ALMOST PERIODICALLY FORCED CIRCLE FLOWS

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Abstract. We study general dynamical and topological behaviors of minimal sets in skew-product circle flows in both continuous and discrete settings, with particular attentions paying to almost periodically forced circle flows. When a circle flow is either discrete in time and unforced (i.e., a circle map) or continuous in time but periodically forced, behaviors of minimal sets are completely characterized by classical theory. The general case involving almost periodic forcing is much more complicated due to the presence of multiple forcing frequencies, the topological complexity of the forcing space, and the possible loss of mean motion property. On one hand, we will show that to some extent behaviors of minimal sets in an almost periodically forced circle flow resemble those of Denjoy sets of circle maps in the sense that they can be almost automorphic, Cantorian, and everywhere non-locally connected. But on the other hand, we will show that almost periodic forcing can lead to significant topological and dynamical complexities on minimal sets which exceed the contents of Denjoy theory. For instance, an almost periodically forced circle flow can be positively transitive and its minimal sets can be Li-Yorke chaotic and non-almost automorphic. As an application of our results, we will give a complete classification of minimal sets for the projective bundle flow of an almost periodic, $sl(2,\mathbb{R})$-valued, continuous or discrete cocycle.

Continuous almost periodically forced circle flows are among the simplest non-monotone, multi-frequency dynamical systems. They can be generated from almost periodically forced nonlinear oscillators through integral manifolds reduction in the damped cases and through Mather theory in the damping-free cases. They also naturally arise in 2D almost periodic Floquet theory as well as in climate models. Discrete almost periodically forced circle flows arise in the discretization of nonlinear oscillators and discrete counterparts of linear Schrödinger equations with almost periodic potentials. They have been widely used as models for studying strange, non-chaotic attractors and intermittency phenomena during the transition from order to chaos. Hence the study of these flows is of fundamental importance to the understanding of multi-frequency-driven dynamical irregularities and complexities in non-monotone dynamical systems.

1. Introduction

Through out the paper, we let $\mathbb{T}=\mathbb{R}$ or $\mathbb{Z}$. Consider a *skew-product circle flow* (*SPCF* for short) $(S^1\times Y,\mathbb{T})=(S^1\times Y,\{\Lambda_t\}_{t\in\mathbb{T}})$ with a compact base (or forcing) flow $(Y,\mathbb{T})=(Y,\{\sigma_t\}_{t\in\mathbb{T}})$, i.e., for each $t\in\mathbb{T}$ the following diagram

$$\begin{array}{ccc} S^1\times Y & \xrightarrow{\Lambda_t} & S^1\times Y \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ Y & \xrightarrow{\sigma_t} & Y \end{array}$$

commutes, where $\pi: S^1\times Y\to Y$ denotes the natural projection. Let $y_0\cdot t=\sigma_t(y_0)$, $\psi(s_0,y_0,t)=\Pi\Lambda_t(s_0,y_0)$, where $\Pi: S^1\times Y\to S^1$ denotes the natural projection. The SPCF can be expressed more explicitly as $\Lambda_t: S^1\times Y\to S^1\times Y$:

$$\Lambda_t(s_0,y_0)=(\psi(s_0,y_0,t),y_0\cdot t),\quad t\in\mathbb{T}. \tag{1.1}$$

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In particular, when $\mathbb{T} = \mathbb{Z}$, the discrete flow Λ_t is generated by the iteration of the skew-product circle map $\Lambda : S^1 \times Y \to S^1 \times Y$:

$$\Lambda(s_0, y_0) = (f(s_0, y_0), T(y_0)), \tag{1.2}$$

where $f(s_0, y_0) = \psi(s_0, y_0, 1)$ and $T(y_0) = y_0 \cdot 1$. Using angular coordinate $\phi_0 \in R^1$ (mod 1), the SPCF Λ_t can be also expressed as

$$\hat{\Lambda}_t(\phi_0, y_0) = (\phi(\phi_0, y_0, t), y_0 \cdot t), \quad t \in \mathbb{T}, \tag{1.3}$$

where $e^{2\pi i \phi(\phi_0, y_0, t)} = \psi(s_0, y_0, t)$ and $e^{2\pi i \phi_0} = s_0$.

The present paper is mainly devoted to the study of dynamical and topological properties of minimal sets in an *almost periodically forced circle flow* (*APCF* for short), i.e., a SPCF $(S^1 \times Y, \mathbb{T})$ with an almost periodic base (or forcing) flow $(Y, \mathbb{T})$. We refer an APCF as *continuous APCF* if $\mathbb{T} = \mathbb{R}$ and as *discrete APCF* if $\mathbb{T} = \mathbb{Z}$. In the case that the SPCF is discrete, we assume that the generating skew-product circle map Λ is homotopic to the identity and fiber-wise monotone, i.e, there is a homeomorphism $\tilde{\Lambda} : R^1 \times Y \to R^1 \times Y$:

$$\tilde{\Lambda}(x_0, y_0) = (\tilde{f}(x_0, y_0), T(y_0)),$$

where for each $y_0 \in Y$, $\tilde{f}(\cdot, y_0)$ is monotone and $\tilde{f}(x_0 + 1, y_0) \equiv \tilde{f}(x_0, y_0) + 1$, such that when both $\tilde{f}$ and x_0 are identified modulo 1 the identified map $\tilde{\Lambda}$ is homeomorphic to Λ.

In the case that the SPCF (1.1) is either discrete and unforced (i.e., a circle map) or continuous but periodically forced, it follows from the classical Poincaré-Birkhoff-Denjoy theory that a minimal set of the SPCF is periodic if the rotation number is rational, and is either almost periodic or of Denjoy type if the rotation number is irrational. It is also known that a Denjoy type of minimal set has a Cantor structure and is almost automorphic.

But even if the SPCF (1.1) becomes an APCF, its minimal dynamics can be far more complex, though it always admits zero topological entropy. Taking the continuous case for example, while the majority existence of quasi-periodic dynamics in a parameter family of quasi-periodically forced ordinary differential equations on the circle is asserted by the Arnold-Moser theorem ([1, 47]) when the forcing frequencies are Diophantine and the forcing functions are sufficiently small, smooth perturbations of a constant plus a deformation parameter, almost periodic dynamics are however not generally expected in a continuous APCF. Even in the continuous APCF generated from the projective bundle flow of a continous, almost periodic, $sl(2, \mathbb{R})$-valued cocycle, it was shown by Johnson ([30]) that if the cocycle is not uniformly hyperbolic and admits two minimal sets, then both minimal sets are almost automorphic which are not necessarily almost periodic, and moreover, if only one minimal set exists, then dynamics of the minimal set can be much more complicated than being almost automorphic. For discrete APCFs, it was recently shown in [28, 29] that even with one forcing frequency the flows can be topologically transitive.

