T), there always exists a $\mu \in \mathcal{M}^e(X,T)$ such that $h_\mu(T) = h_{top}(T) < \infty$ (an asymptotically h-expansive system always admits finite topological entropy).

Corollary 4.12. *Let (X,T) be an asymptotically h-expansive, invertible TDS. Then there exists $x \in X$ for which the following holds.*

1. $h^B_{top}(T|\overline{W^s(x,T)}\cap\overline{W^u(x,T)}) = h_{top}(T)$ *and* $h^B_{top}(T^{-1}|\overline{W^s(x,T)}\cap\overline{W^u(x,T)}) = h_{top}(T)$;
2. *Under the continuum hypothesis, there exists a scrambled set* $S_x \subseteq \overline{W^s(x,T)}\cap\overline{W^u(x,T)}$ *satisfying* $h^B_{top}(T|S_x) = h_{top}(T)$ *and* $h^B_{top}(T^{-1}|S_x) = h_{top}(T)$.

Proof. We only show (2) since the proof of (1) is similar to that of (2). Since (X,T) is an asymptotically h-expansive system, we always have a $\mu \in \mathcal{M}^e(X,T)$ such that $h_\mu(T) = h_{top}(T) < \infty$. Under the continuum hypothesis, we have by Theorem 2 1) that for μ-a.e. $x \in X$ there exists a scrambled set $S_x \subseteq \overline{W^s(x,T)} \cap \overline{W^u(x,T)}$ satisfying $h^B_{top}(T|S_x) \geq h_\mu(T) = h_{top}(T)$ and $h^B_{top}(T^{-1}|S_x) \geq h_\mu(T) = h_{top}(T)$. Since $h^B_{top}(T|E) \leq h_{top}(T)$ and $h^B_{top}(T^{-1}|E) \leq h_{top}(T)$ for any $E \subseteq X$, the proof is complete. □

Corollary 4.13. *Let (X,T) be an asymptotically h-expansive TDS. Then there exists $x \in X$ for which the following holds.*

1. $h^B_{top}(T|\overline{W^s(x,T)}) = h_{top}(T)$;
2. *Under the continuum hypothesis, there exists a scrambled set* $S_x \subseteq \overline{W^s(x,T)}$ *satisfying* $h^B_{top}(T|S_x) = h_{top}(T)$.

Proof. Using Theorem 4.10, the proof is completely similar to that of Corollary 4.12. □

5. Hausdorff dimension of stable sets and scrambled sets

Let (X,d) be a metric space. We first recall the definition of Hausdorff dimension of a set. Fix $t \geq 0$. For each $\delta > 0$ and subset $A \subset X$, define

$$H^{t,\delta}_d(A) = \inf\{\sum_{i=1}^{+\infty} \mathrm{diam}(U_i)^t\},$$

where the infimum is taken over all countable covers $\{U_i : i = 1, 2, \cdots\}$ of A of diameter not exceeding δ. This definition induces an *outer measure* on X, i.e., a function defined on all subsets of X taking values in $[0,+\infty]$ satisfying $H_d^{t,\delta}(\emptyset)=0$, $H_d^{t,\delta}(A)\le H_d^{t,\delta}(B)$ if $A\subseteq B$ and

$$H_d^{t,\delta}(\bigcup_{n=1}^{\infty}A_n)\le\sum_{n=1}^{\infty}H_d^{t,\delta}(A_n)$$

for any countable family $\{A_n:n=1,2,\cdots\}$ of subsets of X.

Since $H_d^{t,\delta}(A)$ increases as δ decreases for any $A\subseteq X$, we can define

$$H_d^t(A)=\lim_{\delta\to 0}H_d^{t,\delta}(A)=\sup_{\delta>0}H_d^{t,\delta}(A).$$

The case $H_d^t(A)=+\infty$ is not excluded. Since all $H_d^{t,\delta}(\cdot)$ are outer measures, $H_d^t(\cdot)$ is also an outer measure. It is well-known that

$$H_d^t(A\cup B)=H_d^t(A)+H_d^t(B)$$

for each pair of *positively separated sets* $A,B\subseteq X$, i.e.,

$$d(A,B)=\inf\{d(x,y):x\in A,y\in B\}>0.$$

The metric outer measure H_d^t is called the *Hausdorff outer measure* associated to t. Its restriction to the σ-algebra of H_d^t-measurable sets, which includes all the Borel sets, is called the *Hausdorff measure* associated to t.

Fix $A\subseteq X$. Since for every $0<\delta\le 1$ the function $t\to H_d^{t,\delta}(A)$ is non-increasing, so is the function $t\to H_d^t(A)$. Moreover, if $0<s<t$, then for every $\delta>0$

$$H_d^{s,\delta}(A)\ge\delta^{s-t}H_d^{t,\delta}(A)$$

which implies that if $H_d^t(A)>0$, then $H_d^s(A)=+\infty$. Thus there is a unique value $H_d(A)\in[0,+\infty]$, which is called the *Hausdorff dimension* of A with respect to the metric d on X, such that

$$H_d^t(A)=\begin{cases}+\infty, & \text{if } 0\le t<H_d(A),\\ 0, & \text{if } H_d(A)<t<\infty.\end{cases}$$

The Hausdorff dimension is a monotone function of sets, i.e., if $A\subseteq B$ then $H_d(A)\le H_d(B)$. Moreover if $\{A_n\}_{n\ge 1}$ is a countable family of subsets of X then

$$H_d(\bigcup_{n=1}^{\infty}A_n)=\sup_{n\ge 1}H_d(A_n).$$

Hence if $E\subset X$ is countable then $H_d(E)=0$.

