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ENTROPY OF DYNAMICAL SYSTEMS WITH REPETITION PROPERTY

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Abstract. The repetition property of a dynamical system, a notion introduced in [2], plays an importance role in analyzing spectral properties of ergodic Schrödinger operators. In this paper, entropy of dynamical systems with repetition property is investigated. It is shown that the topological entropy of dynamical systems with the global repetition property is zero. Minimal dynamical systems having both topological repetition property and positive topological entropy are constructed. This provides a class of ergodic Schrödinger operators with potentials generated by positive entropy minimal dynamical systems that, in contrast to common beliefs, admit no eigenvalues.

1. Introduction

By a *topological dynamical system* (TDS for short) (X, T) we mean a compact metric space X together with a continuous self-map $T : X \to X$. We say that (X, T) is *invertible* if T is a homeomorphism. Recall that (X, T) is *minimal* if for each $x \in X$, the orbit $orb(x, T) = \{x, T(x), T^2(x), \cdots\}$ of x is dense in X. A point $x \in X$ is called a *minimal point* if the subsystem $(\overline{orb(x, T)}, T)$ is a minimal system. Throughout of the paper, we use $M(X, T)$ and $M^e(X, T)$, respectively, to denote the set of all T-invariant Borel probability measures and the set of T-invariant ergodic Borel probability measures on X, respectively, use $\mathbb{N}$ to denote the set of natural numbers, and use $\mathbb{Z}_+$ to denote the set of non-negative integers.

In [2], Boshernitzan and Damanik introduced the following notions.

Definition 1.1. Let (X, T) be a TDS with metric d. $x \in X$ is called a *repetition point*, if for every $\varepsilon > 0$ and $r \in \mathbb{N}$, there exists a $q \in \mathbb{N}$ such that $d(T^n x, T^{n+q} x) < \varepsilon$ for all $n = 0, 1, \cdots, rq$. Let $PRP(X, T)$ denote the set of all repetition points of (X, T). (X, T) is said to have the *topological repetition property* (TRP for short) if $PRP(X, T) \neq \emptyset$, said to have the *global repetition property* (GRP for short) if $PRP(X, T) = X$, and said to have the *metric repetition property* (MRP for short) relative to some $\mu \in M(X, T)$ if $\mu(PRP(X, T)) > 0$.

Obviously, we always have (GRP)⇒(MRP) (relative to any T-invariant measure)⇒(TRP). Also, if (X, T) is minimal, then (TRP) implies that $PRP(X, T)$ is a dense G_δ-set of

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X (see [2, 3]). In [2, 3], the authors showed that there are strictly ergodic examples of (X, T) that satisfy (MRP) but not (GRP), and also there are strictly ergodic examples of (X, T) that satisfy (TRP) but not (MRP).

The repetition property plays an important role in the study of spectral properties of Schrödinger operators $H_x : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$:

$$(H_x\phi)(n) = \phi(n+1) + \phi(n-1) + V_x(n)\phi(n), \tag{1.1}$$

where $V_x(n) = f(T^n x)$, (X, T) is an invertible TDS, and $f : X \to \mathbb{R}$ is a continuous function. In particular, it is shown in [2] that if (X, T) is a minimal invertible TDS having (TRP) (resp., (MRP) relative to some $\mu \in M^e(X, T)$), then there exists a residual set $\mathcal{F} \subset C(X)$ such that for each $f \in \mathcal{F}$, H_x has no eigenvalues for x lying in a dense G_δ (resp. full -measure) subset of X.

With the the above result and the general expectation that a positive entropy system (X, T) should lead to positive Lyapunov exponents and localization for the associated Schrödinger operators at least in an almost-everywhere sense, it is a natural question whether a system having (TRP) (resp. (MRP)) always admits zero topological (resp. metric) entropy. The measure-theoretic affirmation to this question was already suggested by an observation made in [2]: if (X, T) is a subshift over a finite set which has (MRP) relative to a $\mu \in M^e(X, T)$, then the metric entropy $h_\mu(T)$ of (X, T, μ) is zero.

By investigating entropy of dynamical systems with repetition property, we give some general answers to the above question. More precisely, our main results are stated as follows.

Theorem 1. *Let (X, T) be a TDS. Then the following holds.*

1. *If $\mu \in M(X, T)$ with $\mu(PRP(X, T)) = 1$, then $h_\mu(T) = 0$.*
2. *If $\mu \in M^e(X, T)$ and (X, T) has (MRP) relative to μ, then $h_\mu(T) = 0$.*
3. *If (X, T) has (GRP), then the topological entropy $h_{top}(T) = 0$.*

Theorem 2. *There exists a minimal (TRP) invertible TDS having positive topological entropy.*

It is well-known that a discrete, one-dimension Schrödinger operator with periodic potential always admits purely absolutely continuous spectrum. However, it is known that this is not necessary the case for a more complicated forcing function. Also, high complexity of the forcing functions does not necessarily implies the existence of pure point spectra of the corresponding Schrödinger operators either. Indeed, it is shown in [4, Theorem 2] that for each finite set $A \subset \mathbb{R}$ with at least two elements, there exists a minimal subshift $\Omega \subset A^{\mathbb{Z}}$ of two-sided sequences with entropy arbitrarily close to the maximal value such that for a dense G_δ set of $\omega = (\omega(n))_{n\in\mathbb{Z}} \in \Omega$, the Schrödinger operator $H_\omega : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$:

$$(H_\omega\phi)(n) = \phi(n+1) + \phi(n-1) + \omega(n)\phi(n)$$

only admits a purely singular continuous spectrum. We note that in terms of the general setting in (1.1), the operators H_ω correspond to a particular $f \in C(\Omega)$ taking only finitely many values.

