VISCOUS STABILITY OF QUASI-PERIODIC LAGRANGIAN TORI

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Abstract. We consider a smooth Tonelli Lagrangian $L: T\mathbb{T}^n \to \mathbb{R}$ and its viscosity solutions $u(x,P)$ characterized by the cell equation $H(x, P + D_x u(x,P)) = \overline{H}(P)$, where $H: T^*\mathbb{T}^n \to \mathbb{R}$ is the Hamiltonian associated with L. We will show that if P_0 corresponds to a quasi-periodic Lagrangian invariant torus, then $D_x u(x,P)$ is Hölder continuous in P at P_0 with Hölder exponent arbitrarily close to 1, and if both H and the torus are real analytic and the frequency vector of the torus is Diophantine, then $D_x u(x,P)$ is Lipschiz continuous in P at P_0, i.e., there is a constant $C > 0$ such that $\|Du(x,P) - Du(x,P_0)\|_\infty \le C\|P - P_0\|$ as $\|P - P_0\| \ll 1$. Similar P-regularity of the Peierls barrier for L will be obtained, and applications to viscosity solutions near KAM tori in configuration space in a nearly integrable Hamiltonian system will also be considered.

1. Introduction

Consider a *Tonelli* Lagrangian $L: T\mathbb{T}^n \simeq \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$: $(x,v) \mapsto L(x,v)$, i.e., L is of class C^2, strictly convex and super-linear in fibers in the sense that $\frac{\partial^2 L}{\partial v^2}$ is everywhere positive definite and $\lim\limits_{\|v\|\to\infty} \frac{L(x,v)}{\|v\|} = \infty$ for any $x \in \mathbb{T}^n$. Let $H(x,p) = \sup\limits_{v\in R^n} (\langle p, v\rangle - L(x,v))$, $(x,p) \in T^*\mathbb{T}^n \simeq \mathbb{T}^n \times \mathbb{R}^n$, be the associated Hamiltonian. Then H is a *Tonelli* Hamiltonian, i.e., it is at least C^2 and satisfies the analogous convexity and superlinearity that for any $x \in \mathbb{T}^n$ the function $p \to H(x,p)$ is strictly convex on $T_x^*\mathbb{T}^n$ and $\lim_{\|p\|\to\infty} \frac{H(x,p)}{\|p\|} = \infty$. It was shown by Lions, Papanicolaou and Varadhan ([19]) that for any $P \in R^n$ there is a unique real number λ such that the the stationary Hamilton-Jacobi equation

$$H(x, P + D_x u(x)) = \lambda \tag{1.1}$$

admits a *viscosity solution* $u(x)$, i.e., $u: \mathbb{T}^n \to \mathbb{R}$ is a Lipschitz function satisfying (1.1) in the domain of definition $dom(D_x u)$ of $D_x u$, and, for any C^1 function $\phi: \mathbb{T}^n \to \mathbb{R}$, $H(x_0, P + D\phi(x_0)) \le \lambda$ at any maximum point $x_0 \in T^n$ of $u - \phi$ and $H(y_0, P + D\phi(y_0)) \ge \lambda$ at any minimum point y_0 of $u(x) - \phi(x)$. The function $\overline{H}(P) =: \lambda$ is called the *effective Hamiltonian*, and the stationary Hamilton-Jacobi equation (1.1) is usually re-written as

$$H(x, P + D_x u(x,P)) = \overline{H}(P), \tag{1.2}$$

called the *cell equation*. For each $P \in \mathbb{R}^n$, one can thus speak of a viscosity solution $u(\cdot, P)$ of the cell equation (1.2) in the above sense. In fact, it was shown by Fathi ([12]) that a viscosity solution $u(\cdot,P)$ is necessary a weak KAM solution of $L - \langle P, v\rangle$ and vice versa, and moreover, its C^1-graph

$$Graph\big(P + D_x u(x,P)\big) = \big\{(x, P + D_x u(x,P)) | x \in dom(D_x u(\cdot,P))\big\} \subset \mathbb{T}^n \times \mathbb{R}^n$$

is negatively invariant to the associated Hamiltonian flow.

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In [11], Evans and Gomes has asked the following fundamental questions: "To what extent can we employ $\overline{H}$ and u to understand the solutions of the Hamiltonian flow? In particular, how is information about the dynamics 'encoded' into $\overline{H}$?"

The present work aims at giving some partial answers to the above questions in a vicinity of a P_0 corresponding to a non-degenerate, quasi-periodic Lagrangian invariant torus of the Hamiltonian. More precisely, we will show that i) there are positive constants $\Lambda_1, \Lambda_2 > 0$ such that

$$\Lambda_2\|P-P_0\|^2 \le \overline{H}(P) - \overline{H}(P_0) - \langle D\overline{H}(P_0), P-P_0\rangle \le \Lambda_1\|P-P_0\|^2,$$

when $\|P-P_0\| \ll 1$; ii) $P + D_xu(\cdot,P)$ is Hölder continuous at $P = P_0$, where, for each $P \in \mathbb{R}^n$, $u(\cdot,P)$ is the viscosity corresponding solution; and iii) if both the Hamiltonian and the torus are real analytic and the torus is also Diophantine, then $P + D_xu(\cdot,P)$ is actually Lipschiz continuous at $P = P_0$.

Applying these result to a nearly integrable Hamiltonian system with strictly convex unperturbed Hamiltonian, these regularity properties have interesting dynamical implications in term of the distribution of C^1 graphs of viscosity solutions close to a KAM torus in configuration space. Moreover, they are also closely related to the regularity of the Peierls barrier - an important function in analyzing diffusions in Hamiltonian systems.

Let $P_0 \in R^n$ be such that the corresponding viscosity solution $u(x,P_0)$ is unique modular a constant (e.g., when the Mather set $\widetilde{M}_{P_0}$ is uniquely ergodic). We say that $u(x,P_0)$ or its corresponding graph is *viscously stable with stability index* $0 < \chi \le 1$ if there exist positive constants C, $\iota \ll 1$ such that

$$\big\|\big(P + D_xu(\cdot,P)\big) - \big(P_0 + D_xu(\cdot,P_0)\big)\big\|_\infty \le C\|P-P_0\|^\chi \tag{1.3}$$

as $\|P-P_0\| \le \iota$.

We assume the following conditions for the Tonelli Lagrangian $L : T^n \times \mathbb{R}^n \to \mathbb{R}$:

A1) L is of class C^6 whose Euler-Lagrange flow admits a quasi-periodic, invariant n-torus

$$\Gamma_\omega = \{(f(\xi), Df(\xi)\cdot\omega) : \ \xi \in \mathbb{T}^n\},$$

with frequency vector $\omega \in R^n$, where $f : \mathbb{T}^n \to \mathbb{T}^n$ is a C^6 diffeomorphism;
A2) $\eta = \big\langle (Df(\xi))^\top \frac{\partial L}{\partial v}(f(\xi), Df(\xi)\cdot\omega), d\xi\big\rangle$ is a closed 1-form;
A3) $\int_{\mathbb{T}^n} Df^{-1}(\xi)d\xi$ is non-singular.

We note that conditions A1) and A2) essentially says that Γ_ω is a Lagrangian torus and the condition A3) is a relaxed non-degenerate condition. Let $\mathcal{G}_\omega = \tilde{\mathcal{L}}(\Gamma_\omega)$, where $\tilde{\mathcal{L}} : T\mathbb{T}^n \to T^*\mathbb{T}^n$ denotes the Legendre transform. Then $\mathcal{G}_\omega$ is a quasi-periodic invariant n-torus of the Hamiltonian flow associated with H. We will show in the Appendix (Lemma 4.1)that if conditions A1)-A3) are satisfied then $\mathcal{G}_\omega$ is the C^1 graph of a unique (modular a constant) smooth viscosity solution $u(x,P_0)$ for some $P_0 \in \mathbb{R}^n$ determined precisely by ω, f and L.