In the following we investigate the interrelation of Hausdorff dimension and Bowen entropy of a set in some specific TDSs. Let (X,T) be a TDS with metric d. We assume that T is *Lipschitz continuous* with *Lipschitz constant* L, i.e., $d(Tx,Ty)\le Ld(x,y)$ for any $x,y\in X$.

It is easy to see that if $h_{top}(T)>0$, then $L>1$. Indeed, by [2] we know that there exists an uncountable scrambled set S for T. Take $x\ne y\in S$. Then $\{x,y\}$ is a Li-York pair and hence

$$\limsup_{n\to+\infty}d(T^nx,T^ny)>0 \text{ and } \liminf_{n\to+\infty}d(T^nx,T^ny)=0.$$

Thus there exist $i,j\in\mathbb{N}$ with $i<j$ such that $d(T^ix,T^iy)<d(T^jx,T^jy)$. Moreover, we have $d(T^ix,T^iy)<L^{j-i}d(T^ix,T^iy)$ since $d(T^jx,T^jy)\le L^{j-i}d(T^ix,T^iy)$. This implies that $L>1$.

The following result is just Theorem 1 in [24].

Lemma 5.1. *Let (X,T) be a Lipschitz continuous TDS with Lipschitz constant $L>1$ associated to metric d. Then*

$$H_d(Y)\geq \frac{h^B_{top}(T|Y)}{\log L}$$

for any subset $Y\subseteq X$.

Remark 5.2. *Let (X,T) be as in Lemma 5.1. Then $h_{top}(T)\leq H_d(X)\cdot \log L$. Hence when $H_d(X)<\infty$, we always have $h_{top}(T)<\infty$. When $H_d(X)=\infty$, the following example shows that $h_{top}(T)=\infty$ can happen.*

Example 5.3. *Let $X=[0,1]^{\mathbb{N}}$ and X be endowed with the product topology. Then the compact space X is metrizable, and a compatible metric on X can be chosen as*

$$d(x,y)=\sum_{i=0}^{\infty}\frac{d(x_i,y_i)}{2^i},$$

for any $x=(x_0,x_1,\cdots),y=(y_0,y_1,\cdots)\in X$. With the shift map $T:X\to X$:

$$T(x_1,x_2,\cdots)=(x_2,x_3,\cdots),\ \forall (x_1,x_2,\cdots)\in X,$$

it is clear that (X,T) is a Lipschitz continuous TDS with Lipschitz constant $L=2$, and moreover $H_d(X)=\infty$. It is not hard to see that $h_{top}(T)=\infty$ as well.

Lemma 5.4. *Let (X,T) be a TDS with metric d. If there exist $\epsilon>0$ and $L>1$ such that $d(Tx,Ty)\geq Ld(x,y)$ whenever $d(x,y)<\epsilon$, then*

$$H_d(Y)\leq \frac{h^B_{top}(T|Y)}{\log L}$$

for any subset $Y\subseteq X$.

Proof. Let $Y\subseteq X$ be given and $\mathcal{U}$ be a finite open cover of X with $diam(U)<\frac{\epsilon}{2}$ for any $U\in\mathcal{U}$. It is sufficient to show that $H_d(Y)\leq \frac{h^B_{top}(T,\mathcal{U}|Y)}{\log L}$.

Fix $k\in\mathbb{N}$. For any $A\subseteq X$ with $n_{T,\mathcal{U}}(A)\geq k$, it is obvious that $diam(T^i(A))<\epsilon$ for $i=0,1,\cdots,n_{T,\mathcal{U}}(A)-1$. Since $d(Tx,Ty)\geq Ld(x,y)$ when $d(x,y)<\epsilon$, we have

$$diam(A)\leq L^{-n_{T,\mathcal{U}}(A)+1}diam(T^{n_{T,\mathcal{U}}(A)-1}A)\leq L^{-n_{T,\mathcal{U}}(A)+1}\epsilon\leq L^{-k+1}\epsilon. \tag{5.1}$$

Moreover,

$$e^{-sn_{T,\mathcal{U}}(A)}\geq C_{s,\epsilon,L}(diam(A))^{\frac{s}{\log L}} \tag{5.2}$$

for any $s\geq 0$, where $C_{s,\epsilon,L}=(L\epsilon)^{\frac{s}{\log L}}$.

Let $\mathcal{A}=\{A_i\}_{i=1}^{\infty}$ be any cover of Y satisfying $n_{T,\mathcal{U}}(A_i)\geq k$. Then $\mathcal{A}$ is a $L^{-k+1}\epsilon$-cover of Y by (5.1). By (5.2), we have

$$\sum_{i=1}^{\infty}e^{-sn_{T,\mathcal{U}}(A_i)}\geq C_{s,\epsilon,L}\sum_{i=1}^{\infty}(diam(A_i))^{\frac{s}{\log L}}\geq C_{s,\epsilon,L}H_d^{\frac{s}{\log L},L^{-k+1}\epsilon}(Y),\quad s\geq 0.$$

Since $\mathcal{A}$ is arbitrary, we have

$$m_k(T,s,\mathcal{U}|Y)\geq C_{s,\epsilon,L}H_d^{\frac{s}{\log L},L^{-k+1}\epsilon}(Y),\quad s\geq 0.$$

Taking limit $k\to+\infty$ yields that

$$m(T,s,\mathcal{U}|Y)\geq C_{s,\epsilon,L}H_d^{\frac{s}{\log L}}(Y),\quad s\geq 0.$$

This implies that $H_d(Y) \leq \frac{h^B_{top}(T,\mathcal{U}|Y)}{\log L}$. □

We are now ready to prove Theorem 1 2) and Theorem 2 2).