Theorem 2 above implies that the phenomenon of lacking eigenvalues and absolutely continuous spectrum for discrete Schrödinger operators with complicated forcing functions is in fact generic. When (X,T) is a non-periodic, minimal, invertible TDS, it is shown in [1] that there exists a residual set of functions f in $C(X)$ for which the corresponding Schrödinger operators H_x in (1.1) admit no absolutely continuous spectrum for all x (in this case the absolutely continuous spectrum of H_x is independent of x, see [8, Theorem 6.1]). Combing this with the result in [2] on the absence of eigenvalues, Theorem 2 actually implies that there exists a minimal (TRP) invertible TDS having positive topological entropy such that for a residual set of functions f in $C(X)$ the operator H_x in (1.1) admits purely singular continuous spectrum for x lying in a dense G_δ subset of X.

The rest of the paper is devoted to the proof of these theorems. Theorem 1 will be proved in Section 2 along with some background in ergodic theory and TDS. Theorem 2 will be proved in Section 3 via constructing examples in symbolic systems.

2. Repetition property and entropy

We first recall various notions of entropies. Given a TDS (X,T), denote by $\mathcal{B}_X$ the σ-algebra of Borel subsets of X. A *cover* of X is a family of Borel subsets of X whose union is X. An *open cover* is one that consists of open sets. A *partition* of X is a cover of X by pairwise disjoint sets.

We denote the set of finite partitions, finite covers and finite open covers, of X, respectively, by $\mathcal{P}_X$, $\mathcal{C}_X$ and $\mathcal{C}_X^o$, respectively. Given two covers $\mathcal{U},\mathcal{V}$ of X, $\mathcal{U}$ is said to be finer than $\mathcal{V}$ (denote by $\mathcal{U}\succeq\mathcal{V}$) if each element of $\mathcal{U}$ is contained in some element of $\mathcal{V}$. Let $\mathcal{U}\vee\mathcal{V}=\{U\cap V : U\in\mathcal{U}, V\in\mathcal{V}\}$. It is clear that $\mathcal{U}\vee\mathcal{V}\succeq\mathcal{U}$ and $\mathcal{U}\vee\mathcal{V}\succeq\mathcal{V}$. Given non-negative integers M,N with $M\le N$ and $\mathcal{U}\in\mathcal{C}_X$, we use $\mathcal{U}_M^N$ to denote $\bigvee_{n=M}^N T^{-n}\mathcal{U}$.

For $\mathcal{U}\in\mathcal{C}_X$, we define $N(\mathcal{U})$ as the minimum among the cardinalities of the subcovers of $\mathcal{U}$. Then the *topological entropy of* $\mathcal{U}$ with respect to T is defined by

$$h_{\rm top}(T,\mathcal{U})=\lim_{N\to\infty}\frac{1}{N}\log N(\mathcal{U}_0^{N-1}),$$

which is known to be equal to $\inf\limits_{N\in\mathbb{N}}\frac{1}{N}\log N(\mathcal{U}_0^{N-1})$.

The *topological entropy of* (X,T) is defined by

$$h_{\rm top}(T)=\sup_{\mathcal{U}\in\mathcal{C}_X^o}h_{\rm top}(T,\mathcal{U}),$$

which we sometimes denote as $h_{\rm top}(X,T)$.

For any given $\alpha\in\mathcal{P}_X$ and $\mu\in M(X,T)$, let

$$H_\mu(\alpha)=\sum_{A\in\alpha}-\mu(A)\log\mu(A).$$

The *metric entropy relative to* α is defined by

$$h_\mu(T,\alpha) = \lim_{n\to+\infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}),$$

and the *metric entropy of* μ is defined by

$$h_\mu(T) = \sup_{\alpha\in\mathcal{P}_X} h_\mu(T,\alpha).$$

It is well-known that for any $\mu \in M(X,T)$, one can partition X into sets of zero measure boundaries and arbitrarily small diameters. This yields the following result.

Lemma 2.1. *For any* $\mu \in M(X,T)$*,*

$$h_\mu(T) = \sup\{h_\mu(T,\alpha) : \alpha \in \mathcal{P}_X \text{ with } \mu(\partial A) = 0 \text{ for each } A \in \alpha\}.$$

Let $\mu \in M(X,T)$ and $\alpha = \{A_1, A_2, \cdots, A_k\} \in P_X$ with $\mu(\partial A_i) = 0$, $i = 1, 2, \cdots, k$. Then for any $\epsilon > 0$ there exists open cover $\mathcal{V} = \{V_0, V_1, \cdots, V_k\}$ of X such that $\mu(V_0) \leq \epsilon$ and $V_i \subseteq A_i$, $i = 1, 2, \cdots, k$ (some of the V_i's can be empty sets). For instance, $\mathcal{V}$ may consist of the interiors of the sets $A_i \in \alpha$, $i = 1, 2, \cdots, k$, and V_0 can be taken as an open set of small measure containing the union of all boundaries of A_i, $i = 1, 2, \cdots$. A general notion of the same flavor was introduced in [5] as follows.

Definition 2.2. Let $\mu \in M(X,T)$, $\alpha = \{A_1, A_2, \cdots, A_k\} \in \mathcal{P}_X$, and $\epsilon > 0$. An open cover $\mathcal{V}$ of X is said to be *ϵ-inscribed in α with respect to μ* if $\mathcal{V} = \{V_0, V_1, \cdots, V_k\}$, where $\mu(V_0) \leq \epsilon$ and $V_i \subseteq A_i$, $i = 1, 2, \cdots, k$.