Our main result is the following

Theorem. *Assume conditions* A1)-A3) *and let P_0 be associated with ω, f, L as mentioned in the above. Then the following holds.*

a) *There are positive constants $\Lambda_1, \Lambda_2 > 0$ such that*

$$\Lambda_2\|P-P_0\|^2 \le \overline{H}(P) - \overline{H}(P_0) - \langle D\overline{H}(P_0), P-P_0\rangle \le \Lambda_1\|P-P_0\|^2$$

as $\|P-P_0\| \ll 1$.

b) *$u(\cdot, P_0)$ is viscously stable with stability index χ arbitrarily close to* 1.

c) *If, in addition, both L and f are real analytic and ω is Diophantine, then the stability index χ of $u(\cdot, P_0)$ equals* 1.

From part a) of Theorem 1, we easily obtain the following

Corollary. *Assume conditions* A1)-A3) *and let P_0 be as in Theorem 1. If $\overline{H}(P)$ is twice differential at P_0, then the Hessian matrix $D^2\overline{H}(P_0)$ is positive definite. More precisely, $\mu_i \geq 2\Lambda_3$, $i = 1, \cdots, n$, for each eigenvalue μ_i of $D^2\overline{H}(P_0)$, where $0 < \Lambda_3 < \Lambda_2$ is any positive constant and Λ_2 is as in Theorem 1.*

In [11], it is proved by Evans and Gomes that if $\overline{H}$ is twice differentiable at some point $P_0 \in \mathbb{R}^n$, then

$$\int_{\mathbb{T}^n} \left\| \big(P + D_x u(x, P)\big) - \big(P_0 + D_x u(x, P_0)\big) \right\|^2 d\sigma \leq C \|P - P_0\|^2 \tag{1.4}$$

for some positive constant C as $\|P - P_0\| \ll 1$, where σ is the projection of some Mather measure (to $\mathbb{T}^n$). Theorem 1 above implies that if one can have more dynamical information about the projected Mather measure σ, then a better estimate than the above may be obtained. Indeed, under the condition of part c) of Theorem 1, we have

$$\left\| \big(P + D_x u(x, P)\big) - \big(P_0 + D_x u(x, P_0)\big) \right\|_\infty \leq C \|P - P_0\|,$$

which is clearly a better estimate than (1.4).

Theorem 1 is also closely related to the modulus of continuity or regularity of Peierls barrier introduced by Mather in [22]. Let $L(x, v)$ be the Tonelli Lagrangian defined. We consider for each $P \in \mathbb{R}^n$ the Lagrangian $L_P(x, v) = L(x, v) - \langle P, v \rangle$. Then the *Peierls barrier associated with P* is the function $h : \mathbb{T}^n \times \mathbb{T}^n \to \mathbb{R}$ defined by

$$h^P(x, y) = \lim_{t \to +\infty} (h_t^P(x, y) + \overline{H}(P)t), \tag{1.5}$$

where

$$h_t^P(x, y) = \inf_{\gamma} \int_0^t L_P(\gamma(s), \dot{\gamma}(s)) ds \tag{1.6}$$

with the infimum taken over all continuous, piecewise C^1 curves $\gamma : [0, t] \to \mathbb{T}^n$ with $\gamma(0) = x$ and $\gamma(t) = y$. In fact, as shown in ([12]),

$$h^P(x, y) = \sup_{(u_-^P, u_+^P)} u_-^P(y) - u_+^P(x), \tag{1.7}$$

where the supremum is taken over all conjugated weak KAM pairs (u_-^P, u_+^P) associated with P. Following Mather ([22]), we denote

$$B_P(x) = h^P(x, x). \tag{1.8}$$

The Peierls barrier B_P vanishes precisely on the projected Aubry set (hence on the projected Mather set) associated with P ([12]).

Our next result is about the regularity of Peierls barrier $B_P(\cdot)$ at a point P_0 corresponding to a non-degenerate, quasi-periodic Lagrangian invariant torus. A relevant result in monotone twist map was first given by Mather ([21]) who showed that if ω_0 is a Diophantine number of order τ, then the function $\omega \to P_\omega(\xi)$ is Hölder continuous at ω_0 with Hölder exponent $\frac{1}{2\tau}$, i.e., $\left| P_\omega(\xi) - P_{\omega_0}(\xi) \right| \leq const. |\omega - \omega_0|^{\frac{1}{2\tau}}$ as $|\omega - \omega_0| \leq 1$, where $P_\omega(\cdot)$ is the Peierl's barrier function defined for a monotone twist map (see [21]).

Theorem 2. *Assume conditions* A1)-A3) *and let P_0 be as in Theorem 1. Then the Peierls barrier B_P is uniformly Hölder continuous in P at P_0 with Hölder exponent arbitrarily close to* 1*. If, in addition, both L and f are real analytic and ω is Diophantine, then B_P is Lipschiz continuous in P at P_0 uniformly in x.*

Motivated by recent studies on Arnold diffusions especially in a priori unstable, nearly integrable Hamiltonian systems (see [1, 2, 5, 6, 7, 9, 17, 18, 22, 23, 30, 31, 32]), it is well believed that P-regularity of Peierls barrier is of fundamental importance in characterizing Arnold diffusions. However, it is already known that such regularities are not generally expected in higher dimensions even in a weak sense (see [17, 24]). Nevertheless, we will show for a nearly integrable Hamiltonian system that the Peierls barrier is uniformly Lipschiz in P for a nearly full measure set of P's.

Consider a smooth family of nearly integrable, real analytic, fiber-wise convex Lagrangians $L_\varepsilon : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$: $(x, v) \mapsto L_\varepsilon(x, v) = L_0(v) + \varepsilon L_1(x, v, \varepsilon)$, $0 < \varepsilon < 1$. More precisely, we assume that

- **A1')** L_ε is real analytic in a complex neighborhood $D(r_0, s_0) = \{(x, v) : |\mathrm{Im} x| < r_0,\ |v| < s_0\}$ of $\mathbb{T}^n \times \mathbb{R}^n$ for each ε;
- **A2')** L_1 together with its derivatives up to second order are bounded on $\mathbb{T}^n \times \mathbb{R}^n \times (0, 1)$;
- **A3')** There exists a $\delta_0 > 0$ such that

$$\sum_{i,j=1}^{n} \frac{\partial^2 L_0}{\partial v_i \partial v_j}(v)\zeta_i\zeta_j \geq \delta_0\|\zeta\|^2, \quad \forall v, \zeta \in \mathbb{R}^n. \tag{1.9}$$

For fixed $\tau > n - 1$, $\gamma > 0$, $M_0 > 0$, consider the Diophantine set

$$\Omega_{\gamma,\tau} = \{\omega \in \mathbb{R}^n : \ |\omega| \leq M_0,\ |\langle \omega, k\rangle| > \frac{\gamma}{|k|^\tau},\ \ \forall k \in \mathbb{Z}^n \setminus \{0\}\}.$$

It follows from the KAM theorem in configuration space due to Salamon and Zehnder ([27]) that there exists an $\varepsilon_0 = \varepsilon_0(r_0, s_0, \delta_0, M_0, \gamma, \tau) > 0$ such that for each $0 < \varepsilon < \varepsilon_0$ and $\omega \in \Omega_{\gamma,\tau}$ the Euler-Lagrangian flow associated with L_ε admits a real analytic, quasi-periodic, invariant torus $\Gamma^\varepsilon_\omega$ with frequency ω. Moreover, these tori form a continuous family. In fact, the Salamon-Zehnder theorem holds when L_0 is only assumed to have an everywhere positive definite Hessian.

