

BIFURCATIONS IN A STOCHASTIC BUSINESS CYCLE MODEL*

Dongwei Huang[†]

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA

Hongli Wang

School of Mechanical Engineering, Tianjin University, Tianjin, 300072, P.R. China

Yingfei Yi

School of Mathematics, Jilin University, Changchun, 130012, P. R. China

Received (to be inserted by publisher)

We introduce a stochastic business cycle model and study the underlying stochastic Hopf bifurcations with respect to probability densities at different parameter values. Our analysis is based on the calculate of the largest Lyapunov exponent via multiplicative ergodic theorem and the theory of boundary analysis for quasi-non-integrable Hamiltonian systems. Some numerical simulations of the model are performed.

Keywords: stochastic business cycle; Hamiltonian function; stochastic Hopf bifurcation; Lyapunov exponent

1. Introduction

In this work, we introduce a stochastic economy or business model and study its dynamics. Stochastic dynamical frame has become an inevitable tendency of studying economy movements. In a mathematical model describing economy movement, uncertain influential factors can be regarded as random external forcing or excitations to make the economical variables show randomness and non-determinacy. To the contrary, deterministic economy models can neither reflect the real movement of an economy system nor they are suffice to the development of an economical theory.

Let Y_t denote the productive capital at time t , C_t the consumption value at time t , I_t and the investment at time t . Then for a closed economy system, we have that $Y_t = C_t + I_t$. Samuelson and Hicks [Samuelson, 1939; Hicks, 1950] proposed a model based on “linear multiplier-accelerator theory” by assuming that the investment is proportional to the change of incomes between two previous periods and the consumption is proportional to the income in the previous period, i.e., they assume that $I_t = vZ_{t-1}$, where $Z_{t-1} = Y_t - Y_{t-1}$, and $C_t = (1 - s)Y_{t-1}$, where $0 \leq s < 1$. Hence under this model, dynamics of the productive capital is described by the equation $Y_t = (1 + v - s)Y_{t-1} - vY_{t-2}$.

The main weakness of this model is that it fails to describe the sudden changes or recession of the economy which take place in a continuous cycle. So Puu and Sushko [Puu & Sushko, 2004; Puu, 1991, 2000]

*Corresponding author: Dongwei Huang, tjhuangdw@hotmail.com, tjhuandw@163.com

[†]School of Science, Tianjin Polytechnic University, Tianjin, 300160, P.R. China

proposed an improved model by assuming linear-cubic shape for the investment function. More precisely, they assumed that:

$$I_t = v(Y_t - Y_{t-1}) - v(Y_{t-1} - Y_{t-2})^3, \quad (1a)$$

$$Z_t = (v - s)Z_{t-1} - vZ_{t-1}^3, \quad (1b)$$

$$C_t = (1 - s)Y_{t-1} + \epsilon s Y_{t-2}, \quad (1c)$$

where $0 \leq \epsilon < 1$. This model takes into account the real fact that governments tend to distribute infrastructure investment counter-cyclically, partly as a means to fight depression, partly as a means to profit from lower input prices during slumps. Using equations (1a), (1b), (1c) together with the relation $Y_t = C_t + I_t$, Puu and Sushko obtained the equation

$$Z_t = (v - s)Z_{t-1} - vZ_{t-1}^3 - (1 - \epsilon)sY_{t-1}, \quad (2)$$

which, under the rescaling $((1 + v - s)/v)^{1/2}$, becomes

$$Z_t = aZ_{t-1} - (1 + a)Z_{t-1}^3 - bY_{t-1} \quad (3)$$

where $a = v - s > 0$, $v > 1$ is the accelerated number and $b = (1 - \epsilon)s$ represents an eternal rate of saving [Samuelson, 1939; Hicks, 1950].