Besides the presence of multiple forcing frequencies and the topological complexity of the forcing space, much of the dynamical complexity in an APCF (1.1) is governed by the loss of the so-called mean motion property. Let

$$\rho = \lim_{t \to \infty} \frac{\tilde{\phi}(\tilde{\phi}_0, y_0, t)}{t}$$

be the *rotation number* associated with the APCF (1.1) or (1.3), where $\tilde{\phi}(\tilde{\phi}_0, y_0, t)$ denotes the lift of $\phi(\phi_0, y_0, t)$ in R^1 satisfying $\tilde{\phi}(\tilde{\phi}_0 + 1, y_0, t) \equiv \tilde{\phi}(\tilde{\phi}_0, y_0, t) + 1$. The limit exists and is independent of orbits. Differing from the unforced discrete case and periodically forced continuous case, there are general cases in which

$$\sup_{t \in \mathbb{T}} |\tilde{\phi}(\tilde{\phi}_0, y_0, t) - \rho t| = +\infty \tag{1.4}$$

for some $(\tilde{\phi}_0, y_0) \in R^1 \times Y$. We say that the APCF (1.1) admits *mean motion* (or *bounded mean motion*) if

$$\sup_{t \in \mathbb{T}} |\tilde{\phi}(\tilde{\phi}_0, y_0, t) - \rho t| < +\infty \tag{1.5}$$

for all $(\tilde{\phi}_0, y_0) \in R^1 \times Y$. In the opposite case, we say that the APCF (1.1) admits *no mean motion* (or *unbounded mean motion*). It is well-known that if the APCF (1.1) has an almost periodic orbit, then it always admits mean motion. In fact, as suggested by works [28, 29, 58, 67], dynamics of an APCF (1.1) with mean motion should resemble more closely to the unforced discrete case or the periodically forced continuous case, while dynamics of an APCF (1.1) without mean motion should be considerably different than those with mean motion. Indeed, in the case that an APCF (1.1) is continuous in time and admits mean motion, a translation $x = \tilde{\phi} - \rho t$ will readily transform the APCF into an almost periodically forced totally monotone system and an application of results in [60] shows that each of its minimal set is almost automorphic.

In the present paper, we will employee techniques from topological dynamics and ergodic theory to study minimal sets in an APCF (1.1) with particular attentions paying to their general (dynamical and topological) complexities, structures, and topological classifications. We will also investigate in other fundamental dynamics issues for an APCF such as the the existence of almost automorphic minimal sets and sufficient conditions for the validity of the mean motion property. We will obtain a set of general results for an APCF including the following:

a) Each minimal set is either point-distal or residually Li-Yorke chaotic;
b) Any minimal set is either an almost finite to one extension of the base, or the entire phase space, or a Cantorian;
c) If the flow admits more than one minimal set, then each minimal set is an almost finite cover of the base;
d) If the flow admits mean motion, then each minimal set is almost automorphic;
e) If the flow admits no mean motion, then each minimal set is either the entire phase space or is everywhere non-locally connected.

In the case that the forcing is quasi-periodic, the following more concrete results will be obtained:

f) If the rotation number is rationally independent of the forcing frequencies, then the flow admits a unique minimal set and the minimal set is either the entire phase space or is everywhere non-locally connected;
g) If the flow admits no mean motion, then it is positively transitive and admits a unique minimal set, and consequently, if the flow has more than one minimal sets, then it must admit mean motion and each minimal set is almost automorphic.

We remark that, except in those involving rotation number and mean motion, our results above actually hold for a general discrete APCF without assuming its generating skew-product map to be homotopic to identity.

Based on our general results and works [6, 30], we will also give a complete classification of minimal sets for the projective bundle flow $(P^1 \times Y, \mathbb{T})$ of an almost periodic, $sl(2, \mathbb{R})$-valued, continuous (i.e., $\mathbb{T} = \mathbb{R}$) or discrete (i.e, $\mathbb{T} = \mathbb{Z}$) cocycle.

APCFs are among the simplest but fundamental models of non-monotone, multi-frequency systems in which interactions of (both internal and external) frequencies are expected to generate rather complicated dynamics. First of all, APCFs arise naturally in the study of almost periodically forced nonlinear oscillators and their discretizations. Consider an almost periodically forced, damped, nonlinear oscillator

$$x'' + F(x, x', y \cdot t) = 0, \quad x \in R^1, \; y \in Y. \tag{1.6}$$

Such an oscillator admits both internal and external frequencies, and due to damping, its oscillations all lie in a compact global attractor. According to the classical oscillation theory, not only are almost periodic oscillations rare in a such system, but also the global attractor can become complicated especially when the damping is weak. In particular, even in quasi-periodically forced nonlinear oscillators as simple as Van der Pol and Josephson junction, numerical studies have discovered the existence of so-called strange, non-chaotic attractors (SNAs) which are geometrically strange but admit no positive Lyapunov exponent (see e.g., [23, 55]). There have been many theoretical and numerical studies on SNAs with respect to both quasi-periodically forced nonlinear oscillators and their discretizations. It is well believed that a SNA typically lies in the intermittency during the transition from order to chaos. An almost periodically forced, damped, nonlinear oscillator (1.6) can be often reduced through an integral manifolds reduction to an almost periodically forced scalar ordinary differential equation of the form

$$\phi' = f(\phi, y \cdot t), \quad \phi \in R^1, \ y \in Y, \tag{1.7}$$

where $f(\phi+1, y) \equiv f(\phi, y)$ (see [58, 67] for the case of almost periodically forced Van der Pol and Josephson junction oscillators). When both the solution $\phi(\phi_0, y_0, t)$ and the initial value ϕ_0 corresponding to $y = y_0$ are identified modulo 1, the equation (1.7) clearly generates an APCF containing the SNA. In fact, much of the study on SNAs has been made with respect to such reduced quasi-periodically forced circle flows and their discrete counterparts (we refer the readers to [13, 21, 28, 29, 38] for recent progresses on the subject). But besides the case of (1.7) with mean motion property in which minimal sets are known to be almost automorphic ([58, 67]), dynamical and topological structures of SNAs in general situation are yet to be understood.

In the case that an almost periodically forced nonlinear oscillator is damping-free, it becomes an almost periodically forced, one-degree-of-freedom Hamiltonian system of the form

$$x'' + V_x(x, y \cdot t) = 0, \quad x \in R^1, \ y \in Y, \tag{1.8}$$

where $V(x+1, y) \equiv V(x, y)$. Due to the conservative nature, oscillations of a such system spread over the entire phase space. It is known that if (1.8) is quasi-periodically forced with Diophantine frequencies, then associated with high "energy" the system becomes nearly integrable and an application of KAM theory shows the existence of a positive Lebesgue measure set of quasi-periodic invariant tori with Diophantine frequencies. But it is also known that these quasi-periodic tori tend to disappear if either the system become less integrable or the frequencies are close to resonance. Instead, the so-called Mather sets (or Cantori) supporting minimizing measures can be shown to exist in the phase space $R^1 \times S^1 \times Y$ based on the Mather theory ([42, 44]). An application of the Mather theory further shows that dynamics on each projected Mather set in $S^1 \times Y$ is topologically conjugated to that of the corresponding Mather set (see e.g., [27, 43]). More precisely, consider the Lagrangian $L = \frac{p^2}{2} - V(x, y \cdot t)$ associated with (1.8). Then for each $\eta \in R^1$, minimizing measures μ_η exist, i.e., each μ_η is an invariant measure for the skew-product flow $(R^1 \times S^1 \times Y, \mathbb{R})$ generated from (1.8) and satisfies

$$\int_{R^1 \times S^1 \times Y} (L - \eta) d\mu_\eta = \inf_\mu \int_{R^1 \times S^1 \times Y} (L - \eta) d\mu,$$

where the infimum is taken over all Borel probability measures on $R^1 \times S^1 \times Y$. The set $M_\eta = \overline{\cup_{\mu_\eta} \mathrm{supp}\, \mu_\eta}$ is called *Mather set*, which is a compact invariant set of $(R^1 \times S^1 \times Y, \mathbb{R})$. Let $\pi : R^1 \times S^1 \times Y \to S^1 \times Y$ be the natural projection and $\tilde{M}_\eta = \pi M_\eta$ be the *projected Mather set*. Then $\pi : M_\eta \to \tilde{M}_\eta$ is a homeomorphism. It follows that the projected flow on $\tilde{M}_\eta$, as a subflow of an APCF, is topologically conjugated to that defined on M_η. When (1.8) is periodically forced, the projected Mather sets are just the well-known Aubry-Mather sets which have the basic structure of Denjoy sets supporting 2-frequency almost automorphic dynamics. However, not much is known on dynamical and topological properties of Mather sets in situations involving more than two frequencies.