Using this definition, if $\alpha \in \mathcal{P}_X$ consists of sets whose boundaries are of μ-measure zero, then there exist open covers that are ϵ-inscribed in α with respect to μ for every $\epsilon > 0$.

Let $\mathcal{V} \in \mathcal{C}_X$. For $n \in \mathbb{N}$ and $x \in X$, we denote by $\mathcal{V}_0^{n-1}(x)$ the union of all elements of the cover $\mathcal{V}_0^{n-1}$ containing x. We also denote by $R^n_{\mathcal{V}}(x)$ the *first return time* of x to $\mathcal{V}_0^{n-1}(x)$, i.e.,

$$R^n_{\mathcal{V}}(x) = \min\{i > 0 : T^i(x) \in \mathcal{V}_0^{n-1}(x)\}.$$

The following result is Theorem 2 in [5].

Lemma 2.3. *Let* (X,T) *be a TDS,* $\mu \in M^e(X,T)$*, and* $0 < \varepsilon < 1$*. If* $\alpha \in \mathcal{P}_X$ *is of cardinality* k *and* $\mathcal{V}$ *is an open cover of* X *that is* ε*-inscribed in* α *with respect to* μ*, then for* μ*-a.e.*$x \in X$*,*

$$\liminf_{n\to+\infty} \frac{1}{n} \log R^n_{\mathcal{V}}(x) \geq h_\mu(\alpha, T) - H(\varepsilon) - \varepsilon \log k,$$

where $H(\varepsilon) = -\varepsilon\log\varepsilon - (1-\varepsilon)\log(1-\varepsilon)$*.*

We are now ready to prove Theorem 1.

Proof of Theorem 1. (1). Let $\mu \in M(X,T)$ with $\mu(PRP(X,T)) = 1$. Consider the ergodic decomposition $\mu = \int_{M^e(X,T)} \theta dm(\theta)$. Then $h_\mu(T) = \int_{M^e(X,T)} h_\theta(T) dm(\theta)$. Since

$$\int_{M^e(X,T)} \theta(PRP(X,T)) dm(\theta) = \mu(PRP(X,T)) = 1,$$

we have that for m-a.e. $\theta \in M^e(X,T)$, $\theta(PRP(X,T)) = 1$. Thus, to show $h_\mu(T) = 0$, it is sufficient to show that if $\theta \in M^e(X,T)$ with $\theta(PRP(X,T)) = 1$ then $h_\theta(T) = 0$. By Lemma 2.1, this reduces to show that if $\alpha \in \mathcal{P}_X$ with $\mu(\partial A) = 0$ for all $A \in \alpha$, then $h_\theta(T,\alpha) = 0$.

Let $\alpha \in \mathcal{P}_X$ be of cardinality k such that $\mu(\partial A) = 0$ for all $A \in \alpha$. For any $\epsilon > 0$, there exists an open cover $\mathcal{V}$ of X such that $\mathcal{V}$ is an ϵ-inscribed cover in α with respect to θ. Let $\gamma > 0$ be the Lebesgue number of $\mathcal{V}$. By Lemma 2.3, we can find $X_0 \subseteq X$ with $\theta(X_0) = 1$ such that for each $x \in X_0$,

$$h_\theta(T,\alpha) - H(\epsilon) - \epsilon \log k \le \liminf_{n\to+\infty} \frac{1}{n} \log R^n_{\mathcal{V}}(x). \tag{2.1}$$

Since $\theta(X_0 \cap PRP(X,T)) = 1$, $X_0 \cap PRP(X,T) \ne \emptyset$. Let $z \in X_0 \cap PRP(X,T)$ be chosen. Then for each $r \in \mathbb{N}$, we can find $q(r) \in \mathbb{N}$ such that

$$d(T^{n+q(r)}(z), T^n(z)) < \gamma, \ \ n = 0, 1, \cdots, rq(r).$$

Thus, for each $r \in \mathbb{N}$ and $n \in \{0, 1, \cdots, rq(r)\}$, there exists $V(r,n) \in \mathcal{V}$ such that $T^{n+q(r)}(z)$ and $T^n(z)$ are both in $V(r,n)$. Hence $T^{q(r)}(z) \in \mathcal{V}_0^{rq(r)-1}(z)$. This shows that

$$R^{rq(r)}_{\mathcal{V}}(z) \le q(r), \ \ r \in \mathbb{N}. \tag{2.2}$$

Combining (2.2) with (2.1), we have

$$\begin{aligned} h_\theta(T,\alpha) - H(\epsilon) - \epsilon \log k &\le \liminf_{n\to+\infty} \frac{1}{n} \log R^n_{\mathcal{V}}(z) \le \liminf_{r\to+\infty} \frac{1}{rq(r)} \log R^{rq(r)}_{\mathcal{V}}(z) \\ &\le \liminf_{r\to+\infty} \frac{1}{rq(r)} \log q(r) = 0, \end{aligned}$$

i.e., $h_\theta(T,\alpha) \le H(\epsilon) + \epsilon \log k$. By passing limit $\epsilon \to 0$, we have $h_\theta(T,\alpha) = 0$.

(2). Let $\mu \in M^e(X,T)$. Since (X,T) has (MRP) relative to μ and $T(RPR(X,T)) \subseteq RPR(X,T)$, $\mu(PRP(X,T)) = 1$. It follows from (1) that $h_\mu(T) = 0$.