To introduce our stochastic model, we first replace Y_{t-1} , $Z_{t-1} = Y_t - Y_{t-1}$, and $Z_{t-1} = Y_t - Y_{t-1}$ in Eq. (3) by $x(t) \in C^2$, \dot{x} , and \ddot{x} respectively, to obtain the following continuous model:

$$\ddot{x} + (1 - a)\dot{x} + (1 + a)x^3 + bx = 0. \quad (4)$$

By adding an excitation $\Psi(t)$ which may contain the change of productive capital, investment and consumption, we then obtain the following excited business cycle model:

$$\ddot{x} + (1 - a)\dot{x} + (1 + a)x^3 + bx = \Psi(t). \quad (5)$$

Taking into account the randomness and non-determinacy of $\Psi(t)$, we assume it to be a stochastic function. More precisely, we will consider the following stochastic business cycle model:

$$\ddot{x} + (1 - a)\dot{x} + (1 + a)x^3 + bx = \beta x W(t), \quad (6)$$

where β is a parameter, and $W(t)$ is the Gaussian white noise with zero mean value and intensity $2D$; with β and D being small numbers [Zhu, 2006]–[Huang *et al.*, 2008].

The rest of the paper will be devoted to the dynamical study of this stochastic model, for which stability and bifurcation analysis will be conducted, along with some numerical simulations. Some concluding remarks will be made at the end.

2. Stability

Eq. (6) defines a so-called quasi-non-integrable-Hamiltonian system [Zhu, 2006; Zhu & Huang, 1999], i.e., it is a non-integrable Hamiltonian system subject to a weak damping and stochastic stimulation. With $q = x$, $p = \dot{x}$ denoting the generalized displacement and the generalized momentum respectively, we obtain the stochastic differential equations as follows:

$$\dot{q} = p, \quad (7a)$$

$$\dot{p} = -(a + 1)p^3 + (a - 1)p - bq + \beta q W(t). \quad (7b)$$

By leaving out damping and the stochastic stimulation term, the above becomes a Hamiltonian system with the Hamiltonian function

$$H(p, q) = \frac{1}{2}p^2 + \frac{1}{2}bq^2. \quad (8)$$

According to the properties of a quasi-non-integrable Hamiltonian system [Zhu, 2006; Zhu & Huang, 1999], the Hamiltonian function of system (7a), (7b) converges in probability to an one-dimension diffusion process satisfying the following Itô differential equation:

$$dH = m(H)dt + \sigma(H)dB(t), \quad (9)$$

where $B(t)$ is a standard Weiner process, $m(H)$ and $\sigma(H)$ are the drift coefficient and diffusion coefficient of the process. Using the stochastic averaging method [Zhu, 2006; Zhu & Huang, 1999] of quasi-non-integrate Hamiltonian systems, we obtain

$$m(H) = \left(\frac{D\beta^2}{b} + a - 1 \right) H - \frac{3}{2}(a+1)H^2 \quad (10a)$$

$$\sigma^2(H) = \frac{D\beta^2 H^2}{b}. \quad (10b)$$

2.1. Calculation of the Largest Lyapunov Exponent

For the purpose of characterizing the local and global stability of the trivial solution of the stochastic business cycle model, we consider the largest Lyapunov exponent of the linearized system

$$\lambda = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \|Z(t, z_0)\|, \quad (11)$$

where $Z(t, z_0) = [q, p]^T$, $z_0 = z(0)$, is the value of the original state. As $\|Z(t, z_0)\|$ is equivalent to $\|H(t)\|^{1/2}$, we will replace the former by the latter in (11) for computing the largest Lyapunov exponent.

Theorem 1. *If $a < 1 - \frac{D\beta^2}{2b}$, then the trivial solution $H = 0$ of system (9) is stable.*

Proof. From equations (9), (10a), (10b), we can write the linearized system at $H = 0$ as the following linear Itô equation

$$dH = m(0)H dt + \sigma(0)H dB(t). \quad (12)$$

It follows from (12) that

$$H(t) = H(0) \exp \left[\int_0^t \left[m'(0) - \frac{1}{2}(\sigma'(0))^2 \right] ds + \int_0^t \sigma'(0) dB(s) \right] \quad (13)$$

Hence the largest Lyapunov exponent reads

$$\lambda = \lim_{t \rightarrow \infty} \frac{1}{t} \ln H^{1/2}(t) = \frac{1}{2} \left[m(0) - \frac{1}{2}(\sigma(0))^2 \right] = \frac{1}{2} \left[\frac{D\beta^2}{2b} + a - 1 \right]. \quad (14)$$