Theorem 3. *Consider the Lagrangians L_ε satisfying the conditions* A1')-A3') *and let $\Omega_{\gamma,\tau}$ and ε_0 be as in the above. Then the following holds for each $0 < \varepsilon < \varepsilon_0$, $\omega \in \Omega_{\gamma,\tau}$.*

a) *$\tilde{\mathcal{L}}(\Gamma^\varepsilon_\omega)$ is a C^1 (in fact, real analytic)-graph of a unique viscosity solution $u(\cdot, P_0)$.*
b) *There are positive constants $\Lambda_1', \Lambda_2', C$ depending on $r_0, s_0, \delta_0, M_0, \gamma, \tau$ such that*

$$\Lambda_2'\|P - P_0\|^2 \leq \overline{H}_\varepsilon(P) - \overline{H}_\varepsilon(P_0) - \langle D\overline{H}_\varepsilon(P_0), P - P_0\rangle \leq \Lambda_1'\|P - P_0\|^2, \tag{1.10}$$

$$\|Du(x, P) - Du(x, P_0)\|_\infty \leq C\|P - P_0\|, \tag{1.11}$$

$$|B^\varepsilon_P(x) - B^\varepsilon_{P_0}(x)| \leq C\|P - P_0\| \tag{1.12}$$

as $\|P - P_0\| \ll 1$, where $\overline{H}_\varepsilon$ and B^ε_P are the effective Hamiltonian and Peierls barrier, respectively, associated with the Lagrangian L_ε.

As to be seen in the proof, all results above hold if real analyticity of the Lagrangians are replaced by C^r smoothness for r sufficiently large.

Theorem 3 above also gives some information on the strict convexity of the effective Hamiltonian $\overline{H}_\varepsilon$. Let $\mathcal{D} = DL_0^{-1}(\{|\omega| \leq M_0\})$ and $\mathcal{D}_{\gamma,\tau}$ be the set of P_0 corresponding to the frequency set

$\Omega_{\gamma,\tau}$ according to Lemma 4.1. Then for fixed $\tau > n-1$, $|\mathcal{D}\setminus\mathcal{D}_{\gamma,\tau}| = O(\gamma)+O(\sqrt{\varepsilon})$. Since, by Aleksandrov's Theorem ([10]), $\overline{H}_\varepsilon$ as a convex function is (Lebesgue) almost everywhere twice differentiable on D, it follows from the Corollary in the above that it is (Lebesgue) almost everywhere strictly convex on $D_{\gamma,\tau}$, hence strictly convex on a nearly full measure subset of $\mathcal{D}$. We refer the readers to [3, 4, 11, 16, 28, 29] for other interesting properties of effective Hamiltonians, and to [14, 15] for for the ε-regularity of viscosity solutions.

The paper is organized as follows. In Section 2, we show the viscous stability of the trivial torus $\mathbb{T}^n\times\{0\}$ for a Hamiltonian normal form

$$H(x,p) = \langle\omega,p\rangle + \frac{1}{2}\big\langle A(x)p,p\big\rangle + O(\|p\|^3), \qquad \|p\|\ll 1,$$

with the stability index arbitrarily close to 1. In the case that H is real analytic and ω is Diophantine, we average out the angular variable in $A(x)$ and show that the stability index can be improved to 1. In Section 3, we prove these theorems by applying normal form reductions. Some technical lemmas will be placed in the Appendix at the end.

Through the paper, we use $\|\cdot\|$ for l^2-norm and use the same symbol $|\cdot|$ for absolute value of a real number, sup-norm in an Euclidean space, and l^1 norm of integer-valued vectors.

2. Viscous Stability Index of Invariant Tori

We consider a C^3 *Tonelli* Hamiltonian $H(x,p)$, $(x,p)\in\mathbb{T}^n\times\mathbb{R}^n$, which has the following form when $\|p\|$ is small. More clearly,

$$H(x,p) = \langle\omega,p\rangle + \frac{1}{2}\big\langle A(x)p,p\big\rangle + f(x,p), \tag{2.1}$$

where $|f|+|f_x|\le c_*\|p\|^3$, $\|p\|<R_0$, for some positive constants c_* and $R_0\ll 1$, and for all $x\in\mathbb{T}^n$, $A(x)$ are positive definite, symmetric, and satisfy

$$\lambda_2\|v\|^2 \le \big\langle A(x)v,v\big\rangle \le \lambda_1\|v\|^2, \qquad \forall v\in\mathbb{R}^n, \tag{2.2}$$

for some positive constants λ_1, λ_2.

We note that the Hamiltonian system associated with H reads

$$\begin{cases} \dot{x} = & \omega + A(x)p + \frac{\partial f}{\partial p} \\ \dot{p} = & -\frac{1}{2}\big\langle\nabla A(x)p,p\big\rangle - \frac{\partial f}{\partial x}, \end{cases} \tag{2.3}$$

where $\nabla A = (\frac{\partial A}{\partial x_1},\cdots,\frac{\partial A}{\partial x_n})^T$. It is clear that $T_\omega = \mathbb{T}^n\times\{0\}$ is a quasi-periodic invariant torus with frequency vector ω. When $P=0$, the viscosity solution $u(\cdot,0)=0$ is unique modular constants, whose C^1 graph corresponds to the invariant torus T_ω. We will prove the following result.

Theorem 2.1. *For any small positive ϵ_0, there are positive constants $C=C(n,\lambda_1,\lambda_2)$, $\iota = \iota(\lambda_1,\lambda_2,n,\epsilon_0,R_0,f)\ll 1$ such that any viscosity solution $u(x,P)$ with $\|P\|\le\iota$ satisfies*

$$\|D_x u(x,P)\| \le C\|P\|^{1-\epsilon_0}$$

for all $x\in dom(Du(\cdot,P))$.

The theorem implies that the quasi-periodic torus T_ω is viscously stable with stability index $1-$. Define

$$|||u(\cdot,P)-u(\cdot,0)||| = \sup_{x\in\mathbb{T}^n}\inf_c |u(x,P)-u(x,0)-c|.$$

The theorem also yields the following

Corollary 2.1. *For any small positive ϵ_0, there is an $0 < \iota = \iota(\lambda_1, \lambda_2, n, \epsilon_0, R_0, f) \ll 1$ such that any viscosity solution $u(x, P)$ with $\|P\| \le \iota$ satisfies*

$$|||u(\cdot, P) - u(\cdot, 0)||| \le C\|P\|^{1-\epsilon_0},$$

where C is a positive constant depending only on λ_1, λ_2, and n.

The proof of the theorem will be divided into several lemmas as follows.

Since the torus T_ω is uniquely ergodic, it follows from Corollary 4.6 in [13] that the effective Hamiltonian $\overline{H}(P)$ is differentiable at 0.

Lemma 2.1. *As $\|P\| \ll 1$,*

$$\frac{\lambda_2}{4}\|P\|^2 \le \overline{H}(P) - \overline{H}(0) - \langle D\overline{H}(0), P\rangle = \overline{H}(P) - \langle \omega, P\rangle \le \lambda_1\|P\|^2.$$

Proof. We note from [8] that

$$\overline{H}(P) = \inf_{\phi \in C^1(\mathbb{T}^n)} \sup_{x\in\mathbb{T}^n} H(x, P + D_x\phi).$$

By choosing ϕ as a constant function, we have that $\overline{H}(P) \le \sup_{x\in\mathbb{T}^n} H(x, P)$. It follows from (2.1) and (2.2) that

$$\overline{H}(P) \le \sup_{x\in\mathbb{T}^n} H(x, P) \le \langle \omega, P\rangle + \frac{\lambda_1}{2}\|P\|^2 + c_*\|P\|^3 \le \langle \omega, P\rangle + \lambda_1\|P\|^2 \tag{2.4}$$

as $\|P\| \ll 1$,

Let $u(x, P)$, $\|P\| \ll 1$, be a viscosity solution and $x \in dom(Du(\cdot, P))$. It follows from (1.2), (2.1), and (2.2) that

$$\begin{aligned} \overline{H}(P) &= H(x, P + Du(x, P)) \\ &= H(x, P) + D_pH(x, P)Du(x, P) + \frac{1}{2}\Big\langle \frac{\partial^2 H}{\partial p^2}(x, P + \theta Du(x, P))Du(x, P), Du(x, P)\Big\rangle \\ &\ge H(x, P) + D_pH(x, P)Du(x, P) + \frac{\lambda_2}{4}\|Du(x, P)\|^2. \qquad (2.5) \end{aligned}$$