The second, APCFs arise naturally in the spectral theory of 2D linear systems with almost periodic coefficients. For instance, as the projective bundle flows of $sl(2, \mathbb{R})$-valued, almost periodic, continuous or discrete cocycles, they play an essential role in 2D almost periodic Floquet theory and the spectral problem of linear Schrödinger equations/operators and their discrete counterparts (such as Harper's equations and almost Mathieu operators) with almost periodic potentials ([5, 30, 37]). In particular, for the 2D almost periodic Floquet problem, it was a remarkable observation due to Johnson ([30]) that, with the general unavailability of an almost periodic strong Perron transformation which transforms an almost periodic linear differential system into a canonical form, one can often introduce an almost automorphic transformation instead, provided that an almost automorphic minimal set exists in the reduced continuous APCF.

The third, as recently shown by Pliss and Sell ([51]), continuous APCFs arise in oceanic dynamics and climate models through invariant manifolds reductions and high frequency averaging. Hence they can be used as basic models to explain complicated oceanic dynamics in particular the nature of turbulence.

In addition, in the case that the forcing flow is quasi-periodic, (1.1) becomes a toral flow or map whose rotation set is a singleton. Dynamics of toral flows and maps have been extensively studied for cases with convex rotation sets, but the case with "thin" rotation set is more or less open.

Linking to these important problems and applications, our primary goal for the present study is to make a preliminary understanding of frequency-driven dynamical irregularity and complexity in non-monotone, multi-frequency systems. It is our hope that the present study on APCFs will lead to some deep insights on dynamics and structures of SNAs and Mather sets in almost periodically forced, nonlinear oscillators in the damped and damping-free case respectively, on the spectral problem of almost periodic Shrödinder-like operators, on dynamics of toral flows and maps with "thin" rotation sets, on the nature of turbulence in oceanic flows, and on intermittency phenomenon during the transition from order to chaos. We remark that smooth dynamical systems theory plays a less role in the general problems which we are studying. First of all, due to the general almost periodic time dependence, the forcing space of an APCF need not be smooth. Secondly, even if the forcing space is smooth, APCFs arising in applications need not be smooth at all. For instance, for a quasi-periodically forced, damped nonlinear oscillator, it is well-known that the weaker the damping is, the less smoother an integral manifold becomes ([66]). While for a quasi-periodically forced, damping-free nonlinear oscillator, the torus which a projected Mather set is embedded into is only Lipschiz in general even in the periodically forced case ([12, 42]).

The rest of the paper is organized as follows. In Section 2, we will give precise statements of our main results with respect to APCFs along with some discussions. Section 3 is a preliminary section in which we will review basic concepts and results from topological dynamics and ergodic theory to be used in later sections. Our main results will be proved in Sections 4-8 based on some general results which we will prove for compact flow extensions, as well as for general SPCFs. More precisely, in Section 4, we will give an ordering condition under which a compact flow extension preserves topological entropy. In Section 5, we will show that if a minimal flow is a proximal extension of another minimal flow which is not almost 1-1, then it must be Li-Yorke chaotic. In Section 6, we will classify the general topological structures of minimal sets in a SPCF. In Section 7, we will study the nature of minimal sets of a SPCF which are almost finite to one extensions of the base space. In Section 8, we will study dynamical and topological behaviors of minimal sets in a SPCF in connection with the validity of the mean motion property. In particular, we will show that if a SPCF is positively transitive, then it has a unique minimal set. We then consider the case with locally connected base and show that if the SPCF admits no mean motion, then it must be positively transitive. In Section 9, we generalize results in [3, 6, 30] to give a complete classification of minimal dynamics of the projective bundle flow generated from a (continuous or discrete) almost periodic, $sl(2,\mathbb{R})$-valued cocycle of all four basic types: elliptic, parabolic, partially hyperbolic, and hyperbolic.

2. Main results

In this section, we will state our main results with respect to APCFs which are the main motivations for the present study, though most of these results actually hold for more general SPCFs (see Sections 4-8 for more details).

Dynamical and topological structures of minimal sets of an APCF $(S^1 \times Y, \mathbb{T})$ seem to depend on various factors: the number of minimal sets, local connectivity of minimal sets, validity of the mean motion property, and the topological nature of the forcing space. Hence our main results lie in several categories which particularly include cases for general Y as well as for Y being locally connected (e.g., Y is a torus). A special example of the later case is when (1.1) is quasi-periodically forced. As to be seen from the proofs of these results, both the validity of mean motion property and the local connectivity of Y seem to be essential for a minimal set of an APCF to be better behaved in general.

Let $\pi : S^1 \times Y \to Y$ denote the natural projection.

2.1. **General dynamical complexities.** General dynamical complexities of an APCF is characterized in the following two theorems.

Theorem 1. *An APCF always has zero topological entropy.*

Theorem 1 may be proved by using an entropy inequality due to Bowen ([8]) for compact flow extensions. In Section 4 we will give a self-contained proof of this theorem by providing a general result on the preservation of topological entropy between flow extensions under an ordering condition. Such a general result on preservation of topological entropy will also be useful to other skew-product flows having certain fiber-wise order preserving properties.

Theorem 2. *Let M be a minimal set of an APCF. Then precisely one of the following holds:*

a) *M is point-distal;*
b) *M is residually Li-Yorke chaotic.*

Theorem 2 will be proved in Section 5 following a general result which says that if a minimal flow is a proximal extension of another minimal flow that is not almost 1-1, then it must be Li-Yorke chaotic.

The notion of Li-Yorke chaos is introduced based on the well-known work of Li and Yorke ([40]). A compact metric flow $(X, \mathbb{T})$ is called *Li-Yorke chaotic* if X contains an uncountable *scrambled set* S - set in which any pair of distinct points $\{x, y\} \subset S$ is a *Li-Yorke pair*, i.e.,

$$\limsup_{t\to+\infty} d(x \cdot t, y \cdot t) > 0 \text{ and } \liminf_{t\to+\infty} d(x \cdot t, y \cdot t) = 0,$$

where d denotes the metric on X. It is known that if $(X, \mathbb{T})$ admits positive topological entropy, then it is necessarily Li-Yorke chaotic, but not vice versa ([7]).

Residual Li-Yokre chaos is a stronger notion than Li-Yorke chaos. A compact metric flow extension $\pi : (X, \mathbb{T}) \to (Y, \mathbb{T})$ is said to be *residually Li-Yorke chaotic* if there exists a residual (i.e., dense G_δ) subset Y_c of Y such that each fiber $\pi^{-1}(y)$, $y \in Y_c$, admits an uncountable scrambled set.