According to the Oseledec multiplicative ergodic theorem, the necessary and sufficient condition of asymptotic stability with probability 1, of the trivial solution of the system (9), is that the largest Lyapunov exponent λ is negative. We note that if $a < 1 - \frac{D\beta^2}{2b}$, then $\lambda < 0$. This completes the proof. ■

We note that the result of Theorem 1 can only judge the local stability of the trivial solution of the system. With regard to realistic meaning of the variables, it is obvious that $H = 0$ is meaningless. We will therefore assume that $a \geq 1 - \frac{D\beta^2}{2b}$, and discuss whether there are any stable states under this assumption.

2.2. Analysis on the boundaries

As in [Zhu, 2006; Zhu & Huang, 1999], we can classify the boundaries of equation (9) to distinguish the global stability of the trivial solution of the system. The global asymptotic stability, in probability, of the one-dimension diffusion process will be determined by the behaviors of the boundaries of the process: the left boundary $H = 0$ and the right boundary $H \rightarrow \infty$.

Case 1. The boundary $H = 0$. The drift coefficient $m(H)$ and the diffusion coefficient $\sigma(H)$ converge to the following expressions asymptotically when $H \rightarrow 0^+$:

$$m(H) = \left(\frac{D\beta^2}{b} + a - 1 \right) H + o(H) \quad (15a)$$

$$\sigma(H) = \frac{D\beta^2}{b} H^2. \quad (15b)$$

According to the classification of singular boundaries, the left boundary of Eq. (9) belongs to the first kind of singular boundary. Accordingly, the diffusion exponent α_l , the drift exponent β_l and the character value c_l , can be calculated following [Zhu, 2006; Zhu & Huang, 1999]:

$$\alpha_l = 2, \quad \beta_l = 1, \quad c_l = \lim_{H \rightarrow 0^+} \frac{2m(H)H^{\alpha_l - \beta_l}}{\sigma^2(H)} = \frac{2(a-1)b}{D\beta^2} + 2. \quad (16)$$

Thus we obtain the following table

Table 1.

value c_l	$c_l > 1$	$c_l = 1$	$c_l < 1$
parameter a	$a > 1 - \frac{D\beta^2}{2b}$	$a = 1 - \frac{D\beta^2}{2b}$	$a < 1 - \frac{D\beta^2}{2b}$
boundary $H = 0$	exclusive nature	attractive nature threshold	attractive nature

When $a < 1 - \frac{D\beta^2}{2b}$, i.e., the left boundary $H = 0$ is attractive in nature, this implies that as the time increases, $H(t)$ will converge to $H = 0$ asymptotically. This provides another proof of Theorem 1.

Meanwhile, we find that when $a > 1 - \frac{D\beta^2}{2b}$, i.e., the left boundary $H = 0$ is exclusive in nature, this implies that as the time increases, $H(t)$ will move away from $H = 0$ asymptotically; we will investigate this situation later.

Case 2. The boundary $H = +\infty$ ($H \rightarrow +\infty$). The drift coefficient $m(H)$ and the diffusion coefficient $\sigma(H)$ converge to the following expressions asymptotically when $H \rightarrow +\infty$:

$$m(H) = -\frac{3}{2}(a+1)H^2 + o(H^2) \quad (17a)$$

$$\sigma^2(H) = \frac{D\beta^2 H^2}{b}. \quad (17b)$$

According to the classification of singular boundaries, the right boundary of Eq. (9) belongs to the second kind of singular boundary. Accordingly, the diffusion exponent α_r , the drift exponent β_r and the character value c_r can be calculated following [Zhu, 2006; Zhu & Huang, 1999]:

$$\alpha_r = 2, \quad \beta_r = 2, \quad c_r = -\lim_{H \rightarrow +\infty} \frac{2m(H)H^{\alpha_r - \beta_r}}{\sigma^2(H)} = \frac{3(a+1)b}{D\beta^2}. \quad (18)$$

Therefore, when $\beta_r > \alpha_r - 1$ and $m(-\infty) > 0$, $H = +\infty$ belongs to entrance boundary, it implies that with increasing time, $H(t)$ will have a tendency to $H = +\infty$ with some probability. It is clear that value of the probability is rather small, almost zero, for $H(t)$ is an Itô stochastic process governed by Eq. (9) with realistic meaning in the closed business cycle.