(3). By the Variational Principle of Entropy, (3) also follows from (1). □

3. A class of (TRP) TDSs with positive entropy

In this section, we will prove Theorem 2 by constructing a class of (TRP) TDSs with positive entropy.

Let $\Lambda^{\mathbb{Z}_+}$ be the space of one-sided sequences in $\Lambda = \{0,1\}$ endowed with the product topology. Then $\Lambda^{\mathbb{Z}_+}$ is metrizable, and a compatible metric can be chosen as

$$\rho(x,y) = \frac{1}{2^{\min\{i\in\mathbb{Z}_+ : x_i \ne y_i\}}},$$

for $x = (x_i)_{i\in\mathbb{Z}_+}$, $y = (y_i)_{i\in\mathbb{Z}_+} \in \Lambda^{\mathbb{Z}_+}$. With the shift map $\sigma : \Lambda^{\mathbb{Z}_+} \to \Lambda^{\mathbb{Z}_+}$: $\sigma(x)_i = x_{i+1}$, $i \in \mathbb{Z}_+$, $(\Lambda^{\mathbb{Z}_+}, \sigma)$ becomes a TDS. We refer a non-empty, closed, invariant subset of $\Lambda^{\mathbb{Z}_+}$ as a *subshift* of $(\Lambda^{\mathbb{Z}_+}, \sigma)$.

For any two words $A=(a_0,\cdots,a_{k-1})\in\Lambda^k$, $B=(b_0,\cdots,b_{l-1})\in\Lambda^l$, we define the multiplication AB as the extended word $(a_0,\cdots,a_{k-1},b_0,\cdots,b_{l-1})\in\Lambda^{k+l}$. For a single word $A=(a_0,\cdots,a_{k-1})\in\Lambda^k$, we use $|A|=k$ to denote the length of the word A, A^k to denote k-times self multiplication of A, and define $A[i,j]$ as the word $(a_i,a_{i+1},\cdots,a_j)$ for all $0\le i\le j\le k-1$. Similarly, for any $x=(x_0,x_2,\cdots,x_n,\cdots)\in\Lambda^{\mathbb{Z}_+}$ and $0\le i\le j<+\infty$, we define $x[i,j]=(x_i,x_{i+1},\cdots,x_j)$.

Lemma 3.1. *Let $x=(x_0,x_1,\cdots)\in\Lambda^{\mathbb{Z}_+}$. Then $x\in PRP(\Lambda^{\mathbb{Z}_+},\sigma)$ iff for each $k\in\mathbb{N}$ there exists $l_k\in\mathbb{N}$ such that $x[0,kl_k-1]=x[0,l_k-1]^k$.*

Proof. This follows directly from Definition 1.1. □

The above lemma will be useful in computing the Hausdorff dimension of $PRP(\Lambda^{\mathbb{Z}_+},\sigma)$. We first recall the definition of Hausdorff dimension. Let X be a metric space with metric d. For each $t\ge 0$, $\delta>0$, and $A\subset X$, define

$$H_d^{t,\delta}(A)=\inf\{\sum_{i=1}^{+\infty}\operatorname{diam}(U_i)^t\},$$

where the infimum is taken over all countable covers $\{U_i : i=1,2,\cdots\}$ of A of diameter not exceeding δ. Since $H_d^{t,\delta}(A)$ increases as δ decreases, we can define

$$H_d^t(A)=\lim_{\delta\to 0}H_d^{t,\delta}(A)=\sup_{\delta>0}H_d^{t,\delta}(A).$$

Since the function $t\to H_d^{t,\delta}(A)$ is non-increasing, so is the function $t\to H_d^t(A)$. Moreover, if $0<s<t$, then for every $\delta>0$,

$$H_d^{s,\delta}(A)\ge\delta^{s-t}H_d^{t,\delta}(A)$$

which implies that if $H_d^t(A)>0$, then $H_d^s(A)=+\infty$. Thus there is a unique value $H_d(A)\in[0,+\infty]$, which is called the *Hausdorff dimension* of A with respect to the metric d on X, such that

$$H_d^t(A)=\begin{cases}+\infty, & \text{if } 0\le t<H_d(A),\\ 0, & \text{if } H_d(A)<t<\infty.\end{cases}$$

Theorem 3.2. $H_\rho(PRP(\Lambda^{\mathbb{Z}_+},\sigma))=0$.

Proof. Let

$$U_k(w)=\{x\in\Lambda^{\mathbb{Z}_+} : x[im,im+m-1]=w \text{ for } i=0,1,\cdots,k\},\ w\in\Lambda^m,\ k\in\mathbb{N}.$$

It is clear that $\operatorname{diam}(U_k(w))\le 2^{-km}$. By Lemma 3.1, for any $k\in\mathbb{N}$ and $n\in\mathbb{N}$ we have

$$RPR(\Lambda^{\mathbb{Z}_+},\sigma)\subseteq\bigcup_{m\ge n}\bigcup_{w\in\Lambda^m}U_k(w). \tag{3.1}$$

For given $t > 0$, we choose $k_t \in \mathbb{N}$ such that $k_t t > 1$. For any $n \in \mathbb{N}$, by (3.1) we have

$$H_\rho^{t, \frac{1}{2^{k_t n}}}(RPR(\Lambda^{\mathbb{Z}_+}, \sigma)) \le \sum_{m=n}^{\infty} \sum_{w \in \Lambda^m} \operatorname{diam}(U_{k_t}(w))^t$$
$$\le \sum_{m=n}^{\infty} 2^m 2^{-k_t t m} = \frac{2^{-k_t t n}}{1 - 2^{-(k_t t - 1)}}.$$