Remark 1. 1) A point-distal minimal set (including almost automorphic minimal set), though cannot be residually Li-Yorke chaotic, it can well be Li-Yorke chaotic.

2) Minimal sets in a circle map or a periodically forced continuous circle flow can never be Li-Yorke chaotic.

3) Using Theorem 2, one can show that many APCFs admit Li-Yorke chaos. Consider a 2D almost periodic linear system

$$\dot{x} = A(y \cdot t)x, \quad x \in R^2,\ y \in Y, \tag{2.1}$$

where $trA = 0$ and $(Y, \mathbb{R})$ is an almost periodic minimal flow. The system naturally generates a continuous APCF Λ_t^A on the projective bundle $P^1 \times Y$. Let S_0 be the set of continuous matrix-valued function A whose respective linear system (2.1) can be reduced to a system with skew-symmetric coefficient matrix $B(y)$ of zero mean via a strong Perron transformation. It was shown by Johnson ([33]) (see also [45]) that there is a residual subset $\mathcal{S}$ of $\bar{S}_0$ such that for each $A \in \mathcal{S}$, the entire phase space of Λ_t^A is minimal, strictly ergodic, and a proximal extension of Y. Now, for each $A \in \mathcal{S}$, it follows from Theorem 2 that all minimal sets of Λ_t^A are residually Li-Yorke chaotic. We note that with the non-existence of almost automorphic dynamics, the APCFs Λ_t^A with $A \in \mathcal{S}$ admit no mean motion.

We refer the readers to a recent work of Bjerklov and Johnson ([6]) for more concrete discussions on Li-Yorke chaos in continuous, almost periodically forced projective bundle flows and to Section 9 of this paper for some general discussions in this regard.

4) We expect that point-distality of minimal sets stated in Theorem 2 can be replaced by almost automorphy in a generic sense. However, there are minimal sets of APCFs which are point-distal but not almost automorphic. An easy example is as follows. Let $(Y, \mathbb{R})$ be an non-periodic, almost periodic minimal flow and let a be a continuous function on Y with zero mean such that $\int_0^t a(y_0 \cdot s)ds$ is unbounded for some $y_0 \in Y$ (such functions largely exist, see [31]). Then the almost periodically forced circle flow defined by $\Lambda_t^*(\phi, y) = (\phi + \int_0^t a(y \cdot s)ds \bmod 1, y \cdot t)$, is not almost periodic. Hence Λ_t^* is distal (in particular, point-distal) but not almost automrphic, simply because a distal almost automorphic minimal flow must be almost periodic.

5) In [30], concerning the 2D almost periodic Floquet problem, Johnson studied the APCF (projective bundle flow) Λ_t^A generated from (2.1). Minimal sets of the flow were shown to be almost periodic or almost automorphic for most cases except that the flow has only one minimal set M in which no fiber over the base admits a distal pair. Using Theorem 2, we conclude that M is either an almost 1-1 extension of the base (hence almost automorphic, see Theorem 3.2) or residually Li-Yorke chaotic (see also [6]). Similar classifications can be made for the projective bundle flow of a general $sl(2, \mathbb{R})$-valued, almost periodic, continuous or discrete cocycle (see Section 9 for details).

2.2. **Topological classification of minimal sets.** In the case that an APCF is either discrete and unforced or continuous but periodically forced (i.e., $Y = S^1$), it follows from the classical Poincaré-Birkhoff-Denjoy classification that if the rotation number is rational (the resonant case), then each minimal set is a finite to one extension of S^1, and if the rotation number is irrational (the non-resonant case), then a minimal set is either the entire phase space or of Denjoy type (in the continuous case, each of its Poincaré section is a Denjoy Cantor set). Our next result shows that one can have a similar topological classification of minimal sets for a general APCF.

Theorem 3. *Let M be a minimal set of an APCF. Then precisely one of the following holds:*

a) *M is an almost N-1 extension of Y for some positive integer N;*
b) *$M = S^1 \times Y$;*
c) *M is a Cantorian.*

Theorem 3 will be proved in Section 6. A minimal set M of a SPCF $(S^1 \times Y, \mathbb{T})$ is said to be a *Cantorian* if there exists a residual subset Y_0 of Y such that for each $y \in Y_0$, the fiber $M_y = \pi^{-1}(y) \cap M$ is a Cantor set.

Remark 2. 1) Cantorians can arise in APCFs with or without mean motions. The Denjoy type of minimal set in a continuous, periodically forced circle flow is an example of Cantorian in (topologically non-transitive) APCFs with mean motion. An example of Cantorian in (topologically transitive) APCFs without mean motion is constructed in a recent work due to Béguin, Crovisier, Jäger, and Le Roux ([3]).

2) It is clear that a minimal set in the case a) of Theorem 3 cannot be residually Li-Yorke chaotic, hence by Theorem 2 it must be point-distal. Still, such a minimal set can well be Li-Yorke chaotic and topological complicated by being everywhere non-locally connected (see Remark 4 1) below).

3) We believe that a Cantorian minimal set of an APCF is more topologically complicated in the sense that it is not only a Cantorian but also everywhere non-locally connected.

4) According to the Poincaré-Birkhoff-Denjoy theory, dynamics of a minimal set M in a continuous, periodically forced circle flow can be completely classified according to its topological nature: in the resonant case M is periodic, while in the non-resonant case M is either 2-frequency almost periodic if it is the entire phase space or 2-frequency almost automorphic if it is of Denjoy type. To the contrary, dynamics of minimal sets in a general APCF is too complicated to be classified according to their topological natures. For instance, minimal sets in both cases b) and c) of Theorem 3 can be either point-distal or residually Li-Yorke chaotic which need not be almost automorphic. In fact, even in the case a) of Theorem 3 we believe that minimal sets need not be almost automorphic in general.

An almost N-1 extension of an almost periodic minimal flow need not be almost automorphic. A such example can be constructed in symbolic flows as follows. Let τ be the substitution on $\{0,1\}$ with $\tau(0) = 01$ and $\tau(1) = 10$. For any finite word $w = w_0w_1\cdots w_{\ell-1}$ in $\{0,1\}$, define $\tau(w) = \tau(w_0)\tau(w_1)\cdots\tau(w_{\ell-1})$ and $\tau^i(w) = \tau(\tau^{i-1}(w))$, $i \geq 2$. Let $X \subseteq \{0,1\}^{\mathbb{Z}}$ be the set of all bi-infinite binary sequences x in X such that any finite word in x is a sub-word of $\tau^i(0)$ for some $i \in \mathbb{N}$, and let T be the left shift map on X. Then the discrete dynamical system (X,T) is a so-called Morse system which is known to be minimal and an almost 2-1 extension of its maximal almost periodic factor ([41]). As an almost automorphic minimal flow is necessary an almost 1-1 extension of its maximal almost periodic factor (see Theorem 3.2), (X,T) is not almost automorphic. A suspension of (X,T) also leads to an example of continuous flows.

5) Unlike the periodically forced continuous case, the topological classification for general APCFs given in Theorem 3 need not depend on the resonance type. For instance, case a) of Theorem 3 can happen when the rotation number is rationally independent of forcing frequencies, as shown by an example of Johnson ([36]) in which an APCF admits a unique minimal set that is an almost 1-1 extension of the base, but the rotation number is not rationally dependent on the forcing frequencies.