From the analysis above, we see that $H(t)$ will move away from $H = 0$ asymptotically under the condition $a > 1 - \frac{D\beta^2}{2b}$, and it is impossible to have a tendency to $H = +\infty$.

In the next section, we will investigate the tendency of $H(t)$, in order to demonstrate the movement of the closed business cycle subjected to the stochastic excitation.

3. Bifurcation

When $a > 1 - \frac{D\beta^2}{2b}$, we will analyze the transition probability density function of the process $H(t)$, $\rho(H, t | H_0, t_0)$, in order to explain the tendency of $H(t)$ with time increasing. From Eq. (9), $H(t)$ is a temporally homogeneous diffusion process, $\rho(H, t | H_0, t_0) = \rho(H, \tau | H_0)$, $\tau = t - t_0$. The transition probability density

function $\rho(H, \tau | H_0)$ is governed by the Fokker-Planck-Kolmogorov equation with the initial condition as the delta function:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial H} [m(H)\rho] + \frac{1}{2} \frac{\partial^2 [\sigma^2(H)\rho]}{\partial H^2} \quad (19a)$$

$$\rho(H, \tau | H_0) = \delta(t - t_0) \quad (19b)$$

Furthermore, as $H(t)$ is a stationary Markovian process, when $\tau \rightarrow \infty$, i.e. time τ is sufficiently large, $\rho(H, \tau | H_0) \rightarrow \rho(H)$, and $\frac{\partial \rho(H)}{\partial t} = 0$, so we can rewrite (19a) as

$$-\frac{\partial}{\partial H} [m(H)\rho] + \frac{1}{2} \frac{\partial^2 [\sigma^2(H)\rho]}{\partial H^2} = 0. \quad (20)$$

The solution of Eq. (20) can be expressed in the form:

$$\rho(H) = cH^\mu \exp \left[\frac{-3(a+1)b}{D\beta^2} H \right] \quad (21)$$

where c is a normalizing parameter, which satisfies $c = \left(\int_0^{+\infty} H^\mu \exp \left[\frac{-3(a+1)b}{D\beta^2} H \right] dH \right)^{-1}$, and $\mu = \frac{2(a-1)b}{D\beta^2}$. Using Eq. (8), we obtain the following expression for the joint probability density function with respect to the generalized displacement and the generalized momentum of the system:

$$\rho(p, q) = c \frac{\sqrt{b}}{\pi} \left[\frac{1}{2} p^2 + \frac{1}{2} bq^2 \right]^\mu \exp \left[\frac{-3(a+1)b}{D\beta^2} \left(\frac{1}{2} p^2 + \frac{1}{2} bq^2 \right) \right]. \quad (22)$$

As a consequence of $a > 1 - \frac{D\beta^2}{2b}$, we obtain $\mu > -1$. We will now study the maximal probability of the values of the stochastic process $H(t)$. There are two cases to discuss:

Case 1. The probability $\rho(H)$ will reach its maximum at $H = 0$ (or $H \rightarrow 0^+$), if $\mu \leq 0$, i.e. $a \leq 1$.

Since $H = 0$ is equivalent to $p = 0$ and $q = 0$, it implies that under this stochastic excitation, the closed business cycle will terminate with probability.

Case 2. The probability $\rho(H)$ will reach its maximum at $H^* = \frac{2(a-1)}{3(a+1)}$, if $\mu \leq 0$, i.e. $a > 1$.