For fixed P, since $-u(x, P)$ is semi-convex (see [12]), it is differentiable at its maximum point x_0 and $D_{x_0}(-u(x, P)) = 0$. Applying (2.5) with $x = x_0$, we have

$$\overline{H}(P) \ge H(x_0, P) \ge \langle \omega, P\rangle + \frac{\lambda_2}{2}\|P\|^2 - c_*\|P\|^3 \ge \langle \omega, P\rangle + \frac{\lambda_2}{4}\|P\|^2. \tag{2.6}$$

The lemma now follows from (2.4) and (2.6). □

Let L be the Lagrangian associated with H. If $\|v - \omega\|$ small enough, then L has the form

$$L(x, v) = \frac{1}{2}\big\langle A^{-1}(x)(v - \omega), (v - \omega)\big\rangle + g(x, v - \omega).$$

By making R_0 further small if necessary, it is easy to see that there is a constant $c > 0$ such that $|\frac{\partial f}{\partial p}| \le c\|p\|^2$ and $\|\frac{\partial^2 f}{\partial p^2}\| \le c\|p\|$ for all $\|p\| \le R_0$. It follows that there is a constant $c_1 > 0$ such that

$$|g(x, v - \omega)| \le c_1\|v - \omega\|^3, \qquad \Big|\frac{\partial g}{\partial v}(x, v - \omega)\Big| \le c_1\|v - \omega\|^2 \tag{2.7}$$

for all $x \in \mathbb{T}^n$ and v with $\|v - \omega\| \ll 1$. Denote $\hat{L} = L - \langle P, v\rangle + \overline{H}(P)$.

Lemma 2.2. *For any $x,\ y\in\mathbb{T}^n$ and $\|P\|\ll 1$, there exists a curve $\delta:[0,\frac{1}{\|P\|}]\to\mathbb{T}^n$ with $\delta(0)=x,\ \delta(\frac{1}{\|P\|})=y$, such that*

$$S\Big(\delta\Big|_0^{\frac{1}{\|P\|}}\Big)=:\int_0^{\frac{1}{\|P\|}}\hat{L}(\delta,\dot{\delta})ds\le c_2\|P\|,$$

where

$$c_2=\frac{1}{\lambda_2}(2\pi\sqrt{n}+\frac{\lambda_2}{2})^2+\lambda_1. \tag{2.8}$$

Proof. Denote $\|P\|=\epsilon$, $x_0=x+\frac{\omega}{\epsilon}$, and $\delta_0=y-x_0$. Let $\delta(t)=x+(\omega+\delta_0\epsilon)t$. Clearly, $\delta(0)=x$ and $\delta(\frac{1}{\epsilon})=y$. As $\|v-\omega\|\ll 1$ and $\|P\|\le R_0\ll 1$, we have by Lemma 2.1 and (2.7) that

$$\begin{aligned}\hat{L} &\le \frac{1}{2}\langle A^{-1}(x)(v-\omega),(v-\omega)\rangle+g(x,v-\omega)-\langle P,v-\omega\rangle+\lambda_1\|P\|^2\\ &\le \frac{1}{2\lambda_2}\|v-\omega\|^2+c_1\|v-\omega\|^3+\|P\|\|v-\omega\|+\lambda_1\|P\|^2\\ &\le \frac{1}{\lambda_2}(\|v-\omega\|+\frac{\lambda_2}{2}\|P\|)^2+\lambda_1\|P\|^2.\end{aligned}$$

Hence, when $\|P\|=\epsilon$ is sufficiently small, we have

$$S(\delta)=\int_0^{\frac{1}{\epsilon}}\hat{L}(\delta,\dot{\delta})ds\le\int_0^{\frac{1}{\epsilon}}\Big[\frac{1}{\lambda_2}\big[\|\delta_0\|\epsilon+\frac{\lambda_2}{2}\epsilon\big]^2+\lambda_1\epsilon^2\Big]ds\le c_2\epsilon.$$

□

By [12], for any viscosity solution $u(x,P)$ and any $x\in dom(Du(\cdot,P))$, there exists a unique calibrated curve $x(s)$ with $x(0)=x$, satisfying

$$P+Du(x,P)=\frac{\partial L}{\partial v}(x(0),\dot{x}(0)). \tag{2.9}$$

Lemma 2.3. *Let $u(x,P)$ be any viscosity solution associated with H and $x_P(s)$ be the calibrated curve satisfying $x_P(0)=x\in dom(Du(\cdot,P))$ and (2.9). Then for any small positive ϵ_0, there exists an $\iota>0$ such that*

$$\|\dot{x}_P(-s)-\omega\|\le\sqrt{B_1}\|P\|^{1-\epsilon_0},\quad s\ge 0,$$

as $\|P\|\le\iota$, where $B_1=\frac{c_2 2^{10}\lambda_1^3}{\lambda_2^2}+1$ and c_2 is as in (2.8).

Proof. Suppose for contradiction that the lemma is not true. Then for any $0<\iota_0\ll 1$, there exists an $\bar{P}\in R^n$, $\|\bar{P}\|\le\iota_0$ such that calibrated curve $x_{\bar{P}}(s)$ corresponding to some viscosity solution $u(x,\bar{P})$ satisfies

$$\|\dot{x}_{\bar{P}}(-s_0)-\omega\|>\sqrt{B_1}\|\bar{P}\|^\chi \tag{2.10}$$

for some $s_0\ge 0$, where $\chi=1-\epsilon_0$. We note that $(x_{\bar{P}}(s),\dot{x}_{\bar{P}}(s))$, $s\le 0$, actually solves the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L^0}{\partial v}=\frac{\partial L^0}{\partial x},$$

where $L^0=L-\langle\bar{P},v\rangle$ which clearly corresponds to the Hamiltonian $H_0(x,p)=H(x,\bar{P}+p)$. We denote $(x_{\bar{P}}(s),p(s))$ as the solution of the Hamiltonian equation of H_0 corresponding to the calibrated curve $x_{\bar{P}}(s)$. Then $(x_{\bar{P}}(s),w(s))$ with $w(s)=\bar{P}+p(s)$ is the solution of the Hamiltonian equation of H. From Lemma 4.2 in the Appendix and the fact $|f_x|\le c_*\|w\|^3$, we have

$\|\dot{w}(s)\| \le 2c\|w(s)\|^2$, $s \le 0$, for some constant $c > 0$, as long as ι_0 is sufficiently small. Let $r(s) = \langle w(s), w(s)\rangle = \|w(s)\|^2$. Then

$$|\dot{r}(s)| \le 4cr(s)^{\frac{3}{2}}, \qquad s \le 0. \tag{2.11}$$

Since $w(s) = \frac{\partial L}{\partial v}(x_{\bar{P}}(s), \dot{x}_{\bar{P}}(s))$, we have by (2.10) that

$$r(-s_0) \ge \frac{B_1}{4\lambda_1^2}\|\bar{P}\|^{2\chi}. \tag{2.12}$$

Combining (2.11) and (2.12) yields

$$r(t) \ge \frac{B_1}{16\lambda_1^2}\|\bar{P}\|^{2\chi},$$

or equivalently,

$$\|\dot{x}_{\bar{P}}(t) - \omega\| \ge \frac{\lambda_2\sqrt{B_1}}{8\lambda_1}\|\bar{P}\|^{\chi}$$

for all $t \in [-s_0 - \frac{1}{\|\bar{P}\|^{\iota_1}}, -s_0]$, where $\iota_1 = 1 - 2\epsilon_0$. Since by (2.2), (2.7), and Lemma 2.1