Taking limit $n \to +\infty$ in the above yields $H_\rho^t(RPR(\Lambda^{\mathbb{Z}_+}, \sigma)) = 0$. Since this is true for any $t > 0$, we have $H_\rho(PRP(\Lambda^{\mathbb{Z}_+}, \sigma)) = 0$. □

For each $k, \ell \in \mathbb{N}$ with $k \le \ell$ and $A \in \Lambda^\ell$, we let

$$N_k(A) = \#\{A[i, i+k-1] : 0 \le i \le \ell - k + 1\},$$

and for each $x \in \Lambda^{\mathbb{Z}_+}$ and $k \in \mathbb{N}$, we let

$$N_k(x) = \#\{x[i, i+k-1] : i \ge 0\}.$$

It is clear that $1 \le N_k(x) \le 2^k$ and the sequence $\{\log N_k(x)\}$ is sub-additive. Thus the *entropy* $h(x)$ of x is well-defined and equal to

$$\lim_{k \to +\infty} \frac{1}{k} \log N_k(x) = \inf_{k \ge 1} \frac{1}{k} \log N_k(x).$$

Let $X = \overline{orb(x, \sigma)}$. Then (X, σ) is a TDS and

$$h_{\text{top}}(X, \sigma) = h(x) \in [0, \log 2]. \tag{3.2}$$

It is well-known that there exists $x^0 \in \Lambda^{\mathbb{Z}_+}$ such that $h(x^0) = \log 2$.

Lemma 3.3. *Let $x \in \Lambda^{\mathbb{Z}_+}$, $A \in \Lambda^r$ and $0 < \tau < 1$. Then there exists $\ell \in \mathbb{N}$ such that $\frac{1}{r} \log N_r(y[0, (\ell+2)r - 1]) \ge \tau h(x)$ and $h(y) \ge \tau h(x)$, where $y \in \Lambda^{\mathbb{Z}_+}$ satisfies $y[ir(\ell+2), (i+1)r(\ell+2) - 1] = Ax[ir\ell, (i+1)r\ell - 1]A$ for each $i \in \mathbb{Z}_+$.*

Proof. Since $\frac{1}{r} \log N_r(x) \ge h(x)$ and $lim_{t \to +\infty} N_r(x[0, t]) = N_r(x)$, there is $t_0 \in \mathbb{N}$ such that $\frac{1}{r} \log N_r(x[0, t-1]) \ge \tau h(x)$ for all $t \ge t_0$.

Take $\ell \in \mathbb{N}$ with $\ell \ge \max\{t_0, \frac{2\tau}{1-\tau}\}$. We let $y \in \Lambda^{\mathbb{Z}_+}$ be such that

$$y[ir(\ell+2), (i+1)r(\ell+2) - 1] = Ax[ir\ell, (i+1)r\ell - 1]A$$

for all $i \in \mathbb{Z}_+$. Clearly,

$$\frac{1}{r} \log N_r(y[0, (\ell+2)r - 1]) = \frac{1}{r} \log N_r(Ax[0, \ell r - 1]A)$$
$$\ge \frac{1}{r} \log N_r(x[0, \ell r - 1]) \ge \tau h(x).$$

It remains to show that $h(y) \ge \tau h(x)$. For $z \in \Lambda^{\mathbb{Z}_+}$, $0 \le i < k$ and $m \in \mathbb{N}$, let

$$N_m(z; k, i) = \#\{z[i + sk, i + sk + m - 1] : s \in \mathbb{Z}_+\}.$$

Then

$$N_m(z) \le \sum_{i=0}^{k-1} N_m(z;k,i). \tag{3.3}$$

For $0 \le i \le \ell r - 1$ and $q \in \mathbb{N}$,

$$\begin{aligned} N_{q\ell r}(x;\ell r, i) &= \#\{x[i+s\ell r, i+s\ell r+q\ell r-1] : s \in \mathbb{Z}_+\} \\ &= \#\{y[i+r+s(\ell+2)r, i+r+s(\ell+2)r+q(\ell+2)r-1] : s \in \mathbb{Z}_+\} \\ &= N_{q(\ell+2)r}(y;(\ell+2)r, i+r) \\ &\le N_{q(\ell+2)r}(y), \end{aligned}$$

i.e.,

$$N_{q\ell r}(x;\ell r, i) \le N_{q(\ell+2)r}(y). \tag{3.4}$$

Finally, by (3.3), (3.4), and the fact that $\ell \ge \frac{2\tau}{1-\tau}$, we have

$$\begin{aligned} h(x) &= \lim_{q\to+\infty} \frac{1}{q\ell r} \log N_{q\ell r}(x) \le \limsup_{q\to+\infty} \frac{1}{q\ell r} \log\left(\sum_{i=0}^{\ell r-1} N_{q\ell r}(x;\ell r,i)\right) \\ &\le \limsup_{q\to+\infty} \frac{1}{q\ell r} \log(\ell r N_{q(\ell+2)r}(y)) = \limsup_{q\to+\infty} \frac{1}{q\ell r} \log N_{q(\ell+2)r}(y) \\ &= \frac{\ell+2}{\ell} \limsup_{q\to+\infty} \frac{1}{q(\ell+2)r} \log N_{q(\ell+2)r}(y) = \frac{\ell+2}{\ell} h(y), \end{aligned}$$