Remark: Let $\frac{d\rho(H)}{dH} = 0$, we can obtain the unique solution $H^* = \frac{2(a-1)}{3(a+1)}$, and we also obtain $\left. \frac{d^2\rho(H)}{dH^2} \right|_{H=H^*} = \frac{-9(a+1)^3}{4(a-1)^2} \mu e^{\mu(\ln H^* - 1)} > 0$. As H^* is the unique stationary point, we can find that the probability function $\rho(H)$ has the unique maximum:

$$\rho(H^*) = c \left(\frac{2(a-1)}{3(a+1)} \right)^{\frac{2(a-1)b}{D\beta^2}} \exp \left[\frac{-2(a-1)b}{D\beta^2} \right]. \quad (23)$$

Thus we find that after a sufficiently long time, the stochastic process $H(t)$ governed by Eq. (9) will obtain values around H^* , with a probability. If $a \leq 1$, $\rho(H)$ will get its maximum at $H = 0$. So we see that $a = 1$ is not only the bifurcation point of the stability at $H = 0$ but also the bifurcation point of stochastic Hopf bifurcation of the system governed by (9), with probability.

Since $H(p, q) = \frac{1}{2} p^2 + \frac{1}{2} bq^2$, we can draw a phase graph of (q, p) , and the density graph of its probability. We obtain a stochastic Hopf bifurcation with probability. We recall that q stands for Y_{t-1} , the productive value of last period, should be positive. We will present some numerical simulations in the next section.

4. Numerical simulations

We demonstrate the results of our analysis by some numerical simulations. For the sake of illustration, we will only present our numerical results for the parameter values $\epsilon = 0.1$, $s = 0.2$, $v = 1.88$, i.e., $a = 1.68$, $b = 0.0802$, $\beta = 0.6$, $D = 0.36$. In Fig. 1, we show graphs of the simulations with respect to equations (9, 19a, 19b, 21, 22) for the function $H(t)$, including the situation $H(t) \leq 0$. We observe that the values of $H(t)$ form an elliptical belt with a high probability. Considering the realistic meaning of the variables, we know that $H(t)$ should be positive. Without loss of generality, we assume $q(t) \geq 1$ (i.e. $y_{t-1} \geq 1$), for $b > 0$, so $H(t) > 0$, and present the results for the function $H(t)$ and its probability function in Fig. 2. We also compare the results of the system, with and without stochastic excitation in Fig. 3.

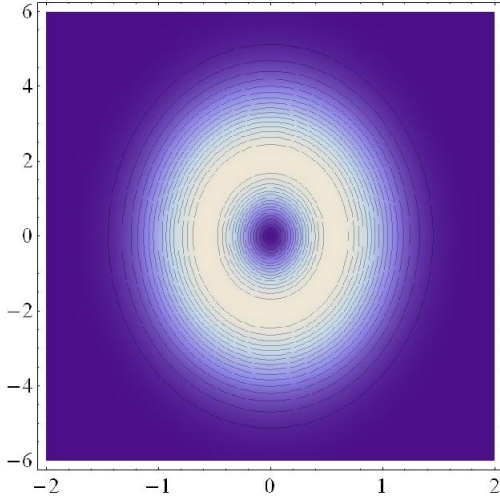
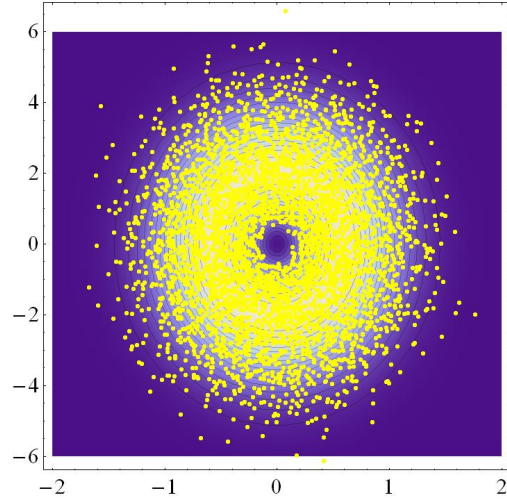
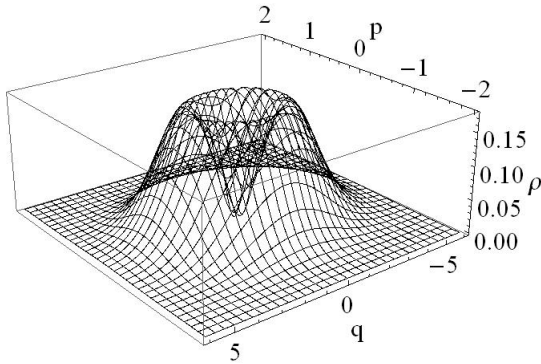
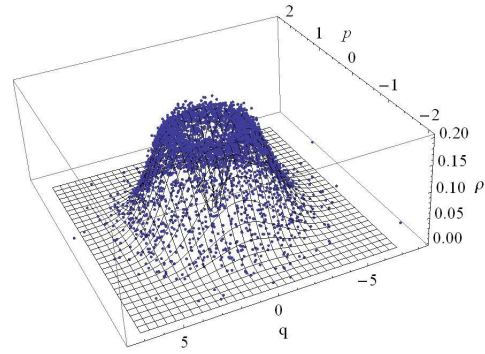
(a) graph of $H(t)$ corresponding to the probability function(b) graph of $H(t)$ corresponding to the Eq. (9)(c) graph of the probability function $\rho(H) = \rho(p, q)$ from Eq. (22)(d) graph of the probability function $\rho(H) = \rho(p, q)$ from Eq. (19a, 19b) compared with the result from Eq. (21).