2.3. **Almost finite to one extensions.** We would like to exam cases of APCFs in which an almost N-1 extension of the base becomes almost automorphic. We first give a structural characterization of a minimal set if there are more than one minimal sets presented in an APCF.

Theorem 4. *Suppose that an APCF $(S^1 \times Y, \mathbb{T})$ has more than one minimal sets. Then the following holds.*

1) *There exists a positive integer N such that each minimal set of $(S^1 \times Y, \mathbb{T})$ is an almost N-1 extension of Y.*
2) *If one minimal set of $(S^1 \times Y, \mathbb{T})$ is almost automorphic, then so are others.*
3) *If Y is locally connected, then all minimal sets of $(S^1 \times Y, \mathbb{T})$ are almost automorphic.*

In the case that an APCF $(S^1 \times Y, \mathbb{T})$ admits more than one minimal sets, we feel that the local connectivity of Y is essential for the minimal sets to be almost automorphic. In the case that it only admits one minimal set which is an almost N-1 extension of Y for some $N > 1$, we believe that even local connectivity of Y would not be sufficient for the minimal set to be almost automorphic. However, we do have the following result.

Theorem 5. *Let M be a minimal set of an APCF $(S^1 \times Y, \mathbb{T})$ which is an almost N-1 extension of Y for some $N \geq 1$. If M is not everywhere non-locally connected, then it is almost automorphic, and moreover, for any $y \in Y$, each fiber $\pi^{-1}(y) \cap M$ consists of exactly N connected components which are either singletons or closed intervals, if it is not homeomorphic to S^1.*

Theorems 4, 5 will be proved in Section 7.

Remark 3. 1) A Denjoy type of minimal set in a continuous, periodically forced circle flow is an almost 1-1 extension of a 2-torus with two points on each non-residual fiber. Hence by Theorem 5 it is everywhere non-locally connected.

2) In [35], Johnson constructed an example of continuous, quasi-periodically forced circle flow which has a unique minimal set M with the following properties: i) M is an almost 1-1 extension of the base torus (hence it is almost automorphic); ii) M is locally connected at every points on singleton fibers; iii) M is not locally connected at all points; vi) there is a full (Haar) measure set in the base torus over which all fibers are non-degenerate intervals. This gives an example of Theorem 5. In fact, by Theorem 5, all non-singleton fibers in M are non-degenerate intervals.

2.4. **Mean motion and dynamics.** In the next two results, we describe the behavior and structure of a minimal set of an APCF in both the cases with and without mean motion. Theorem 6 below is more or less known in the continuous case ([58, 67]) but unknown in the discrete case.

Theorem 6. *Suppose that an APCF $(S^1 \times Y, \mathbb{T})$ admits mean motion. Then the following holds.*

1) *Each minimal set of* $(S^1 \times Y, \mathbb{T})$ *is almost automorphic whose frequency module is generated by the rotation number and the forcing frequencies.*
2) *If a minimal set of* $(S^1 \times Y, \mathbb{T})$ *is an almost* N-1 *extension of* Y *for some positive integer* N*, then* N *is the smallest positive integer whose multiplication to the rotation number is contained in the frequency module of the forcing.*

Theorem 7. *Suppose that an APCF* $(S^1 \times Y, \mathbb{T})$ *admits no mean motion. Then the following holds.*

1) *Each minimal set of* $(S^1 \times Y, \mathbb{T})$ *is either the entire phase space* $S^1 \times Y$ *or is everywhere non-locally connected.*
2) *If* Y *is locally connected, then* $(S^1 \times Y, \mathbb{T})$ *is positively transitive and has only one minimal set.*

Theorems 6, 7 will be proved in Section 8 based on some general results on the connections between the lacking of mean motion, positive transitivity, and the uniqueness of minimal set. Theorem 7 2) is partially known for a quasi-periodically forced circle map with one forcing frequency ([28, 29]). But arguments in [28, 29], being crucially depending on the one-dimensional forcing space, does not extend to the general situation completely.

Corollary. *Consider an APCF* $(S^1 \times Y, \mathbb{T})$ *with* Y *being locally connected. Then the following holds.*

1) *If* $(S^1 \times Y, \mathbb{T})$ *has more than one minimal set, then it admits mean motion.*
2) *If the entire phase space* $S^1 \times Y$ *is not minimal, then each minimal set of* $(S^1 \times Y, \mathbb{T})$ *is either everywhere non-locally connected or almost automorphic.*
3) *If the rotation number is rationally independent of the forcing frequencies, then* $(S^1 \times Y, \mathbb{T})$ *has a unique minimal set.*

In the above Corollary, 1) follows immediately from Theorem 7 2), 2) follows immediately from Theorem 6 1) and Theorem 7 1), and 3) follows immediately from Theorem 7 2), Theorem 4 1) and Theorem 6 2).

Remark 4. 1) Consider the example of Johnson ([36]) in which an APCF admits a unique minimal set that is an almost 1-1 extension of the base, but the rotation number is not rationally dependent on the forcing frequencies. By Theorem 6, this example admits no mean motion, and, by Theorem 7, the unique almost automorphic minimal set is everywhere non-locally connected.

Also consider the quasi-periodically forced circle flow constructed by Johnson ([35]) in which the unique minimal set is not everywhere non-locally connected. Theorem 7 implies that this flow does admit mean motion.

2) If an APCF has a minimal set which is residually Li-Yorke chaotic, then the minimal set cannot be almost automorphic and hence by Theorem 6 the APCF admits no mean motion.

3) An almost automorphic minimal set often occurs as intermediate dynamics in a parameter family of quasi-periodic forced circle flows. Consider a smooth family of quasi-periodically forced equations

$$\phi' = \lambda + \varepsilon f(\phi, y \cdot t), \qquad \phi \in R^1, \tag{2.2}$$

where $f : R^1 \times T^k \to R^1$ is sufficiently smooth, $f(\phi + 1, y) \equiv f(\phi, y)$, $y \cdot t = y + \omega t$, $\omega \in R^k$ is Diophantine, and λ, ε are bounded parameters. We let Σ be the set of (λ, ε) whose corresponding equation (2.2) is smoothly reducible to a pure rotation according to Arnold-Moser theorem ([1, 47]), i.e., there is a smooth, near identity transformation $\phi = \psi + h_{\lambda,\varepsilon}(\psi, y)$ with $\|h_{\lambda,\varepsilon}\|_\infty < 1$, such that the transformed equation becomes $\psi' = \lambda$ (hence the corresponding quasi-periodically forced circle flow is quasi-periodic and Diophantine). Now consider a boundary point $(\lambda_0, \varepsilon_0)$ of Σ, i.e., there is a sequence $(\lambda_n, \varepsilon_n) \in \Sigma \to (\lambda_0, \varepsilon_0)$. Let $y_0 \in T^k$, $\psi_0 \in [0, 1)$ be given, and $h_n(t) = h_{\lambda_n,\varepsilon_n}(\psi_0, y_0 + \omega t)$ converges uniformly on compact sets to some $h_\infty(t)$ according to the Ascoli theorem. Then $\phi_n(t) = \psi_0 + \lambda_n t + h_n(t)$ converges uniformly on compact sets to $\phi_\infty = \psi_0 + \lambda_\infty t + h_\infty(t)$ which is a solution of (2.2) corresponding to $(\lambda_0, \varepsilon_0)$. Since $\|h_\infty\|_\infty \le 1$, it follows that the quasi-periodically forced circle flow (2.2) corresponding to $(\lambda_0, \varepsilon_0)$ admits mean motion (see Theorem 8.2 in Section 8) and hence by Theorem 6 all its minimal sets are almost automorphic. The rotation number associated with $(\lambda_0, \varepsilon_0)$ may well depend on ω in a joint Liousville way so that the frequencies of the almost automorphic minimal sets need not be Diophantine. Dynamics of the flow associated with (λ, ε) lying in the complement of $\bar{\Sigma}$ are expected to be more complicated due to the possible loss of mean motion property. Similar intermittency phenomenon can be observed in the spectral problem of an almost periodic Schrödinger operator (see Section 9 for detail).