i.e., $h(y) \ge \frac{\ell}{\ell+2} h(x) \ge \tau h(x)$. □

If $u \in \Lambda^k$, $v \in \Lambda^n$ and $x \in \Lambda^{\mathbb{Z}_+}$, u is said to *occur at* i *in* v $(i < n)$ if $v[i, i+|u|-1] = u$; u is said to *occur at* i *in* x $(i \in \mathbb{Z}_+)$ if $x[i, i+|u|-1] = u$. We define the *occurring time* u in x by

$$N(u,x) = \{i \in \mathbb{Z}_+ : u \text{ occurs at } i \text{ in } x\}.$$

Recall that an infinite subset $S = \{s_1 < s_2 < \cdots\}$ of $\mathbb{Z}_+$ is *syndetic* if there exists $\ell \in \mathbb{N}$ such that $s_{i+1} - s_i \le \ell$ for each $i \in \mathbb{N}$. For a TDS (X,T) and $x \in X$, x is a minimal point of (X,T) iff for each neighborhood U of x, $N(x,U) = \{n \in \mathbb{Z}_+ : T^n(x) \in U\}$ is syndetic [7]. Particularly, we have the following result.

Lemma 3.4. *$x \in \Lambda^{\mathbb{Z}_+}$ is a minimal point of $(\Lambda^{\mathbb{Z}_+}, \sigma)$ iff there exists $1 \le t_1 < t_2 < \cdots$ such that $N(x[0, t_n - 1], x)$ is syndetic for each $n \in \mathbb{N}$.*

Theorem 3.5. *For each $0 < h < \log 2$, there exists $x \in \Lambda^{\mathbb{Z}_+}$ such that (X,σ) is a minimal (TRP) TDS with $h_{top}(X,\sigma) \ge h$, where $X = \overline{orb(x,\sigma)}$.*

Proof. Let $0 < h < \log 2$ be given and pick a $p \in \mathbb{N}$ such that $(1 - \frac{1}{p-1}) \log 2 \ge h$. We will construct $x \in \Lambda^{\mathbb{Z}_+}$ via induction. To do so, we will construct a sequence of finite words A_i inductively so that A_{i+1} begins with A_i and x is the limit of A_i.

Initially, take $x^0 \in \Lambda^{\mathbb{Z}_+}$ with $h(x^0) = \log 2$. Let $A_1 = 1$, $r_1 = |A_1|$ and $\tau_1 = \frac{p-1}{p}$. Then by Lemma 3.3, there exists $\ell_1 \in \mathbb{N}$ such that

$$\frac{1}{r_1} \log N_{r_1}(x^1[0, r_1(\ell_1+2)-1]) \geq \tau_1 h(x^0)$$

and $h(x^1) \geq \tau_1 h(x^0)$, where $x^1 \in \Lambda^{\mathbb{Z}_+}$ satisfies

$$x^1[ir_1(\ell_1+2), (i+1)r_1(\ell_1+2)-1] = A_1 x^0[ir_1\ell_1, (i+1)r_1\ell_1 - 1]A_1$$

for each $i \in \mathbb{Z}_+$.

Let $A_2 = x^1[0, r_1(\ell_1+2)-1]$, $r_2 = 2|A_1|$ and $\tau_2 = \frac{p^2-1}{p^2}$. Then by Lemma 3.3, there exists $\ell_2 \in \mathbb{N}$ such that

$$\frac{1}{r_2} \log N_{r_2}(x^2[0, r_2(\ell_2+2)-1]) \geq \tau_2 h(x^1)$$

and $h(x^2) \geq \tau_2 h(x^1)$, where $x^2 \in \Lambda^{\mathbb{Z}_+}$ satisfies

$$x^2[ir_2(\ell_2+2), (i+1)r_2(\ell_2+2)-1] = A_2A_2 x^1[ir_2\ell_2, (i+1)r_2\ell_2 - 1]A_2A_2$$

for each $i \in \mathbb{Z}_+$.

Suppose that A_k, r_k, ℓ_k and x^k have been constructed for some $k \geq 2$. Let

$$A_{k+1} = x^k[0, r_k(\ell_k+2)-1],$$

$r_{k+1} = (k+1)|A_{k+1}|$ and $\tau_{k+1} = \frac{p^{k+1}-1}{p^{k+1}}$. Then by Lemma 3.3, there exists $\ell_{k+1} \in \mathbb{N}$ such that

$$\frac{1}{r_{k+1}} \log N_{r_{k+1}}(x^{k+1}[0, r_{k+1}(\ell_{k+1}+2)-1]) \geq \tau_{k+1} h(x^k)$$

and $h(x^{k+1}) \geq \tau_{k+1} h(x^k)$, where $x^{k+1} \in \Lambda^{\mathbb{Z}_+}$ satisfies

$$\begin{aligned} & x^{k+1}[ir_{k+1}(\ell_{k+1}+2), (i+1)r_{k+1}(\ell_{k+1}+2)-1] \\ &= \underbrace{A_{k+1}\cdots A_{k+1}}_{k+1 \text{ times}} x^k[ir_{k+1}\ell_{k+1}, (i+1)r_{k+1}\ell_{k+1}-1] \underbrace{A_{k+1}\cdots A_{k+1}}_{k+1 \text{ times}} \end{aligned}$$

for each $i \in \mathbb{Z}_+$.