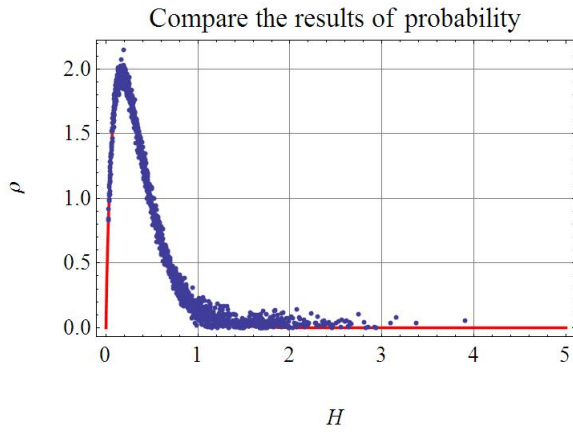
Fig. 1. Graphs of the function $H(t)$ and its probability function $\rho(H)$ or $\rho(p, q)$ with parameters $\epsilon = 0.1$, $s = 0.2$, $v = 1.88$, i.e. $a = 1.68$, $b = 0.0802$, and $\beta = 0.6$, $D = 0.36$.

5. Discussions

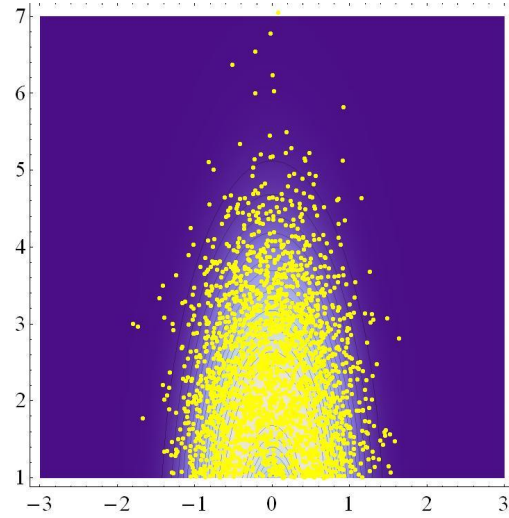
There exist different behaviors of bifurcations between stochastic and deterministic systems. The bifurcations of the stochastic system may occur, with some probability when certain conditions caused by stochastic excitations are satisfied. However, even if these conditions are satisfied, the bifurcations will not necessarily occur. The probability of the bifurcation only reflects the possibility of the bifurcation's occurrence, from which one can learn the complexity of the stochastic business system.

According to the results of our analysis, under the stochastic excitation we assumed, we show that a is the bifurcation parameter for the stability of $H(t) = 0$, and is also the bifurcation parameter for the occurrence of stochastic Hopf bifurcation of the system (9), with probability.