Proof of Theorem 2 2). Let μ be a T-invariant ergodic measure with $h_\mu(T) > 0$. Under the continuum hypothesis, we have by Theorem 4.9 that for μ-a.e $x \in X$ there exists a scrambled set $S_x \subseteq \overline{W^s(x,T)} \cap \overline{W^u(x,T)}$ for both T and T^{-1} such that $h^B_{top}(T|S_x) \geq h_\mu(T)$. Moreover, since $d(Tx,Ty) \leq Ld(x,y)$ for any $x, y \in X$, we have by Lemma 5.1 that $H_d(S_x) \geq \frac{h^B_{top}(T|S_x)}{\log L} \geq \frac{h_\mu(T)}{\log L}$ for μ-a.e. $x \in X$. The proof of the remaining part of Theorem 2 2) uses a similar argument. □

Proof of Theorem 1 2). By using Theorem 4.10 and Lemma 5.1, the proof is completely similar to that of Theorem 2 2). □

Using Remark 5.2 and Theorem 1 2), we have the following result.

Theorem 5.5. *Let (X,T) be a TDS with metric d such that $H_d(X) < \infty$ and T be a Lipschitz continuous self-map with Lipschitz constant $L > 1$. If μ is a T-invariant ergodic measure with $h_\mu(T) > 0$, then the following holds.*

1. *$H_d(\overline{W^s(x,T)}) \geq \frac{h_\mu(T)}{\log L}$ for μ-a.e $x \in X$.*
2. *Under the continuum hypothesis, for μ-a.e $x \in X$ there exists a scrambled set $S_x \subseteq \overline{W^s(x,T)}$ for T such that $H_d(S_x) \geq \frac{h_\mu(T)}{\log L}$.*

As a direct consequence of the above theorem, we have the following results.

Corollary 5.6. *Let (X,T) be a TDS with metric d such that $h_{top}(T) < \infty$ and T be a Lipschitz continuous self-map with Lipschitz constant $L > 1$. If there exists $\mu \in \mathcal{M}^e(X,T)$ such that $h_\mu(T) = h_{top}(T)$, then the following holds.*

1. *$H_d(\overline{W^s(x,T)}) \geq \frac{h_{top}(T)}{\log L}$ for μ-a.e. $x \in X$.*
2. *Under the continuum hypothesis, for μ-a.e $x \in X$ there exists a scrambled set $S_x \subseteq \overline{W^s(x,T)}$ for T satisfying $H_d(S_x) \geq \frac{h_{top}(T)}{\log L}$.*

Corollary 5.7. *Let (X,T) be an asymptotically h-expansive TDS with metric d and T be a Lipschitz continuous self-map with Lipschitz constant $L > 1$. Then there exists $x \in X$ for which the following holds.*

1. *$H_d(\overline{W^s(x,T)}) \geq \frac{h_{top}(T)}{\log L}$.*
2. *Under the continuum hypothesis, there exists a scrambled set $S_x \subseteq \overline{W^s(x,T)}$ for T satisfying $H_d(S_x) \geq \frac{h_{top}(T)}{\log L}$.*

Proof. Since (X,T) is an asymptotically h-expansive system, there exists a $\mu \in \mathcal{M}^e(X,T)$ such that $h_\mu(T) = h_{top}(T) < \infty$. Hence the corollary follows from Corollary 5.6. □

Let $A = \{0, 1, \cdots, N-1\}$ for some integer $N \geq 2$ endowed with the discrete metric d, and $\Sigma_+(N)$ be the space of one-sided sequences in A endowed with the product topology. Then $\Sigma_+(N)$ is metrizable, and a compatible metric ρ on $\Sigma(N)$ can be chosen as

$$\rho(x,y) = \sum_{i=0}^{\infty} \frac{d(x_i,y_i)}{N^i}, \;\; x = (x_0, x_1, \cdots), \; y = (y_0, y_1, \cdots) \in \Sigma_+(N).$$

We consider the shift map $\sigma : \Sigma_+(N) \to \Sigma_+(N)$: $\sigma(x)_i = x_{i+1}$, $i = 0, 1, \cdots$. If X is a σ-invariant non-empty closed subset of $\Sigma_+(N)$, then we say that (X, σ) is a *subshift* of $(\Sigma_+(N), \sigma)$.

Theorem 5.8. *Let (X, σ) be a subshift of $(\Sigma_+(N), \sigma)$ with a metric ρ as above. Then there exists $\mu \in \mathcal{M}^e(X, \sigma)$ with $h_\mu(\sigma) = h_{top}(\sigma, X)$ for which the following holds.*

(1) *$H_\rho(\overline{W^s(x, \sigma)}) = \frac{h_{top}(\sigma,X)}{\log N}$ for μ-a.e. $x \in X$.*
(2) *Under the continuum hypothesis, for μ-a.e. $x \in X$ there exists a scrambled set $S_x \subseteq \overline{W^s(x, \sigma)}$ for σ satisfying $H_\rho(S_x) = \frac{h_{top}(\sigma,X)}{\log N}$.*

Proof. Since the entropy map $\theta \in \mathcal{M}(X, T) \mapsto h_\theta(\sigma, X)$ is upper semi-continuous, there exists $\mu \in \mathcal{M}^e(X, T)$ such that $h_\mu(\sigma) = h_{top}(\sigma, X)$.

To finish the proof, we only need to show (2) since the proof of (1) is similar to that of (2). Since $\rho(\sigma x, \sigma y) \le N\rho(x, y)$ for any $x, y \in X$, it follows from Theorem 5.5 that for μ-a.e $x \in X$ there exists a scrambled set $S_x \subseteq \overline{W^s(x, \sigma)}$ for σ satisfying $H_\rho(S_x) \ge \frac{h_\mu(\sigma)}{\log N} = \frac{h_{top}(\sigma)}{\log N}$.