$$\begin{aligned} L(x,v) - \langle P, v\rangle + \overline{H}(P) &\ge \frac{1}{2}\langle A^{-1}(x)(v-\omega,(v-\omega))\rangle + g(x, v-\omega) - \langle P, v-\omega\rangle \\ &\ge \frac{1}{4\lambda_1}\left[\|v-\omega\| - \lambda_1\|P\|\right]^2 - \lambda_1\|P\|^2, \end{aligned}$$

we have that

$$S(x_{\bar{P}}) = \int_{-s_0 - \frac{1}{\|\bar{P}\|^{\iota_1}}}^{-s_0} L(x_{\bar{P}}, \dot{x}_{\bar{P}}) - \langle \bar{P}, \dot{x}_{\bar{P}}\rangle + \overline{H}(\bar{P})ds \ge \frac{\lambda_2^2 B_1}{2^{10}\lambda_1^3}\|\bar{P}\|,$$

as ι_0 sufficiently small. But by Lemma 2.2 there exists a curve $\delta : [0, \frac{1}{\|\bar{P}\|}] \to \mathbb{T}^n$ with $\delta(0) = x_{\bar{P}}(-s_0 - \frac{1}{\|\bar{P}\|^{\iota_1}})$ and $\delta(\frac{1}{\|\bar{P}\|}) = x_{\bar{P}}(-s_0)$, which satisfies

$$S(\delta) = \int_0^{\frac{1}{\|\bar{P}\|}} \hat{L}(\delta, \dot{\delta})ds \le c_2\|\bar{P}\|,$$

as ι_0 sufficiently small. Since $S(x_{\bar{P}}) \le S(\delta)$ according to the minimality of calibrated curves, we have that $\frac{\lambda_2^2 B_1}{2^{10}\lambda_1^3} \le c_2$, which is a contradiction to the choice of B_1. □

Proof of Theorem 2.1. For any $\|P\| \ll 1$ and $x \in dom(Du(\cdot, P))$, we let $\gamma(s)$ be the calibrated curve satisfying $\gamma(0) = x$ and (2.9). Applying Lemma 2.3 to $\gamma(s)$ easily yields the theorem. □

For the real ball B_{R_1} of radius R_1 in $\mathbb{R}^n$ centered at the origin, we denote

$$D(R_1, R_2, \sigma) = \{(p,x) \in \mathbb{C}^n \times \mathbb{C}^n/(2\pi\mathbb{Z}^n) : \ dist(p,\ B_{R_1}) \le R_2,\ |Imx| < \sigma\}. \tag{2.13}$$

For the rest of the section, we assume that

H1) ω is Diophantine of type (τ, γ) for some constants $\tau \ge n-1$ and $\gamma > 0$, i.e., $\big|\langle k, \omega\rangle\big| \ge \gamma|k|^{-\tau}$, $k \in \mathbb{Z}^n \setminus \{0\}$.

We also assume that

H2) the Hamiltonian (2.1) is real analytic in the domain $D(R_0, 3R_0, \sigma_0)$ for some constants $0 < R_0 \ll 1$ and $\sigma_0 > 0$.

The following result implies that the viscous stability index of T_ω is optimal under these conditions.

Theorem 2.2. *Under assumptions H1), H2), there are positive constants C, ι depending on $\lambda_1, \lambda_2, n, \tau, \sigma_0, R_0, M, \omega, \gamma$ in which ι also depends on f such that any viscosity solution $u(x, P)$ with $\|P\| \le \iota$ satisfies*

$$\|D_x u(x, P)\| \le C\|P\|$$

for all $x \in dom(Du(\cdot, P))$.

Theorem 2.2 also yields the following

Corollary 2.2. *Under assumptions H1), H2), any viscosity solution $u(x, P)$ with $\|P\| \le \iota$ satisfies*

$$\||u(\cdot, P) - u(\cdot, 0)\|| \le C\|P\|,$$

where C, ι are as in Theorem 2.2.

The proof of Theorem 2.2 will use the minimality of calibrated curves and the superexponential stability of KAM tori ([25]). A main ingredient in the proof is to average out the angular dependence in the matrix A via performing one step of standard KAM iterations (see e.g., [26, 20]). For this purpose, we rewrite the Hamiltonian H as

$$H = N + G + f,$$

where $N = \langle \omega, p\rangle$ and $G = \frac{1}{2}\big\langle A(x)p, p\big\rangle$. For any real analytic function

$$h(x) = \sum_{k \in Z^n} h_k e^{\sqrt{-1}\langle k, x\rangle}$$

defined in the domain $\{x \in \mathbb{C}^n/2\pi\mathbb{Z}^n : \ |\mathrm{Im}\ x| < \sigma\}$, we denote

$$\|h\|_\sigma = \sum_{k \in Z^n} |h_k| e^{|k|\sigma},$$

and for any real analytic function $h(x, p)$ on $D(R_0, R, \sigma)$, we define

$$|h|_{D(R_0,R,\sigma)} = \sup_{(x,p)\in D(R_0,R,\sigma)} |h(x, p)|.$$

We let

$$\begin{aligned} C_\tau &= (10\tau)^\tau e^{-\tau}, \\ c_0 &= \sup_{0<R\le R_0} \frac{|f|_{D(R,3R,\sigma_0)}}{R^3}, \\ M &= \max_{1\le i,j\le n} \{|a_{ij}|_{\sigma_0}\}, \\ R &= \frac{1}{2}\min\{\frac{\gamma\sigma_0^{\tau+1}}{10C_\tau M}, \frac{n^2 M}{c_0}, R_0\}, \end{aligned}$$

where $a_{i,j}$'s are entries of A.

Lemma 2.4. *If $\sigma_0 < \left(\frac{\|\omega\|}{\gamma}\right)^{\frac{1}{\tau+1}}$, then there exists a symplectic, real analytic transformation $\Phi : D(R, R, \frac{7}{10}\sigma_0) \to D(R, 2R, \frac{4}{5}\sigma_0)$ such that*

$$H_* = H \circ \Phi = N + \widehat{G} + f + G_*,$$

where $\widehat{G}$ is the average of G in x and G_* satisfies

$$|G_*|_{D(R,R,\frac{7}{10}\sigma_0)} \le \frac{C_\tau M^2\|\omega\|R^3}{\gamma^2\sigma_0^{2\tau+2}}. \tag{2.14}$$

Proof. Consider

$$F = \frac{1}{2}\sum_{i,j=1}^{n} F_{ij}(x)p_i p_j,$$

where

$$F_{ij}(x) = \sum_{k\neq 0} \frac{\sqrt{-1}a_{ij,k}e^{\sqrt{-1}\langle k,x\rangle}}{\langle k,\omega\rangle}.$$

Then F solves the homological equation

$$\{N,F\} + G = \widehat{G},$$

where

$$\widehat{G} = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij,0}p_i p_j$$

in which $a_{i,j,0}$ is the average of $a_{i,j}(x)$ for each i,j. Let Φ be the time-1 map X_F^1 of the Hamiltonian flow X_F^t associated with the Hamiltonian F. Then

$$H_* = H\circ\Phi = N + \{N,F\} + G + f + G_* = N + \widehat{G} + f + G_*,$$

where

$$G_* = \int_0^1 \big\{(1-t)\{N,F\} + G + f, F\big\}\circ X_F^t dt.$$

Since

$$\|F_{ij}\|_{\frac{9}{10}\sigma_0} \le \frac{C_\tau}{\gamma\sigma_0^\tau}\|a_{ij}\|_{\sigma_0},$$

the choice of σ_0, R and Cauchy estimates easily yield that

$$|F|_{D(R,3R,\frac{9}{10}\sigma_0)} \le \frac{C_\tau M R^2}{\gamma\sigma_0^\tau}, \tag{2.15}$$

$$|F_x|_{D(R,3R,\frac{4}{5}\sigma_0)} \le \frac{C_\tau M R^2}{\gamma\sigma_0^{\tau+1}}, \tag{2.16}$$

$$|F_p|_{D(R,2R,\frac{4}{5}\sigma_0)} \le \frac{C_\tau M R}{\gamma\sigma_0^\tau}. \tag{2.17}$$