For a general parameter family of APCFs, we believe that almost automorphic intermittency (or bifurcation) may occur at the critical value when either almost periodicity is lost, or mean motion property becomes invalid, or when Li-Yorke chaos tends to appear (order to chaos).

2.5. **An extended Denjoy theorem.** From Theorems 4, 7 and the Corollary above, we see that local connectivity of Y plays an important role in the dynamics and topological structures of minimal sets in the corresponding APCF $(S^1 \times Y, \mathbb{T})$. The case when Y is locally connected of course includes that of a quasi-periodically forced circle flow. In fact, when Y is a torus, one can have a more complete characterization on the topological structure of a minimal set. The following result can be regarded as a quasi-periodic extension of the classical Denjoy theorem with respect to the topological structure of minimal sets.

Theorem 8. *Consider an APCF $(S^1 \times Y, \mathbb{T})$ with Y being a torus (e.g., $(Y, \mathbb{T})$ is quasi-periodic) and suppose that the rotation number is rationally independent of the forcing frequencies. Then $(S^1 \times Y, \mathbb{T})$ has a unique minimal set M and M is either the entire phase space $S^1 \times Y$ or is everywhere non-locally connected. If, in addition, the APCF admits mean motion, then M is almost automorphic, and moreover, M is either the entire phase space $S^1 \times Y$ or an everywhere non-locally connected Cantorian.*

Theorems 8 will also be proved in Section 8.

Remark 5. 1) In light of Theorem 2, under the condition of Theorem 8, an everywhere non-locally connected minimal set can be either a finite to one extension of the base or a Cantorian.

2) Our results give some information on possible topological and dynamical complexity of a SNA in a quasi-periodically forced, damped nonlinear oscillator and Mather sets in a quasi-periodically forced, damping-free nonlinear oscillator.

Consider a quasi-periodically forced, damped, nonlinear oscillator (1.6) in which a SNA exists. If the damping is not too weak, then the attractor lies in a quasi-periodically forced circle flow through an integral manifolds reduction. The complexity of a SNA is often reflected by that on its minimal sets, because, using arguments in [59], such an attractor is made up by minimal sets and their "connecting orbits". Due to the geometric strangeness, the SNA is however not the entire phase space if it is globally attracting. It follows from Theorem 3 that each minimal set in the SNA is either an almost finite cover of the forcing space (a torus) or a Cantorian, which, by part 2) of the Corollary, is almost automorphic and/or everywhere non-locally connected. In particular, if the rotation number is rationally independent of the forcing frequencies, then it follows from Theorem 8 that the SNA contains a unique minimal set which is everywhere non-locally connected (which can be a Cantorian carrying almost automorphic dynamics). Of course, minimal dynamics in a SNA can well be Li-Yorke chaotic or even residually Li-Yorke chaotic according to Theorem 2 (as shown in [21], a SNAs can exhibit certain mild chaotic behavior). All these actually suggest that topologically a minimal set in a SNA should typically be everywhere non-local connected and be either an almost finite cover of the forcing space or a Cantorian; and dynamically a minimal set in a SNA should essentially be either almost automorphic or residually Li-Yorke chaotic (see Section 9 for more discussions in this regard on almost periodically forced projective bundle flows).

We remark that the kind of complexity of SNAs described above for a quasi-periodically forced, damped, nonlinear oscillator is particularly significant when the damping is weak, in which case the reduced flow on the integral manifold becomes a less/non-smooth, quasi-periodically forced circle flow. To the contrary, when the damping is strong, one can well have cases in which topological complexity of minimal sets plays a less role to the geometric complexity of a SNA in comparing with its measure-theoretic complexity.

For a quasi-periodically forced, damping-free nonlinear oscillator (1.8), we note that the flow on each projected Mather set is a (not necessarily smooth) skew-product flow lying in $S^1 \times Y$ for which all our results above are applicable. Hence if dynamics on a projected Mather set is not quasi-periodic, then similar topological and dynamical structures are expected for its minimal sets.

3. Preliminary

For simplicity, we assume that all $\mathbb{T}$-flows, for $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$, to be considered in the rest of the paper are defined on complete separable metric spaces. We will use the same symbol $|\cdot|$ to denote the cardinality of a set, the absolute value of a number, and the norm of a matrix or a function. Also, for a compact metric space X, we let the set 2^X of compact subsets of X be endowed with the Hausdroff metric.

We say that a flow $(X, \mathbb{T})$ is *compact* if the phase space X is a compact metric space. Recall that a nonempty compact invariant subset M of a flow $(X, \mathbb{T})$ is *minimal* if it contains no nonempty, proper, closed invariant subset. A compact flow $(X, \mathbb{T})$ is said to be *minimal* if X itself is a minimal set, to be *strictly ergodic* if it is both minimal and uniquely ergodic (i.e., it admits an unique invariant probability measure), and to be *positively transitive* if for each pair of nonempty open subsets U, V of X there exists $t \geq 0$ such that $U \cdot t \cap V \neq \emptyset$. If $(X, \mathbb{T})$ is positively transitive, then the set $Tran^+(X)$ of positive transitive points of X is a residual subset of X, and moreover,

$$Tran^+(X) = \bigcap_{n=1}^{\infty} \bigcup_{t \geq 0} U_n \cdot t,$$

where $\{U_n\}_{n=1}^{\infty}$ is a countable basis of X.