Since $\rho(\sigma x, \sigma y) \ge N\rho(x, y)$ for any $x, y \in X$ with $\rho(x, y) < \frac{1}{N}$, Lemma 5.4 implies that $H_\rho(S_x) \le \frac{h_{top}(\sigma|S_x)}{\log N} \le \frac{h_{top}(\sigma)}{\log N}$ for μ-a.e $x \in X$. Hence $H_\rho(S_x) = \frac{h_{top}(\sigma,X)}{\log N}$ for μ-a.e $x \in X$. □

Remark 5.9. *1) In* [12] *Furstenberg proved that for a subshift (X, σ) of $(\Sigma_+(N), \sigma)$, $H_\rho(X) = \frac{h_{top}(\sigma,X)}{\log N}$. Hence for any $E \subseteq X$, we always have $H_\rho(E) \le \frac{h_{top}(\sigma,X)}{\log N}$ (see also Lemma 5.4).*

2) Let (X, σ) be a subshift of $(\Sigma_+(N), \sigma)$. For any $x \in X$, it is clear that $W^s(x, \sigma)$ is a countable set. Hence $h^B_{top}(\sigma|W^s(x, \sigma)) = 0$ and $H_\rho(W^s(x, \sigma)) = 0$ (in fact, the former is true for any TDS (X, T)). Thus taking closures of the stable sets in the statement of Theorems 1-2 is necessary.

We end this section by posting the following questions:

Question 5.10. *Let (X, T) be an invertible TDS and μ be a T-invariant ergodic measure on X with $h_\mu(T) > 0$. Then whether the following statements hold for μ-a.e $x \in X$:*

(1) *$h^B_{top}(T|W^u(x, T)) \ge h_\mu(T)$ and $h^B_{top}(T^{-1}|W^s(x, T)) \ge h_\mu(T)$?*
(2) *$(W^s(x, T) \cap W^u(x, T)) \setminus \{x\} \ne \emptyset$?*

6. C^1 self-maps on Riemannian manifold

Let M be a smooth Riemannian manifold. The Riemannian structure on M induces a natural norm $\|\cdot\|_x$ on each tangent space T_xM which we simply denote by $\|\cdot\|$ if there is no confusion. M turns out to be a metric space with the metric ρ:

$$\rho(x, y) = \inf\{\int_a^b \|\dot\gamma(t)\|dt : \gamma : [a, b] \to M \text{ is a } C^1 \text{ map with } \gamma(a) = x, \gamma(b) = y\},$$

for any $x, y \in M$. For $x \in M$ and $r > 0$, let $B(x, r) = \{y \in M : \rho(x, y) < r\}$ denote the ball centered at x of radius r. Any C^1 self-map g on M gives rise to the tangent map Dg on the tangent bundle TM, which is a linear operator $Dg_x : T_xM \to T_{gx}M$, $x \in M$, with norm $\|\cdot\|$ defined by

$$\|Dg_x\| = \max_{u \in T_xM : \|u\|=1} \|Dg_x(u)\|.$$

We consider the *cut-norm* of Dg_x at $x \in M$ defined by $\|Dg_x\|_+ = \max\{1, \|Dg_x(u)\|\}$. Obviously $\|Dg_x\|_+ \geq 1$ and $\log \|Dg_x\|_+ = \max\{0, \log \|Dg_x(u)\|\} \geq 0$ on M. This cut-norm $\|Dg_{\{\cdot\}}\|_+$ will play an important role in the following Lemma to overcome the obstruction caused by the set $\{x \in M : \|Dg_x\| = 0\}$.

Lemma 6.1. *Let g be a C^1 self-map on M, and Γ be a non-empty compact g-invariant subset. Then for any $\delta > 0$ there exists $\varepsilon = \varepsilon(\delta, g, \Gamma) > 0$ such that $\rho(gx, gy) \leq \rho(x, y)\|Dg_x\|_+ e^{\delta}$ for any $x \in \Gamma$ and $y \in B(x, \varepsilon)$.*

Proof. For each $x \in M$, we consider the exponential map $\exp_x : T_xM \to M$: $\exp_x(v) = \gamma_v(1)$, where $\gamma_v(t)$ is the geodesic with $\gamma_v(0) = x, \dot{\gamma}_v(0) = v$. The following properties are well-known:

1. For $x \in M$, $\exp_x(0_x) = x$ and $D(\exp_x)_{0_x} = Id_{T_xM}$.
2. There exist a positive number $R = R(\Gamma) > 0$ and an open neighborhood U of Γ whose closure $\overline{U}$ is compact, such that for each $x \in U$, $\exp_x : T_xM(R) \to M$ is embedding and
$$\rho(\exp_x(v), x) = \|v\| \text{ for any } v \in T_xM(R),$$
where $T_xM(s) \doteq \{v \in T_xM : \|v\| < s\}$ for $s > 0$. Moreover $\exp_x(T_xM(r)) = B(x, r) \subset M$ for any $x \in U$ and $r \in (0, R)$.

Since Γ is compact and g-invariant, there exist a positive number $r_0 = r_0(g, \Gamma) \in (0, R)$ and an open neighborhood $W \subseteq U$ of Γ such that for any $x \in W$ and $y \in B(x, r_0)$ we have $gx \in U$ and $\rho(gx, gy) < R$.