It follows that $\Phi = X_F^1 : D(R,R,\frac{7}{10}\sigma_0) \to D(R,2R,\frac{4}{5}\sigma_0)$ is well defined. The bounds on G_* in (2.14) follow from (2.15)-(2.17) and Cauchy estimates. □

Lemma 2.5. *Let $(x_*(t),p_*(t))$ be a solution of the Hamiltonian system associated with H_* satisfying $\|p_*(t)\| \le R$ for all $t \le 0$, and let $(x(s),p(s)) = \Phi(x_*(s),p_*(s))$ be the corresponding solution of the Hamiltonian H, where Φ is as in the proof of Lemma 2.4. Then the following holds for given $B \ge (\frac{60c_3}{\sigma_0})^2$, where*

$$c_3 = c_0 + \frac{C_\tau M^2\|\omega\|}{\gamma^2\sigma_0^{2\tau+2}}. \tag{2.18}$$

.

a) *If $\|p_*(-s_0)\| \ge \frac{\sqrt{B}}{2}\|P\|$ for some $s_0 \ge 0$, and $\|P\| \ge R^{\frac{3}{2}}$, then*

$$\|p_*(t)\| \ge \frac{\sqrt{B}}{3}\|P\|$$

for all $t \in [-s_0 - \frac{1}{\|P\|}, -s_0]$. If, in addition, $\|P\| \le \frac{36R}{7\sqrt{B}}$, then $\|p(t)\| \ge \frac{\sqrt{B}}{4}\|P\|$ for all $t \in [-s_0 - \frac{1}{\|P\|}, -s_0]$.

b) *If $\|P\| \le \frac{36R}{7\sqrt{B}}$ and $\|p(-s_0)\| \ge \sqrt{B}\|P\|$ for some $s_0 \ge 0$, then $\|p_*(-s_0)\| \ge \frac{\sqrt{B}}{2}\|P\|$.*

Proof. For a), we write $W = f + G_*$. From the estimates $|f|_{D(R,3R,\sigma_0)} \leq c_0 R^3$ and (2.14), we have that $|W|_{D(R,R,\frac{7}{10}\sigma_0)} \leq c_3 R^3$. It follows from Cauchy estimate that $|\frac{\partial W}{\partial x}|_{D(R,R,\frac{6}{10}\sigma_0)} \leq \frac{10 c_3 R^3}{\sigma_0}$. Let $t \in [-s_0 - \frac{1}{\|P\|}, -s_0]$. It then follows that

$$\|p_*(t)\| \geq \|p_*(-s_0)\| - \int_t^{-s_0} \|\dot{p}_*(s)\| ds \geq \frac{\sqrt{B}}{2}\|P\| - \int_t^{-s_0} \frac{10 c_3 R^3}{\sigma_0} ds \geq \frac{\sqrt{B}}{3}\|P\|.$$

The rest of proof of a) follows from the fact that $|p(t)| \geq 3R - |p_*(t)|$.

The proof of b) is similar to that of a). □

Lemma 2.6. *Let $u(x,P)$ be any viscosity solution associated with H and $x_P(s)$ be the calibrated curve satisfying $x_P(0) = x \in dom(Du(\cdot,P))$ and (2.9). Then there exists a constant $\iota = \iota(\lambda_1, \lambda_2, n, \tau, M, \omega, \gamma, \sigma_0, R_0, f) > 0$ such that*

$$\|\dot{x}_P(-s) - \omega\| \leq 2\lambda_1 \sqrt{B}\|P\|, \quad s \geq 0$$

as $\|P\| \leq \iota$, where $B = 2\max\{\frac{2^9\lambda_1}{\lambda_2^2}(\frac{\lambda_1}{2} + c_2) + 1, (\frac{60c_3}{\sigma_0})^2\}$ with c_2, c_3 defined in (2.8), (2.18) respectively.

Proof. Suppose for contradiction that the lemma is not true. Then for any $0 < \iota_0 \ll 1$, there exists an $\bar{P} \in R^n$, $\|\bar{P}\| \leq \iota_0$ such that the calibrated curve $x_{\bar{P}}(s)$ corresponding to some viscosity solution $u(x,\bar{P})$ satisfies

$$\|\dot{x}_{\bar{P}}(-s_0) - \omega\| > 2\lambda_1\sqrt{B}\|\bar{P}\| \tag{2.19}$$

for some $s_0 \geq 0$. Using (2.19), the same argument as in the proof of Lemma 2.3 yields that

$$\|p(-s_0)\| \geq \sqrt{B}\|\bar{P}\|. \tag{2.20}$$

Denote $H_*(x_*, p_*) = H \circ \Phi(x_*, p_*) = H(x,p)$. From Theorem 2.1, we have $\|\bar{P} + Du(x,\bar{P})\| \leq C\|\bar{P}\|^{\frac{11}{12}}$. It follows that there is a constant $C > 0$ such that $\|p(s)\| \leq C\|\bar{P}\|^{\frac{11}{12}}$ as $s \leq 0$. Hence

$$\|p_*(s)\| \leq C\|\bar{P}\|^{\frac{11}{12}} \leq \|\bar{P}\|^{\frac{5}{6}}, \ s \leq 0, \tag{2.21}$$

for some constant $C > 0$. Now, Lemma 2.5 and (2.20) imply that

$$\|p_*(-s_0)\| \geq \frac{\sqrt{B}}{2}\|\bar{P}\|. \tag{2.22}$$

With (2.21), (2.22), an application of Lemma 2.5 to $R = \|\bar{P}\|^{\frac{5}{6}}$ yields $\|p_*(s)\| \geq \frac{\sqrt{B}}{3}\|\bar{P}\|$ for $s \in [-s_0 - \frac{1}{\|\bar{P}\|}, -s_0]$. Applying Lemma 2.5 again, we have $\|p(s)\| \geq \frac{\sqrt{B}}{4}\|\bar{P}\|$ as $s \in [-s_0 - \frac{1}{\|\bar{P}\|}, -s_0]$. Thus $\|\dot{x}_{\bar{P}}(t) - \omega\| \geq \frac{\sqrt{B}\lambda_2}{8}\|\bar{P}\|$ for any $t \in [-s_0 - \frac{1}{\|\bar{P}\|}, -s_0]$. Since $B \geq \frac{2^8\lambda_1^2}{\lambda_2^2}$, an argument similar to that in the proof of Lemma 2.3 yields that

$$S(x_{\bar{P}}) = \int_{-s_0 - \frac{1}{\|\bar{P}\|}}^{-s_0} \hat{L}(x_{\bar{P}}(s), \dot{x}_{\bar{P}}(s)) ds \geq (\frac{\lambda_2^2 B}{2^9\lambda_1} - \frac{\lambda_1}{2})\|\bar{P}\|.$$

But from Lemma 2.2, there exists a curve $\delta : [0, \frac{1}{\|\bar{P}\|}] \to \mathbb{T}^n$ with $\delta(0) = x_{\bar{P}}(-s_0 - \frac{1}{\|\bar{P}\|})$ and $\delta(\frac{1}{\|\bar{P}\|}) = x_{\bar{P}}(-s_0)$ such that

$$S(\delta) = \int_0^{\frac{1}{\|\bar{P}\|}} \hat{L}(\delta, \dot{\delta}) ds \leq c_2\|\bar{P}\|.$$

As in the proof of Lemma 2.3, one has that $S(x_{\bar{P}}) \leq S(\delta)$, which leads to a contradiction to the definition of B. □

Proof of Theorem 2.2. For any $\|P\| \ll 1$ and $x \in dom(Du(\cdot,P))$, we let $\gamma(s)$ be the calibrated curve satisfying $\gamma(0) = x$ and (2.9). Applying Lemma 2.6 to $\gamma(s)$ easily yields the theorem. □

3. Proof of main results

We let $L: \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ be a Lagrangian whose Euler-Lagrange flow admits a quasi-periodic, invariant torus $\Gamma_\omega = \{(f(\xi), Df(\xi)\omega) : \xi \in \mathbb{T}^n\}$ with frequency vector $\omega \in \mathbb{R}^n$, for which the conditions A1), A2), A3) are satisfied.