By the above construction,

$$A_{k+1} = x^k[0, (\ell_k+2)r_k - 1] = \underbrace{A_k\cdots A_k}_{k \text{ times}} x^{k-1}[0, r_k\ell_k - 1] \underbrace{A_k\cdots A_k}_{k \text{ times}} \text{ and} \tag{3.5}$$

$$\frac{1}{\tau_k} \log N_{r_k}(A_{k+1}) \geq \tau_k h(x^{k-1}) \geq \tau_k\tau_{k-1} h(x^{k-2} \geq \cdots \geq (\prod_{i=1}^{k} \tau_i) h(x^0) \tag{3.6}$$

for each $k \in \mathbb{N}$. Let $x = \lim_{k\to+\infty} A_k$ and $X = \overline{orb(x,\sigma)}$.

Claim. For each $k \in \mathbb{N}$, $N(A_k, x)$ is syndetic.

For $k_1, k_2 \in \mathbb{N}$ with $k_2 \leq k_1$, $u \in \Lambda^n$ with $k_1 | n$, and $y \in \Lambda^{\mathbb{Z}_+}$, we define

$$\mathcal{A}(u; k_1, k_2) = \{u[ik_1, ik_1 + k_2 - 1] : 0 \leq i < \frac{n}{k_1}\} \text{ and}$$
$$\mathcal{A}(y; k_1, k_2) = \{y[ik_1, ik_1 + k_2 - 1] : i \geq 0\}.$$

We wish to show that

$$\mathcal{A}(x; |A_{k+1}|, |A_k|) = A_k, \tag{3.7}$$

i.e., A_k occurs at $i|A_{k+1}|$ in x for each $i \geq 0$. Clearly, (3.7) implies that $N(A_k, x)$ is syndetic. By (3.5), $\mathcal{A}(A_{k+1}; |A_{k+1}|, |A_k|) = \{A_k\}$. Since

$$x^k[i|A_{k+1}|, (i+1)|A_{k+1}| - 1] = \underbrace{A_k \cdots A_k}_{k \text{ times}} x^{k-1}[ir_k\ell_k, (i+1)r_k\ell_k - 1] \underbrace{A_k \cdots A_k}_{k \text{ times}}$$

for each $i \in \mathbb{Z}_+$, we have $\mathcal{A}(x^k; |A_{k+1}|, |A_k|) = \{A_k\}$. Next we inductively show that $\mathcal{A}(A_{j+1}; |A_{k+1}|, |A_k|) = \{A_k\}$ and $\mathcal{A}(x^j; |A_{k+1}|, |A_k|) = \{A_k\}$ for each $j \geq k$.

Suppose that $\mathcal{A}(A_{j+1}; |A_{k+1}|, |A_k|) = \{A_k\}$ and $\mathcal{A}(x^j; |A_{k+1}|, |A_k|) = \{A_k\}$ for each $k \leq j \leq t$. For $j = t+1$, since

$$\begin{aligned} &x^j[ir_{t+1}(\ell_{t+1}+2), (i+1)r_{t+1}(\ell_{t+1}+2) - 1] \\ =& \underbrace{A_{t+1} \cdots A_{t+1}}_{t+1 \text{ times}} x^t[ir_{t+1}\ell_{t+1}, (i+1)r_{t+1}\ell_{t+1} - 1] \underbrace{A_{t+1} \cdots A_{t+1}}_{t+1 \text{ times}} \end{aligned}$$

for each $i \in \mathbb{Z}_+$, we have

$$\begin{aligned} &\mathcal{A}(x^j; |A_{k+1}|, |A_k|) \\ =& \bigcup_{i \geq 0} \mathcal{A}(x^j[ir_{t+1}(\ell_{t+1}+2), (i+1)r_{t+1}(\ell_{t+1}+2) - 1]; |A_{k+1}|, |A_k|) \\ =& \mathcal{A}(A_{t+1}; |A_{k+1}|, |A_k|) \cup \bigcup_{i \geq 0} \mathcal{A}(x^t[ir_{t+1}\ell_{t+1}, (i+1)r_{t+1}\ell_{t+1} - 1]; |A_{k+1}|, |A_k|) \\ =& \mathcal{A}(A_{t+1}; |A_{k+1}|, |A_k|) \cup \mathcal{A}(x^t; |A_{k+1}|, |A_k|) = \{A_k\}. \end{aligned}$$

Moreover, by (3.5),

$$\mathcal{A}(A_{j+1}; |A_{k+1}|, |A_k|) = \mathcal{A}(A_j; |A_{k+1}|, |A_k|) \cup \mathcal{A}(x_j[0, r_j(\ell_j+2) - 1]; |A_{k+1}|, |A_k|) = \{A_k\}.$$

Now

$$\mathcal{A}(x; |A_{k+1}|, |A_k|) = \bigcup_{j \geq k+1} \mathcal{A}(A_j; |A_{k+1}|, |A_k|) = \{A_k\},$$

i.e., (3.7) holds.

It now follows from the above Claim and Lemma 3.4 that x is a minimal point, i.e., (X, σ) is a minimal TDS. By (3.5), for each $k \in \mathbb{N}$,

$$x[0, k|A_k| - 1] = \underbrace{A_k \cdots A_k}_{k \text{ times}} = x[0, |A_k| - 1]^k.$$

Thus $x \in PRP(\Lambda^{\mathbb{Z}_+}, \sigma)$ by Lemma 3.1. This implies that $x \in PRP(X, \sigma)$.