Consider the map $\tilde{g} : T_WM(r_0) \to T_UM(R)$:
$$\tilde{g}(v) = \exp_{gx}^{-1} \circ g \circ \exp_x(v) \text{ for } v \in T_xM(r_0) \text{ and } x \in W,$$
where $T_WM(r_0) = \{v \in T_WM : \|v\| < r_0\}$ and $T_UM(R) = \{u \in T_UM : \|u\| < R\}$. Since the exponential map is sufficiently smooth and $\tilde{g}$ is as smooth as g, $\tilde{g}$ is a C^1 map. The partial derivative D_2 of $\tilde{g}$ with respect to the fiber variable (or the second variable) is defined by
$$D_2\tilde{g}_v = D(\tilde{g}|_{T_xM(r_0)})_v : T_v(T_xM) \to T_{\tilde{g}(v)}(T_{gx}M)$$
for any $x \in W$ and $v \in T_xM(r_0)$. Using the isometries between $T_v(T_xM)$ and T_xM and between $T_{\tilde{g}(v)}(T_{gx}M)$ and $T_{gx}M$, we can simply regard $D_2\tilde{g}_v : T_xM \to T_{gx}M$ for any $v \in T_xM(r_0)$ and $x \in W$. At $v = 0_x$ we have $\tilde{g}(0_x) = 0_{gx}$, $D(\exp_{gx}^{-1})_{gx} = Id_{T_{gx}M}$ and
$$D_2\tilde{g}_{0_x} = D(\exp_{gx}^{-1} \circ g \circ \exp_x)_{0_x} = Id_{T_{gx}M} \circ Dg_x \circ Id_{T_xM} = Dg_x.$$

For any $\delta > 0$, since $\Gamma \subseteq W$ is compact, there exists $\varepsilon = \varepsilon(\delta, g, \Gamma) \in (0, r_0)$ such that
$$\|D_2\tilde{g}_u - D_2\tilde{g}_v\| < \delta \text{ for any } x \in \Gamma, u, v \in T_xM(r_0) \text{ with } \|u - v\| < \varepsilon. \tag{6.1}$$

Let $x \in \Gamma$ and $y \in B(x, \varepsilon)$ and consider the curve $\gamma(t) = t \exp_x^{-1}(y) \in T_xM$. Then
$$\begin{aligned} \exp_{gx}^{-1}(gy) &= \tilde{g}(\exp_x^{-1}(y)) = \int_0^1 \frac{d(\tilde{g} \circ \gamma(t))}{dt} dt \\ &= \int_0^1 D_2\tilde{g}_{\gamma(t)}(\dot{\gamma}(t))dt = \int_0^1 D_2\tilde{g}_{\gamma(t)}(\exp_x^{-1}(y))dt. \end{aligned}$$

Since $\|\gamma(t)\| = t\|\exp_x^{-1}(y)\| = t\rho(x,y) < t\varepsilon \le \varepsilon$ for any $0 \le t \le 1$, we have by (6.1) that

$$\|\exp_{gx}^{-1}(gy) - D_2\tilde{g}_{0_x}(\exp_x^{-1}(y))\| = \|\int_0^1 (D_2\tilde{g}_{\gamma(t)} - D_2\tilde{g}_{0_x})(\exp_x^{-1}(y))dt\|$$

$$\le \int_0^1 \|D_2\tilde{g}_{\gamma(t)} - D_2\tilde{g}_{0_x}\| \cdot \|\exp_x^{-1}(y)\|dt \le \delta \cdot \|\rho(x,y)\|.$$

By noting that $D_2\tilde{g}_{0_x} = Dg_x$, we have

$$\|\exp_{gx}^{-1}(gy) - Dg_x(\exp_x^{-1}(y))\| \le \delta\rho(x,y).$$

Hence

$$\begin{aligned} \rho(gx,gy) &= \|\exp_{gx}^{-1}(gy)\| \le \|Dg_x(\exp_x^{-1}(y))\| + \delta\rho(x,y) \\ &\le \|Dg_x\|\|\exp_x^{-1}(y))\| + \delta\rho(x,y) \\ &= \rho(x,y)(\|Dg_x\| + \delta) \le \rho(x,y)(\|Dg_x\|_+ + \delta) \\ &\le \rho(x,y)\|Dg_x\|_+(1+\delta) \le \rho(x,y)\|Dg_x\|_+e^{\delta}. \end{aligned}$$

□

For a g-invariant ergodic measure ν on Γ, recall that the generic set of ν with respect to g is defined by (see (4.3))

$$G_\nu = \{x \in \Gamma : \lim_{n\to+\infty} \frac{1}{n}\sum_{i=0}^{n-1} \phi(g^i x) = \int_\Gamma \phi\, d\nu \text{ holds for any } \phi \in C(\Gamma;\mathbb{R})\}.$$

If ν is ergodic, we have by Birkhoff Pointwise Ergodic Theorem that $\nu(G_\nu) = 1$.

Lemma 6.2. *Let g be a C^1 self-map on M, Γ be a compact g-invariant subset, and ν be a g-invariant ergodic measure on Γ. Then $h_{top}^B(g|Y) \le H_\rho(Y)\cdot\chi_\nu^+$ for any $Y \subseteq G_\nu$, where $\chi_\nu^+ = \chi_\nu^+(g) = \int_M \log\|Dg_x\|_+ d\nu(x)$.*

Proof. Let $Y \subseteq G_\nu$. It is sufficient to show that

$$h_{top}^B(g,\mathcal{U}|Y) \le (H_\rho(Y)+\delta)\cdot(\chi_\nu^+ + 2\delta)$$

for any finite open cover $\mathcal{U}$ of Γ and $\delta > 0$.