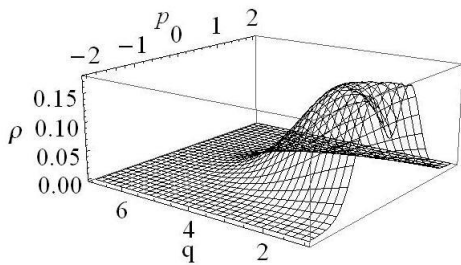
The meaning of the stability of $H(t) = 0$ implies that the movement of the business cycle will stop, with high probability. Whereas the meaning of the occurrence of stochastic Hopf bifurcation implies that there exists a periodic-like behavior of this closed business cycle, with high probability. In fact, most business cycles have their own periods. The main reason why we can't predict the business period correctly is the existence of the stochastic excitations of the system and the fact that the period is not regular as shown in Fig. 3(a); a comparison of the deterministic and the stochastic results is shown in Fig. 3(b). The results



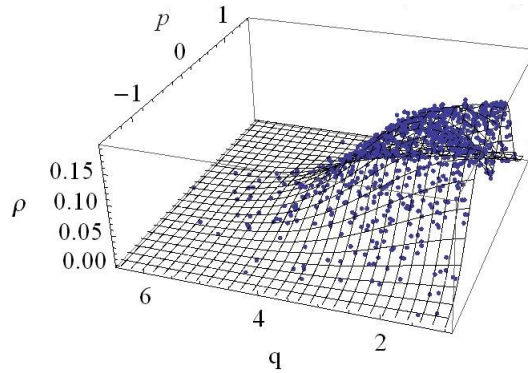
(a) graph of probability function $\rho(H)$ of $H(t)$ from Eq. (19a, 19b) compared with the result from Eq. (21)



(b) graph of $H(t)$ corresponding to the Eq. (9) with $q(t) \geq 1$



(c) graph of the probability function $\rho(H)$ from Eq. (22) with $q(t) \geq 1$



(d) graph of the probability function $\rho(H) = \rho(p, q)$ from Eq. (19a, 19b) with $q(t) \geq 1$, compared with the result from Eq. (21)

Fig. 2. Graphs of the function $H(t)$ and its probability function $\rho(H)$ or $\rho(p, q)$ with parameters $\epsilon = 0.1$, $s = 0.2$, $v = 1.88$, i.e. $a = 1.68$, $b = 0.0802$, and $\beta = 0.6$, $D = 0.36$, with $q(t) \geq 1$.

we have obtained can be used to estimate the movement of business cycle with probability. We plan to consider other types of stochastic excitations in our further research.