Consider the Lagrangian

$$L_*: \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R} :, (\xi, q) \mapsto L(f(\xi), Df(\xi)q).$$

For any ξ, since $f(\xi + \omega t)$ satisfies the Euler-Lagrange equation associated with L, $\xi + \omega t$ is a solution of the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L_*}{\partial q} = \frac{\partial L_*}{\partial \xi}. \tag{3.1}$$

It follows that

$$d = L_*(\xi, \omega) - \langle \frac{\partial L_*}{\partial q}(\xi, \omega), \omega \rangle$$

is independent of ξ. Let

$$L_{**} = L_* - \eta - d \tag{3.2}$$

and $\frac{\partial^2 L_{**}}{\partial q^2}(\xi, \omega) = A^{-1}(\xi)$, where $\eta = \langle \frac{\partial L_*}{\partial q}(\xi, \omega), d\xi \rangle$. Clearly, $A^{-1} > 0$ and L_{**} is of the class C^5. It follows that there exist positive constants λ_1' and λ_2' such that

$$\lambda_2' \|v\|^2 \leq \langle A(\xi)v, v \rangle \leq \lambda_1' \|v\|^2, \ v \in \mathbb{R}^n,$$

and

$$L_{**}(\xi, q) = \frac{1}{2}\langle A^{-1}(\xi)(q - \omega), (q - \omega) \rangle + g(\xi, q).$$

From Taylor's theorem, it is easy to see that there is a constant $c > 0$ such that

$$\begin{aligned} |g(\xi, q)| &\leq c\|q - \omega\|^3, \ |g_q(\xi, q)| \leq c\|q - \omega\|^2, \ |g_\xi(\xi, q)| \leq c\|q - \omega\|^3, \\ \Big|\frac{\partial^2 g}{\partial q_i \partial q_j}\Big| &\leq c\|q - \omega\|, \ \Big|\frac{\partial^2 g}{\partial \xi_i \partial q_j}\Big| \leq c\|q - \omega\|^2, \end{aligned}$$

for all $i, \ j = 1, \cdots, n$.

Hence, by the implicit function theorem, the Hamiltonian $H_{**}(\xi, p)$ associated with L_{**} has the form

$$H_{**}(\xi, p) = \langle \omega, p \rangle + \frac{1}{2}\langle A(\xi)p, p \rangle + f(\xi, p), \tag{3.3}$$

where $|f(\xi, p)| + |f_\xi(\xi, p)| \leq c\|p\|^3$ as $\|p\| \ll 1$, for some constant $c > 0$.

Proof of Theorem 1. By A2), $\eta = \langle \frac{\partial L_*}{\partial q}(\xi, \omega), d\xi \rangle$ is a closed 1-form. Since for each $P' \in \mathbb{R}^n$,

$$\eta_{P'} = \langle (Df^{-1}(\xi))^\top P', d\xi \rangle$$

is also a closed 1-form, there are smooth functions $f_*(\xi)$, $F_*(\xi, P')$ such that

$$\begin{aligned} \eta &= \langle P_* + Df_*(\xi), d\xi \rangle, \\ \eta_{P'} &= \langle \mathcal{F}P' + D_\xi F_*(\xi, P'), d\xi \rangle, \end{aligned}$$

where $P_* = \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} \frac{\partial L_*}{\partial q}(\xi, \omega) d\xi$, $\mathcal{F} = \frac{1}{(2\pi)^n)}\int_{T^n} (Df^{-1}(\xi))^T d\xi$. Let $P = \mathcal{F}P' \in \mathbb{R}^n$ be sufficiently close to P_0 and $u(\cdot, P)$ be a viscosity solution associated with P. Then $u(x, P)$ is a weak KAM solution associated with the Lagrangian $L(x, v) - \langle P, v \rangle$. It follows easily that

$u_*(\xi,P) \equiv u(f(\xi),P) - F_*(f(\xi),P') - f_*(\xi)$ is a weak KAM solution associated with the Lagrangian $L_{**}(\xi,q) - \langle P'-P_*, q\rangle + d$. Hence it is a viscosity solution of the cell equation

$$H_{**}(\xi, P'-P_* + D_\xi u_*(\xi,P)) = \overline{H}_{**}(P'-P_*) = \overline{H}_*(P') + d = \overline{H}(P) + d.$$

Now, part a) of the theorem follows from the above equality and Lemma 2.1. Since $P_0 = \mathcal{F}P_*$ (Lemma 4.1), parts b), c) of the theorem follow from applications of Theorems 2.1, 2.2 with respect to $D_\xi u_*$ and H_{**}. □

Proof of Theorem 2. For any $P \in R^n$ sufficiently close to P_0, we let $u_-(\cdot,P),\ u_+(\cdot,P)$ be conjugated weak KAM pairs associated with P. Consider the projected Mather set $\mathcal{M}_P$ and fix a point $x_0 \in \mathcal{M}_P$. Then

$$u_-(x_0,P) = u_+(x_0,P),\ u_-(x,P_0) = u_+(x,P_0),\ \ x \in \mathbb{T}^n.$$

It follows that

$$\begin{aligned} & |u_-(x,P) - u_+(x,P)| \\ = \ & |(u_-(x,P) - u_-(x,P_0)) - (u_-(x_0,P) - u_-(x_0,P_0)) \\ - \ & (u_+(x,P) - u_+(x,P_0)) + (u_+(x_0,P) - u_+(x_0,P_0))| \\ \leq \ & |(u_-(x,P) - u_-(x,P_0) - (u_-(x_0,P) - u_-(x_0,P_0))| \\ + \ & |(u_+(x,P) - u_+(x,P_0)) - (u_+(x_0,P) - u_+(x_0,P_0))|. \end{aligned}$$

Since $B_{P_0} \equiv 0$, the Theorem 2 immediately follows from Theorem 1. □

Proof of Theorem 3. By the Salamon-Zehnder Theorem ([27]), the quasi-periodic tori $\{\Gamma^\varepsilon_\omega\}$ of the Euler-Lagrangian flow are graphs of real analytic toral diffeomorphisms $x = f^\varepsilon_\omega(\xi)$ satisfying $\|D_\xi f^\varepsilon_\omega - I\| = O(\epsilon)$ and $\mathcal{D}(L)_v(f^\varepsilon_\omega, \mathcal{D}f^\varepsilon_\omega) = (L)_x(f^\varepsilon_\omega, \mathcal{D}f^\varepsilon_\omega)$, where $\mathcal{D} = \sum_{j=1}^n \omega_j \frac{\partial}{\partial \xi_j}$. Hence the conditions A1), A3) are satisfied. As the quasi-periodic, invariant tori $\tilde{\mathcal{L}}(\Gamma^\varepsilon_\omega)$ of the Hamiltonian flow are Lagrangian submanifolds under the standard symplectic structure, A3) is also satisfied. Theorem 3 now follows from Theorems 1, 2. □

4. Appendix

Let $\Gamma_\omega = \{(f(\xi), Df(\xi)\cdot\omega) :\ \ \xi \in \mathbb{T}^n\}$ be the quasi-periodic invariant torus of the Euler-Lagrangian flow associated with L that satisfies the conditions A1)-A3). We denote $L_*(\xi,q) = L(f(\xi), Df(\xi)q)$.