For any finite open cover $\mathcal{U}$ of Γ and $\delta > 0$, we let $\eta = \eta(\mathcal{U}) > 0$ be the Lebesgue number of $\mathcal{U}$. By Lemma 6.1, there exists $\varepsilon \in (0, \frac{\eta}{2})$ such that

$$\rho(gu,gv) \le \rho(u,v)\|Dg_u\|_+e^{\delta} \tag{6.2}$$

for any $u \in \Gamma$ and $v \in B(u,\varepsilon)$.

Claim. Let $n \ge 2$ and $x \in E \subseteq \Gamma$. If $\mathrm{diam}(E) < \varepsilon e^{-n\delta - \sum_{k=0}^{n-1}\log\|Dg_{g^k x}\|_+}$, then $g^i(E) \prec \mathcal{U}$ for $0 \le i \le n$, i.e., $n_{g,\mathcal{U}}(E) \ge n+1$.

Since $g^i(E) \subset \Gamma$ for $0 \le i \le n$ and η is the Lebesgue number of $\mathcal{U}$, to prove the Claim it is sufficient to show that for any $y \in E$,

$$\rho(g^i x, g^i y) < \varepsilon e^{-(n-i)\delta - \sum_{k=i}^{n-1}\log\|Dg_{g^k(g^i x)}\|_+} \le \varepsilon < \frac{\eta}{2} \tag{6.3}$$

for all $0 \le i \le n$. This can be shown via induction on i. For $i = 0$,

$$\rho(x,y) \le \mathrm{diam}(E) < \varepsilon e^{-n\delta - \sum_{k=0}^{n-1}\log\|Dg_{g^k x}\|_+} \le \varepsilon < \frac{\eta}{2}$$

since $\sum_{k=0}^{n-1}\log\|Dg_{g^kx}\|_+\geq 0$. Now we assume (6.3) holds for $i=\ell<n$. Then for $i=\ell+1$,

$$\rho(g^\ell x,g^\ell y)<\varepsilon e^{-(n-\ell)\delta-\sum_{k=\ell}^{n-1}\log\|Dg_{g^k(g^\ell x)}\|_+}\leq\varepsilon<\frac{\eta}{2}.$$

Since $g^\ell x\in\Gamma$, we have by (6.2) that

$$\begin{aligned}&\rho(g^{\ell+1}x,g^{\ell+1}y)\leq\rho(g^\ell x,g^\ell y)e^\delta\|Dg_{g^\ell x}\|_+\\ \leq\ &\varepsilon e^{-(n-(\ell+1))\delta+\sum_{k=\ell+1}^{n-1}\log\|Dg_{g^k(g^{\ell+1}x)}\|_+}\leq\varepsilon<\frac{\eta}{2}.\end{aligned}$$

This proves the Claim.

For each $r\in\mathbb{N}$, we consider the following set

$$Y_r=\{x\in Y:\frac{1}{n}\sum_{k=0}^{n-1}\log\|Dg_{g^kx}\|_+\leq\chi_\nu^++\delta\text{ for any }n\geq r\}.$$

By the continuity of $\log\|Dg_{\{\cdot\}}\|_+$ and the fact that $Y\subseteq G_\nu$, we know that $\bigcup_{r\in\mathbb{N}}Y_r=Y$.

Now fix a $r\in\mathbb{N}$. Let $t=H_\rho(Y)+\delta$ and $K=\max\{\log\|Dg_u\|_+:u\in\Gamma\}$. Then $0\leq K<\infty$ and $0=H_\rho^t(Y_r)=\lim\limits_{s\to0+}H_\rho^{t,s}(Y_r)$. Hence there exists $s_*>0$ such that, for all $s\in(0,s_*)$,

$$H_\rho^{t,s}(Y_r)=\inf\{\sum_{i=1}^\infty\operatorname{diam}(U_i)^t:\bigcup_{i=1}^\infty U_i\supset Y_r\text{ and }\operatorname{diam}(U_i)<s,\ \forall i\in\mathbb{N}\}<\frac{1}{2}.$$

Consider the sequence $s_n=\varepsilon e^{-n(\delta+K)}$, $n\geq N$, where $N>r$ is a fixed integer such that $\varepsilon e^{-N(\delta+K)}<s_*$. Since $H_\rho^{t,s_n}(Y_r)<\frac{1}{2}$ and $s_n<s_*$, we can choose a countable cover $\{U_i^n:i\in\mathbb{N}\}$ of Y_r such that $U_i^n\subseteq Y_r$, $\operatorname{diam}(U_i^n)<s_n$ for each $i\in\mathbb{N}$ and $\sum\limits_{i=1}^\infty\operatorname{diam}(U_i^n)^t<1$. For each U_i^n we pick a point $x_i^n\in U_i^n\cap Y_r\subseteq\Gamma$. Since $\operatorname{diam}(U_i^n)<s_n\leq\varepsilon e^{-(n\delta+\sum_{k=0}^{n-1}\log\|Dg_{g^kx_i^n}\|_+)}$, we have by the above Claim that $n_{g,\mathcal{U}}(U_i^n)\geq n+1$.