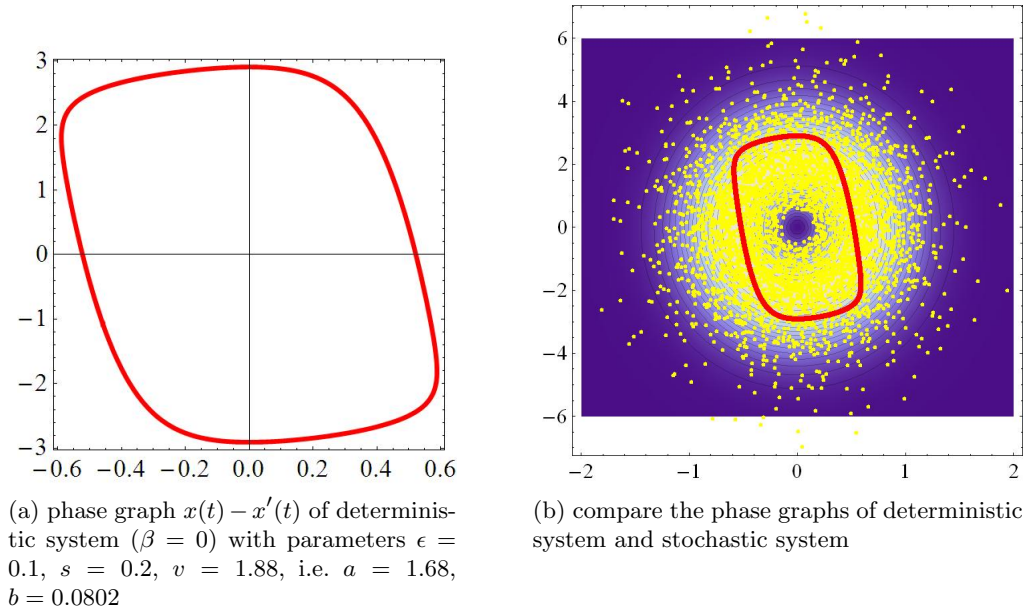


Fig. 3. Phase graphs of deterministic system ($\beta = 0$) and stochastic system ($\beta = 0.6$, $D = 0.36$) with parameters $\epsilon = 0.1$, $s = 0.2$, $v = 1.88$; i.e. $a = 1.68$, $b = 0.0802$.

Acknowledgements

The first author was partially supported by Project of Advanced Study Abroad for the Young and Middle Aged Key Faculty of Colleges in Tianjin, and NNSFC Grant 10732020. The second author was partially supported by NNSFC Grant 10732020. The third author was partially supported by NSF grant DMS0708331, NSFC Grant 10428101, and a Changjiang Scholarship from Jilin University.

References

- Brock, W. A. and Sayers, C. [1988] "Is the business cycles characterized by deterministic chaos?," *Journal of Monetary Economics* **22** 71–80.
- Chian, A. C. L., Borotto, F. A., Rempel, E. L. and Rogers, C. [2005] "Attractor merging crisis in chaotic business cycles," *Chaos Solitons and Fractals* **24** 869–875.
- Grebogi, C., Ott, E., York, J. A. [1983] "Crises, sudden changes in chaotic attractors, and transient chaos," *Physica D* **7** 181–200.
- Helal, M. A. [2002] "Soliton solution of some nonlinear partial differential equations and its applications in fluid mechanics," *Chaos Solitons and Fractals* **13** (9) 1917–1929.
- Hicks, J. R. [1950] *A Contribution to the Theory of the Trade Cycle* (Oxford University Press)
- Huang, D. W., Wang, H. L., Feng, J. F., Zhu, Z. W. [2006] "Hopf bifurcation of the stochastic model on HAB nonlinear stochastic dynamics," *Chaos Solitons and Fractals* **27** (4) 1072–1079.
- Huang, D. W., Wang, H. L., Feng, J. F., Zhu, Z. W. [2008] "Modelling algal densities in harmful algal blooms (HAB) with stochastic dynamics," *Applied Mathematical Modelling* **32** (7) 1318–1326.
- Imoto, A. [2003] "An example of nonlinear endogenous business cycle model: build in the trade union," *Economics Letters* **81** 117–124.
- Puu, T. [1991] "Chaos in business cycles," *Chaos Solitons and Fractals* **1** 457–73.
- Puu, T. [2000] *Attractors, Bifurcations, and Chaos-Nonlinear Phenomena in Economics* (Springer-Verlag).
- Puu, T., Sushko, I. [2004] "A business cycle model with cubic nonlinearity," *Chaos Solitons and Fractals* **19** 597–612.
- Samuelson, P. A. [1939] "Interactions between the multiplier analysis and the principle of acceleration," *Rev. Economic Stat.* **21** 75–78.
- Solomonovich, M. and Apedaile, L.P. [1997] "A dynamical economic model of sustainable agriculture and the ecosphere," *Applied Mathematics and Computation* **84** 221–246.

- Wagg, D. J. [2003] “Periodic and chaotic dynamics in an asymmetric elastoplastic oscillator,” *Chaos Solitons and Fractals* **16** (5) 779–786.
- Zhu, W. Q. [2006] “Nonlinear stochastic dynamics and control in Hamiltonian formulation,” *Applied Mechanics Reviews* **59** 230–248.
- Zhu, W.Q. and Huang, Z. L. [1999] “Stochastic Hopf bifurcation of quasi-nonintegrable-Hamiltonian systems,” *International Journal of Non Linear Mechanics* **34** 437–447.