Lemma 4.1. *Up to translations of constants, there is a unique smooth viscosity solution $u(x,P_0)$ such that*

$$\mathcal{G}_\omega = \tilde{\mathcal{L}}(\Gamma_\omega) = \bigcup_{x\in\mathbb{T}^n} \big(x, P_0 + Du(x,P_0)\big),$$

where $P_0 = \mathcal{F}P_*$ with

$$\begin{aligned} \mathcal{F} &= \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} \big(Df^{-1}(x)\big)^\top dx, \\ P_* &= \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} \frac{\partial L_*}{\partial q}(\xi,\omega)d\xi. \end{aligned}$$

Moreover,

$$Du(x,P_0) = \big(Df^{-1}(x)\big)^\top P_* - \mathcal{F}P_* + \big(Df^{-1}(x)\big)^\top \big(\frac{\partial L_*}{\partial q}(f^{-1}(x),\omega) - P_*\big).$$

Proof. Let $v_*(\xi, P)$ be a weak KAM solution associated with the Lagrangian $L_* - \langle P, q\rangle$, $v(x, P_0) = v_*(f^{-1}(x), P_*)$, f_*, F_* be as in the proof of Theorem 1, and

$$u(x, P_0) = F_*(x, P_*) + v(x, P_0).$$

It follows from (3.2) that $v_*(\xi, P)$ is also a weak KAM solution associated with the Lagrangian $L_{**} + d - \langle P - P_* - Df_*, q\rangle$. Hence $v_*(\xi, P) - f_*$ is the viscosity solution of the cell equation

$$H_{**}(\xi, P - P_* + D(v_*(\xi, P) - f_*)) = \overline{H}_{**}(P - P_*). \tag{4.1}$$

By (3.3), we have

$$D_\xi v_*(\xi, P_*) = Df_*. \tag{4.2}$$

It follows that

$$D_x v(x, P_0) = (Df^{-1}(x))^\top D_\xi v_*(\xi, P_*) = (Df^{-1}(x))^\top D_\xi f_*.$$

Since

$$P_* + Df_* = \frac{\partial L_*}{\partial q}(\xi, \omega) = (D_\xi f)^\top \frac{\partial L}{\partial q}(f(\xi), Df(\xi)\omega),$$

we have

$$\begin{aligned} \frac{\partial L}{\partial q}(f(\xi), Df(\xi)\omega) &= (D_x f^{-1})^\top (P_* + D_\xi f_*) = (D_x f^{-1})^\top P_* + D_x v(x, P_0) \\ &= P_0 + D_x F_*(x, P_*) + D_x v(x, P_0) = P_0 + Du(x, P_0). \end{aligned}$$

Now, by (4.1) and (4.2),

$$H_{**}(\xi, D(v_*(\xi, P_*) - f_*)) = \overline{H}_{**}(0),$$

i.e., $v_*(\xi, P_*)$ is a weak KAM solution associated with the Lagrangian $L_{**} + d + \langle Df_*, q\rangle$. It follows that $v(f(\xi), P_0) = v_*(\xi, P_*)$ is a weak KAM solution associated with the Lagrangian $L_* - \langle P_*, q\rangle$. Hence $v(x, P_0)$ is a weak KAM solution associated with the Lagrangian $L - \eta_x^{P_0}$, i.e., $u(x, P_0)$ is a viscosity solution of the cell equation

$$H(x, P_0 + Du(x, P_0)) = \overline{H}(P_0).$$

□

We note that from the proof above Lemma 4.1 is also true if L is of class C^3 and $f : \mathbb{T}^n \to \mathbb{T}^n$ is a C^3 diffeomorphism.

Lemma 4.2. *Let $H(x, p)$, $(x, p) \in \mathbb{T}^n \times \mathbb{R}^n$, be a C^2 Tonelli Hamiltonian such that $H(x, 0) \equiv 0$, and, for each $P \in \mathbb{R}^n$, denote by $u(x, P)$ as the viscosity solution satisfying the cell equation*

$$H(x, P + D_x u(x, P)) = \overline{H}(P).$$

Then $\lim_{P\to 0} \|P + D_x u(x, P)\| = 0$, for all $x \in dom(Du(x, P))$.

Proof. Suppose the lemma is not true, then there exists an $\epsilon_0 > 0$ such that for any sequence $\delta_n = \frac{1}{n} \to 0$ there are sequences P_n, $u_n(x, P_n)$, and $x_n \in dom(Du_n(x, P_n))$ such that $\|P_n\| \le \delta_n$ and

$$\|P_n + D_x u_n(x_n, P_n)\| \ge \epsilon_0. \tag{4.3}$$

For each n, since u_n is a viscosity solution, there exists a $(u_n, L - \langle P_n, v\rangle, \overline{H}(P_n))$-calibrated curve γ_n with $\gamma_n(0) = x_n$ such that

$$u_n(\gamma_n(0)) - u_n(\gamma_n(-t)) = \int_{-t}^{0} [L - \langle P_n, v\rangle] ds + \overline{H}(P_n)t, \ \ t > 0, \tag{4.4}$$

and,

$$P_n + D_x u_n(x_n, P_n) = \frac{\partial L}{\partial v}(\gamma_n(0), \dot\gamma_n(0)), \tag{4.5}$$

where L is the Lagrangian associated with H. By (4.5) and (**??**), we see that the sequence $\|(\gamma_n(0), \dot\gamma_n(0))\|$ is bounded. Without loss of generality, we assume that $(\gamma_n(0), \dot\gamma_n(0))$ converges, say, to some $(x_0, v_0) \in \mathbb{T}^n \times \mathbb{R}^n$. Denote

$$(\gamma_0(t), \dot\gamma_0(t)) = \phi^L_{-t}(x_0, v_0), \ \ t > 0,$$

where ϕ^L_t is the Lagrangian flow associated with L. For each $t > 0$, since

$$\lim_{n\to\infty} \phi^L_{-t}(\gamma_n(0), \dot\gamma_n(0)) = \phi^L_{-t}(x_0, v_0) = (\gamma_0(-t), \dot\gamma_0(-t)), \tag{4.6}$$

we have

$$\lim_{n\to\infty} (\gamma_n(-t), \dot\gamma_n(-t)) = (\gamma_0(-t), \dot\gamma_0(-t)). \tag{4.7}$$

Let $[u_n](x) = u(x, P_n) - \min_{x\in\mathbb{T}^n} u_n(x, P_n)$. Since

$$H(x, P_n + Du_n(x, P_n)) = \overline{H}(P_n), \tag{4.8}$$

the sequence $\{u(\cdot, P_n)\}$, hence $\{[u_n]\}$, is equicontinuous. Hence there exists a $u \in C^0(\mathbb{T}^n, \mathbb{R})$ such that $[u_n]$ converges to u in the C^0 topology uniformly on $\mathbb{T}^n$. Since

$$\lim_{n\to\infty} \overline{H}(P_n) = \overline{H}(0) = 0,$$

we have by letting $n \to \infty$ in (4.4) and (4.7) that

$$u(\gamma_0(0)) - u(\gamma_0(-t)) = \int_{-t}^0 L(\gamma_0(s), \dot\gamma_0(s))ds, \ \ t > 0. \tag{4.9}$$

From (4.8) and Theorem 8.1.1 in [12], we see that u is a viscosity solution satisfying $H(x, D_x u) = 0$. Hence u is a constant. It follows that

$$D_x u(\gamma_0(0)) = \frac{\partial L}{\partial v}(\gamma_0(0), \dot\gamma_0(0)) = 0.$$

But since $\|P_n + D_x u_n(x_n, P_n)\| = \|\frac{\partial L}{\partial v}(\gamma_n(0), \dot\gamma_n(0))\| \ge \epsilon_0$, we have by (4.7) that

$$\|\frac{\partial L}{\partial v}(\gamma_0(0), \dot\gamma_0(0))\| \ge \epsilon_0.$$

This is a contradiction. □

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