Next we show that

$$\varepsilon e^{-n_{g,\mathcal{U}}(U_i^n)(\chi_\nu^++2\delta)}\leq\operatorname{diam}(U_i^n)\tag{6.4}$$

for all $i\in\mathbb{N}$, $n\geq N$. If (6.4) is not true, then there exist $i\in\mathbb{N}$ and $n\geq N$ such that $\operatorname{diam}(U_i^n)<\varepsilon e^{-n_{g,\mathcal{U}}(U_i^n)(\chi_\nu^++2\delta)}$. Since $x_i^n\in Y_r$, we clearly have $n+1\leq n_{g,\mathcal{U}}(U_i^n)<\infty$ and

$$\operatorname{diam}(U_i^n)<\varepsilon e^{-n_{g,\mathcal{U}}(U_i^n)(\chi_\nu^++2\delta)}\leq\varepsilon e^{-n_{g,\mathcal{U}}(U_i^n)\delta-\sum_{k=0}^{n_{g,\mathcal{U}}(U_i^n)-1}\log\|Dg_{g^kx_i^n}\|_+}.$$

Using the fact $x_i^n\in U_i^n\subseteq\Gamma$, we have by the above Claim that $n_{g,\mathcal{U}}(U_i^n)\geq n_{g,\mathcal{U}}(U_i^n)+1$, a contradiction.

It now follows from (6.4) that

$$m_{n+1}\left(g,(\chi_\nu^++2\delta)t,\mathcal{U}|Y_r\right)\leq\sum_{i=1}^\infty e^{-n_{g,\mathcal{U}}(U_i^n)(\chi_\nu^++2\delta)t}\leq\sum_{i\in\mathbf{N}}\frac{\operatorname{diam}(U_i^n)^t}{\varepsilon^t}\leq\frac{1}{\varepsilon^t},\ \ n\geq N.$$

Since the last term in the above is independent of the choice of $n\geq N$, we have

$$m(g,(\chi_\nu^++2\delta)t,\mathcal{U}|Y_r)\leq\frac{1}{\varepsilon^t}.$$

Hence $h^B_{top}(g,\mathcal{U}|Y_r) = \inf\{s \geq 0 : m(g,s,\mathcal{U}|Y_r) < \infty\} \leq (\chi^+_\nu + 2\delta)t$. Since $r \in \mathbb{N}$ is arbitrary, we have $h^B_{top}(g,\mathcal{U}|Y) = \sup_{r\in\mathbb{N}} h^B_{top}(g,\mathcal{U}|Y_r) \leq (\chi^+_\nu + 2\delta)t = (\chi^+_\nu + 2\delta)(H_\rho(Y) + \delta)$. □

Now let f be a C^1 self-map on M and μ be an f-invariant ergodic measure with a compact support $\Lambda \subset M$. Denote $\varphi_n(x) = \log \|Df^n_x\|$, $x \in M$, $n \in \mathbb{N}$. Then $\{\varphi_n\}_{n=1}^\infty$ is a family of continuous functions on M and $\varphi_{n'+n}(x) \leq \varphi_{n'}(f^n x) + \varphi_n(x)$ for any $x \in M$ and $n', n \in \mathbb{N}$. Since Λ is compact, $D = \{\|Df_x\| : x \in \Lambda\} < +\infty$. Hence the integral $\int_M \log \|Df_x\| d\mu(x)$ is well defined and $\int_M \log \|Df_x\| d\mu(x) \in [-\infty, \log D]$. By subadditivity we have $\int_M \log \|Df^n_x\| d\mu(x) \leq n \log D$ for any $n \in \mathbb{N}$, and by Kingman Sub-additive Ergodic Theorem (see [17] or [31, Theorem10.1]), the following limit exists for μ-a.e. $x \in M$:

$$\chi^1_x = \lim_{n\to+\infty} \frac{1}{n} \log \|Df^n_x\|.$$

Clearly the limit function $\chi^1_{\{\cdot\}}$ is f-invariant. Since μ is ergodic, $\chi^1_{\{\cdot\}}$ is μ-a.e. a constant χ^1_μ, called the *top Lyapunov exponent of f with respect to μ*. More precisely, $\chi^1_x = \chi^1_\mu$ for μ-a.e. $x \in M$ (see the remark of [31, Theorem 10.2]). Moreover, by Dominated Convergence Theorem and subadditivity we have

$$\chi^1_\mu = \lim_{n\to+\infty} \frac{1}{n} \int_M \log \|Df^n_x\| d\mu(x) = \inf_{n\geq 1} \frac{1}{n} \int_M \log \|Df^n_x\| d\mu(x). \tag{6.5}$$

We note that $\chi^1_\mu = -\infty$ is not excluded. It is well-known that $h_\mu(f) \leq \max\{0, \chi^1_\mu\} \cdot dim(M)$ by Ruelle inequality [28]. Hence when $h_\mu(T) > 0$ we have $\chi^1_\mu > 0$.

Since (Λ, μ, f) is ergodic, it is well-known that for each $k \in \mathbb{N}$ there exist a factor p_k of k and a f^k-invariant ergodic measure ν_k with $\mathrm{supp}(\nu_k) \subseteq \Lambda$ such that $f^{p_k}\nu_k = \nu_k$, $\mu = \frac{1}{p_k}\sum_{j=0}^{p_k-1} f^j\nu_k$, $f^i\nu_k \neq f^j\nu_k$ for $0 \leq i < j \leq p_k - 1$, where $f^j\nu_k(A) = \nu_k(f^{-j}A)$ for any Borel subset A of M. Since Λ is f-invariant and ν_k is a f^k-invariant ergodic measure supporting on Λ, $f^j\nu_k$ is also a f^k-invariant ergodic measure and $\mathrm{supp}(f^j\nu_k) \subseteq \Lambda$ for $j = 0, 1, \cdots, p_k - 1$. Consider sets $G^k = \bigcup_{j=0}^{k-1} G_{f^j\nu_k}$. Clearly $\mu(G^k) = 1$ for each $k \geq 1$. Let $G^\infty_\mu = \bigcap_{k=1}^{+\infty} G^k$. Then $\mu(G^\infty_\mu) = 1$.