3.1. **Proximality, distality, and almost automorphy.**

Let $(X, \mathbb{T})$ be a flow and d be the metric on X. Two points $x, y \in X$ are said to be *positively proximal* if $\liminf_{t \to +\infty} d(x \cdot t, x' \cdot t) = 0$; *proximal* if $\liminf_{t \to \infty} d(x \cdot t, x' \cdot t) = 0$. For any $x \in X$, we define

$$PR_+(x) = \{x' \in X : x, x' \text{ are positively proximal}\};$$
$$PR(x) = \{x' \in X : x, x' \text{ are proximal}\}.$$

Now assume that $(X, \mathbb{T})$ is a compact flow and consider the flow maps $\Pi_t : X \to X$: $\Pi_t(x) = x \cdot t$, $t \in \mathbb{T}$. Then $\{\Pi_t : t \in \mathbb{T}\} \subset X^X$ - the compact Hausdorff space of self-maps of X endowed with the topology of pointwise convergence. The space X^X is also a semigroup under the composition of maps on which the right multiplication $p \to pp_0$ is continuous for all $p_0 \in X^X$ and the left multiplication $p \to p_0 p$ is continuous only if p_0 is a continuous map. The *Ellis semigroup* $E(X, \mathbb{T})$ of X is simply defined as $E(X, \mathbb{T}) = \overline{\{\Pi_t : t \in \mathbb{T}\}}$, where the closure is taken under the topology of pointwise convergence. Hence $E(X, \mathbb{T})$ is compact and a sub-semigroup of X^X with identity e - the identity map, on which the right multiplication is continuous. We note that the flow $(X, \mathbb{T})$ also induces a natural compact flow $(E(X, \mathbb{T}), \mathbb{T})$ on $E(X, \mathbb{T})$: $\gamma \cdot t \equiv \Pi_t \gamma$, $\gamma \in E(X, \mathbb{T})$, $t \in \mathbb{T}$. Let $\omega(e)$ be the ω-limit set of the identity e in $(E(X, \mathbb{T}), \mathbb{T})$. It is clear that two points $x, y \in X$ are proximal (resp. positively proximal) iff there exists $p \in E(X, \mathbb{T})$ (resp. $p \in \omega(e)$) such that $p(x) = p(y)$.

$x \in X$ is called a *positive distal point* (resp. *distal point*) if $PR_+(x) = \{x\}$ (resp. $PR(x) = \{x\}$). A minimal flow $(X, \mathbb{T})$ is called *point-distal* if it contains a distal point. It is well-known that if $(X, \mathbb{T})$ is point-distal, then the set X_d of distal points of X is a residual subset ([63]).

It is clear that a distal point in a flow must be a positive distal point. In the following, we show that the converse is also true.

Proposition 3.1. *Let $(X, \mathbb{T})$ be a minimal flow. Then a positive distal point in X is also a distal point. In particular, if $(X, \mathbb{T})$ is not point-distal, then for any $x \in X$, $PR_+(x) \setminus \{x\} \neq \emptyset$.*

Proof. Let $\omega(e)$ be the ω-limit set of e in the flow $(E(X,\mathbb{T}),\mathbb{T})$. Using continuity of the right multiplication and the fact that e is the identity of $E(X,\mathbb{T})$, we have that $E(X,\mathbb{T})\omega(e)=\omega(e)$ and $\omega(e)$ is a sub-semigroup of $E(X,\mathbb{T})$.

Let x be a positive distal point. Since $(X,\mathbb{T})$ is minimal, it is easy to see that for any $y\in X$, $\omega(e)y=\{p(y):p\in\omega(e)\}=X$. Let $y\in X\setminus\{x\}$ and consider $\omega_y=\{p\in\omega(e):p(y)=y\}$. Since $\omega(e)y=X$, ω_y is nonempty. Since ω_y is a closed sub-semigroup of $\omega(e)$ on which the right multiplication is continuous, it follows from a general result due to Namakura [49] (see also [10, Lemma 1]) that ω_y contains an idempotent point u, i.e., $u^2=u$. Clearly $u(y)=y$.

Since $u(x)=u(u(x))$ and x is a positive distal point, $x=u(x)$. Hence for any $p\in E(X,\mathbb{T})$,

$$p(x)=pu(x) \text{ and } p(y)=pu(y). \tag{3.1}$$

Since $pu\in E(X,\mathbb{T})\omega(e)=\omega(e)$ and x,y are not positively proximal, $pu(x)\neq pu(y)$. It follows from (3.1) that $p(x)\neq p(y)$. Since p is arbitrary, x,y are not proximal. This shows that $PR(x)=\{x\}$, i.e., x is a distal point. □

A function $f\in C(\mathbb{T},X)$, where X is a complete separable metric space, is said to be *almost automorphic* if whenever $\{t_n\}$ is a sequence such that $f(t_n+t)\to g(t)\in C(\mathbb{T},X)$ uniformly on compact sets, then also $g(t-t_n)\to f(t)$ uniformly on compact sets, as $n\to\infty$. An almost automorphic function valued in a separable Banach space admits well-defined Fourier series which are however not necessarily unique and only converge point-wise in general in term of Bochner-Fejer summation ([61, 62]). But one can uniquely define the *frequency module* of an almost automorphic function in the usual way as the smallest additive sub-group of $\mathbb{R}$ containing a Fourier spectrum ([61]). In this sense, both almost periodic and almost automorphic functions can be viewed as natural generalizations to the periodic ones in the strongest and the weakest sense respectively.

A point x in a flow $(X,\mathbb{T})$ is said to be *almost automorphic* if the orbit $\{x\cdot t\}$ is an almost automorphic function in t. A flow $(X,\mathbb{T})$ is called *almost automorphic minimal* if X is the closure of an almost automorphic orbit. An almost automorphic minimal flow is compact, minimal, point-distal, and contains residually many almost automorphic points which are precisely the distal points ([63]). Unlike an almost periodic minimal flow, an almost automorphic one can be non-uniquely ergodic, topologically complicated, can admit positive topological entropy, and its general measure-theoretic characterization can be completely random (see [4, 14, 16, 61, 67] and references therein). Hence, though an almost automorphic minimal flow resembles an almost periodic one harmonically, it can have certain dynamical, topological, and measure-theoretic complexities which significantly differ from an almost periodic one.

Let $(X,\mathbb{T})$ be a flow. A Δ*-set* S of $\mathbb{T}$ is the set of all increasing differences in a sequence $\{s_n\}_{n=1}^{\infty}$, i.e., $S=\{s_n-s_m:n>m\}$, and a Δ^**-set* is a subset of $\mathbb{T}$ which has nonempty intersection with each Δ-set. A point $x\in X$ is called Δ^**-recurrent* if for every neighborhood V of x, the *recurrent time set*

$$N(x,V)=\{t\in\mathbb{T}:x\cdot t\in V\}$$

is a Δ^*-set.

Almost automorphic points can be characterized by Δ^*-recurrency as follows.

Theorem 3.1. *A point x in a compact flow $(X,\mathbb{T})$ is almost automorphic iff it is Δ^*-recurrent.*

Proof. The theorem is a classical result of Furstenberg ([15]) when $\mathbb{T}=\mathbb{Z}$. For $\mathbb{T}=\mathbb{R}$, the proof is completely similar. □

3.2. **Flow extensions.** A *flow extension* (or a *flow homomorphism* or a *factor map*) $\pi:(X,\mathbb{T})\to(Y,\mathbb{T})$ is a continuous onto map $\pi:X\to Y$ which preserves the flows. If such a flow extension exists, then $(X,\mathbb{T})$ (or X) is called an *extension* of $(Y,\mathbb{T})$ (or Y) and $(Y,\mathbb{T})$ (or Y) is called a *factor* of $(X,\mathbb{T})$ (X).