Finally, using (3.6), we have

$$\begin{aligned} h_{\text{top}}(X,\sigma) = h(x) &= \lim_{k\to+\infty} \frac{1}{r_k} \log N_{r_k}(x) \geq \liminf_{k\to+\infty} \frac{1}{r_k} \log N_{r_k}(A_{k+1}) \\ &\geq \liminf_{k\to+\infty} (\prod_{i=1}^{k} \tau_i) h(x^0) = (\prod_{i=1}^{+\infty} \tau_i) \log 2 = (\prod_{i=1}^{+\infty} (1-\frac{1}{p^i})) \log 2 \\ &\geq (1-\sum_{i=1}^{+\infty} \frac{1}{p^i}) \log 2 = (1-\frac{1}{p-1}) \log 2 \geq h. \end{aligned}$$

This completes the proof. □

Let $X = \overline{PRP(\Lambda^{\mathbb{Z}_+},\sigma)}$. Then by Theorem 3.5, $h_{\text{top}}(X,\sigma) = \log 2$ - the maximal entropy of the system. Since (X,σ) is a subshift of $(\Lambda^{\mathbb{Z}_+},\sigma)$, $h_{\text{top}}(X,\sigma) = H_\rho(X)\log 2$ ([6]). Hence $H_\rho(X) = 1$, though by Theorem 3.2 $H_\rho(PRP(\Lambda^{\mathbb{Z}_+},\sigma)) = 0$.

For a TDS (X,T) with T being surjective, let

$$\widetilde{X} = \{(x_1,x_2,\cdots) : T(x_{i+1}) = x_i, x_i \in X, i \in \mathbb{N}\}$$

be a subspace of the product space $X^{\mathbb{N}} = \Pi_{i=1}^{\infty} X$ and $\widetilde{T} : \widetilde{X} \to \widetilde{X}$ be the shift homeomorphism, i.e., $\widetilde{T}(x_1,x_2,\cdots) = (T(x_1),x_1,x_2,\cdots)$. It is clear that $(\widetilde{X},\widetilde{T})$ is an invertible TDS, called the *inverse limit* of (X,T). For each i, let $\pi_i : \widetilde{X} \to X$ be the natural projection to the i-th coordinate. Then $\pi_i : (\widetilde{X},\widetilde{T}) \to (X,T)$ is a factor map, i.e., $\pi_i \circ \widetilde{T} = T \circ \pi_i$.

Lemma 3.6. *Let (X,T) be a minimal (TRP) TDS and $(\widetilde{X},\widetilde{T})$ be the inverse limit of (X,T). Then $(\widetilde{X},\widetilde{T})$ is a minimal (TRP) invertible TDS.*

Proof. It is well-known that $(\widetilde{X},\widetilde{T})$ is a minimal invertible TDS. For a compatible metric d on X, we define a metric ρ on $\widetilde{X}$ as

$$\rho(\widetilde{x},\widetilde{y}) = \sum_{n=1}^{\infty} \frac{d(x_i,y_i)}{2^i}, \;\; \widetilde{x} = (x_1,x_2,\cdots), \widetilde{y} = (y_1,y_2,\cdots) \in \widetilde{X}.$$

For each i, since π_i is a factor map between two minimal flows, it is semi-open (i.e., if U is a non-empty open subset of $\widetilde{X}$ then $\pi_i(U)$ has non-empty interior in X) by a classical result of Auslander ([9]). Thus if A is a dense G_δ subset of X then $\pi_i^{-1}(A)$ is a dense G_δ subset of $\widetilde{X}$. By Lemma 2 in [3], $PRP(X,T)$ is a dense G_δ subset of (X,T). Hence $W := \bigcap_{i=1}^{\infty} \pi_i^{-1}(PRP(X,T))$ is a dense G_δ subset of $\widetilde{X}$.

Let $\widetilde{x} = (x_1,x_2,\cdots) \in W$. Then $x_i \in PRP(X,T)$ for $i \in \mathbb{N}$. Fix $\epsilon > 0$ and $r \in \mathbb{N}$. There exist $N \in \mathbb{N}$ and $\delta > 0$ such that $\sum_{\ell=N}^{\infty} \frac{\text{diam}(X)}{2^\ell} < \frac{\epsilon}{2}$, and moreover, when $d(x,y) < \delta$, there holds $\sum_{\ell=0}^{N-1} \frac{d(T^{N-\ell}x,T^{N-\ell}y)}{2^\ell} < \frac{\epsilon}{2}$. Since $x_N \in PRP(X,T)$, there exists $q \in \mathbb{N}$ such that $d(T^n x_N, T^{n+q} x_N) < \delta$ for $n = 0,1,\cdots,rq$. Now for each

$n \in \{0, 1, \cdots, rq\}$, we have

$$\rho(\widetilde{T}^n \widetilde{x}, \widetilde{T}^{n+q}\widetilde{x}) = \sum_{\ell=0}^{N-1} \frac{d(T^{N-\ell}T^n x_N, T^{N-\ell}T^{n+q}x_N)}{2^\ell} + \sum_{\ell=N}^{\infty} \frac{d(T^n x_\ell, T^{n+q}x_\ell)}{2^\ell}$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This implies $\widetilde{x} \in PRP(\widetilde{X}, \widetilde{T})$. Thus $(\widetilde{X}, \widetilde{T})$ is a minimal (TRP) invertible TDS. □

Proof of Theorem 2. By Theorem 3.5, there exists a positive entropy minimal (TRP) subshift (X, σ). Since (X, σ) is minimal, σ is surjective. Thus by Lemma 3.6, the inverse limit of (X, σ) is a minimal (TRP) invertible TDS with positive entropy. □

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