Proposition 6.3. *For any $Y \subseteq G^\infty_\mu$, $h^B_{top}(f|Y) \leq \max\{0, \chi^1_\mu\} \cdot H_\rho(Y)$*

Proof. Let $Y \subseteq G^\infty_\mu$. Then $Y \subseteq G^k$ for $k \in \mathbb{N}$. For convenience we view the index set $\mathbf{Z}_{p_k} = \{0, \cdots, p_k - 1\}$ as a finite additive group. It is not hard to see that $f^i G_{f^j\nu_k} \subseteq G_{f^{i+j}\nu_k}$ for all $i, j \in \mathbf{Z}_{p_k}$. Denote $Y_{j,k} = Y \cap G_{f^j\nu_k}$ and consider $Y^k_j = \bigcup_{i=0}^{p_k-1} f^i Y_{j-i,k} \subseteq G_{f^j\nu_k}$ – the jth copy of Y in $G_{f^j\nu_k}$. By Proposition 3.2 (1) and (2), we have, for any $E \subset \Lambda$ and $i \in \mathbf{Z}_{p_k}$,

$$h^B_{top}(f^k|f^k E) = h^B_{top}(f^k|E) \text{ and } h^B_{top}(f^k|f^i E) \geq h^B_{top}(f^k|f^{i+1}E).$$

Thus $h^B_{top}(f^k|f^i E) = h^B_{top}(f^k|E)$ for any $E \subset \Lambda$ and $i \in \mathbf{Z}_{p_k}$. It follows that for each $j \in \mathbf{Z}_{p_k}$,

$$h^B_{top}(f^k|Y^k_j) = \max_{0\leq i\leq p_k-1} h^B_{top}(f^k|f^i Y_{j-i,k}) = \max_{0\leq i\leq p_k-1} h^B_{top}(f^k|Y_{j-i,k}) = h^B_{top}(f^k|Y).$$

Since f is differentiable and Λ is compact and f-invariant, f and hence f^i is uniformly Lipschitz continuous on Λ for any $i \in \{0, \cdots, p_k - 1\}$. Then $H_\rho(f^i E) \leq H_\rho(E)$ for any $E \subset \Lambda$ and $i \in \{0, \cdots, p_k - 1\}$. This implies that

$$H_\rho(Y^k_j) = \max_{0\leq i\leq p_k-1} H_\rho(f^i Y_{j-i,k}) \leq \max_{0\leq i\leq p_k-1} H_\rho(Y_{j-i,k}) = H_\rho(Y).$$

Applying Lemma 6.2 to $(\Lambda, f^j\nu_k, f^k)$ with Y_j^k and $\chi^+_{f^j\nu_k}(f^k)$, we have,

$$h^B_{top}(f^k|Y) = h^B_{top}(f^k|Y_j^k) \leq \chi^+_{f^j\nu_k}(f^k)\cdot H_\rho(Y_j^k) \leq \chi^+_{f^j\nu_k}(f^k)\cdot H_\rho(Y). \tag{6.6}$$

By Proposition 3.2 (3), we also have $h^B_{top}(f^k|Y) = k\cdot h^B_{top}(f|Y)$. Summing up (6.6) over $j\in\{0,\cdots,p_k-1\}$ and dividing both hand sides by $k\cdot p_k$ yields

$$\begin{aligned} h^B_{top}(f|Y) &= \frac{1}{k}h^B_{top}(f^k|Y) \leq \frac{1}{k\cdot p_k}\sum_{j=0}^{p_k-1}\chi^+_{f^j\nu_k}(f^k)\cdot H_\rho(Y) \\ &= H_\rho(Y)\cdot\frac{1}{k}\int_M \log\|Df^k\|_+ d\left(\frac{1}{p_k}\sum_{j=0}^{p_k-1} f^j\nu_k\right) \qquad (6.7)\\ &= H_\rho(Y)\cdot\frac{1}{k}\int_M \log\|Df^k\|_+ d\mu. \end{aligned}$$

By Kingman Sub-additive Ergodic Theorem [17] as in equation (6.5), we have

$$\lim_{k\to+\infty}\frac{1}{k}\log\|Df^k_x\| = \chi^1_\mu \text{ for } \mu\text{-a.e. } x\in\Lambda.$$

It follows that

$$\lim_{k\to+\infty}\frac{1}{k}\log\|Df^k_x\|_+ = \max\{\chi^1_\mu, 0\} \text{ for } \mu\text{-a.e. } x\in\Lambda.$$

Thus by Dominated Convergence Theorem we have

$$\lim_{k\to+\infty}\frac{1}{k}\int_M \log\|Df^k_x\|_+ d\mu(x) = \max\{\chi^1_\mu, 0\}. \tag{6.8}$$

By (6.7) and (6.8), we finally have

$$h^B_{top}(f|Y) \leq \lim_{k\to+\infty} H_\rho(Y)\cdot\frac{1}{k}\int_M \log\|Df^k\|_+ d\mu = H_\rho(Y)\cdot\max\{0,\chi^1_\mu\}.$$

This completes the proof. □

Proof of Theorem 3. We only need to show 2) since the proof of 1) is similar to that of 2). Consider the system (Λ, f) with the metric ρ on Λ. Since $h_\mu(f) > 0$, Ruelle inequality ([28]) implies that $\chi^1_\mu > 0$. Since μ is f-invariant and ergodic, $\mu(G^\infty_\mu) = 1$. We also note that $h_{top}(f,\Lambda) < \infty$ (see for example Theorem 7.15 in [31]). Part 2) of the theorem now follows from Theorem 4.9 and Proposition 6.3.

Using Theorem 4.10 and Proposition 6.3, part 1) of the theorem follows from a similar argument. □

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