Let $\pi:(X,\mathbb{T})\to(Y,\mathbb{T})$ be an extension between compact flows. π is called *proximal* (resp. *distal*) if for each $y\in Y$ and any two points on $\pi^{-1}(y)$ are proximal (resp. distal), called *almost*

N-1 (resp. *almost finite to one*), if there exists a residual subset $X_0 \subset X$ such that for any $x \in X_0$ $\pi^{-1}\pi(x)$ consists of N points (resp. $\pi^{-1}\pi(x)$ is a finite set), called N-1 (resp. *finite to one*) if $\pi^{-1}\pi(x)$ consists of N points (resp. $\pi^{-1}\pi(x)$ is a finite set) for all $x \in X$, called *open* if $\pi : X \to Y$ is an open map, and called *semi-open* if $\pi : X \to Y$ is a semi-open map, i.e., for any nonempty open subset U of X the image $\pi(U)$ has nonempty interior in Y. An 1-1 flow extension is also called a *flow isomorphism.*

Let $R_\pi = \{(x_1, x_2) \in X \times X : \pi(x_1) = \pi(x_2)\}$. The flow $(X, \mathbb{T})$ induces a natural flow $(R_\pi, \mathbb{T})$ on R_π. π is called *positively weakly mixing* if the flow $(R_\pi, \mathbb{T})$ is positively transitive.

Using the ω-limit sets of $(R_\pi, \mathbb{T})$, it is easy to see that if π is a proximal extension, then it must be a positive proximal extension, i.e., for any $x \in X$, any two points in $\pi^{-1}\pi(x)$ are positively proximal.

The general structure of an almost automorphic minimal flow is characterized by the following structure theorem due to Veech ([62]).

Theorem 3.2. *A compact flow is almost automorphic minimal iff it is an almost* 1-1 *extension of an almost periodic minimal flow.*

By the above structure theorem, almost automorphic points in an almost automorphic minimal flow are precisely those lying in singleton fibers of the corresponding almost 1-1 extension of an almost periodic minimal flow. Hence an almost automorphic minimal set becomes almost periodic iff every point in the set is an almost automorphic point.

Proposition 3.2. *Let $\pi : (X, \mathbb{T}) \to (Y, \mathbb{T})$ be an extension between minimal flows. Then the following holds.*

1) *If π is almost finite to one, then there is a positive integer N such that π is an almost N-1 extension.*
2) *If π is open and finite to one, then π is a distal and N-1 extension for some positive integer N. If, in addition, $(Y, \mathbb{T})$ is point-distal, then so is $(X, \mathbb{T})$.*

Proof. 1) We let $Y_* = \{y \in Y : |\pi^{-1}(y)| < +\infty\}$. Since $\pi^{-1} : Y \to 2^X$: $y \mapsto \pi^{-1}(y)$ is upper semi-continuous, the set Y_0 of all continuity points of π^{-1} is a residual subset of Y. For any given $y_* \in Y_*$ and $y_0 \in Y_0$, we let $\{t_n\} \subset \mathbb{T}$ be a sequence such that $y_* \cdot t_n \to y_0$. It follows from the continuity of π^{-1} at y_0 that $|\pi^{-1}(y_0)| \le |\pi^{-1}(y_* \cdot t_n)| = |\pi^{-1}(y_*)| < +\infty$. Hence $Y_0 \subset Y_*$. Now, for any $y_1, y_2 \in Y_0$, the above argument yields that $|\pi^{-1}(y_1)| \le |\pi^{-1}(y_2)|$ and $|\pi^{-1}(y_2)| \le |\pi^{-1}(y_1)|$, i.e., the map $Y_0 \to \mathbb{N}$: $y \mapsto |\pi^{-1}(y)|$ is a constant, say N.

2) Since π is open, π^{-1} is continuous, i.e., $Y_0 = Y$. It follows from the proof of 1) above that the map $Y \to \mathbb{N}$: $y \mapsto |\pi^{-1}(y)|$ is a constant N. The continuity of π^{-1} also implies that there cannot be any proximal pair on each fiber, for otherwise the number of points on some fiber would be smaller than N.

A distal extension of a point-distal flow is easily seen to be point-distal. □

Proposition 3.3. *If $\pi : (X, \mathbb{T}) \to (Y, \mathbb{T})$ is an open and proximal extension between minimal flows, then it is a positively weakly mixing extension.*

Proof. We follow the arguments of the proof of Theorem 6.3, in [18]. We first show the following

Claim: If $x_1, x_2, \cdots, x_n \in X$ are such that $\pi(x_1) = \pi(x_2) = \cdots = \pi(x_n)$, then for any $x \in X$ there is a positive increasing sequence $\{t_m\} \subset \mathbb{T}$ such that

$$\lim_{m\to\infty} x_i \cdot t_m = x, \text{ for all } 1 \le i \le n. \tag{3.2}$$

Since $(X, \mathbb{T})$ is minimal, the Claim clearly holds for $n = 1$. By induction, we assume that the Claim is true for some $n = k$. Then for any $x \in X$ there exists a positive increasing sequence $\{t_m\}$ such that $\lim_{m\to\infty} x_i \cdot t_m = x$ for all $1 \le i \le k$. Without loss of generality, we let $x_{k+1} \cdot t_m$ be convergent, say to some $x' \in X$. Then $\pi(x) = \pi(x')$, and hence $\liminf_{t\to+\infty} d(x\cdot t, x'\cdot t) = 0$. Using minimality of $(X, \mathbb{T})$ we let $\{s_j\}$ be a positive increasing sequence such that $\lim_{j\to\infty} x \cdot s_j = x$ and $\lim_{j\to\infty} x' \cdot s_j = x$, i.e.,

$$\lim_{j\to\infty} \lim_{m\to\infty} x_i \cdot (t_m + s_j) = x, \; i = 1, 2, \cdots, k+1.$$

It follows that we can take a positive increasing sequence $\{r_j = s_j + t_{m(j)}\}$ for sufficiently large $\{m(j)\}$ such that

$$\lim_{j\to\infty} x_i \cdot r_j = x, \; i = 1, 2, \cdots, k+1.$$

This proves the Claim.

Let W, W' be two nonempty open subsets of $R_\pi = \{(x_1, x_2) \in X \times X : \pi(x_1) = \pi(x_2)\}$. Since π is an open map, there exist nonempty open sets $U, V; U', V'$ of X such that $\pi(U) = \pi(V)$, $\pi(U') = \pi(V')$, $W \supseteq (U \times V) \cap R_\pi \neq \emptyset$, and $W' \supseteq (U' \times V') \cap R_\pi \neq \emptyset$.

For a fixed $z_0 \in \pi(U)$, we have by minimality of $(X, \mathbb{T})$ that there are $t_1, t_2, \cdots, t_n \in \mathbb{T}$ such that $\bigcup_{i=1}^n V \cdot t_i = X$. By relabeling the t_i's if necessary, we assume without loss of generality that there is an integer $1 \le m \le n$ such that $V \cdot t_i \cap \pi^{-1}(z_0) \neq \emptyset$ for all $i = 1, 2, \cdots, m$ and $\bigcup_{i=1}^m V \cdot t_i \supset \pi^{-1}(z_0)$. For each $i = 1, 2, \cdots, m$, we let $v_i \in V$ be such that $v_i \cdot t_i = y_i \in \pi^{-1}(z_0)$. Since $\pi(U) = \pi(V)$, there is a point $u_i \in U$ with $\pi(u_i) = \pi(v_i)$. Denote $x_i = u_i \cdot t_i$, $i = 1, 2, \cdots, m$. Then it is clear that $\pi(x_i) = \pi(u_i \cdot t_i) = \pi(v_i \cdot t_i) = \pi(y_i) = z_0$, i.e., $x_i \in \pi^{-1}(z_0)$, $i = 1, 2, \cdots, m$.

By the Claim, there exists $t > 0$ such that $x_i \cdot t \in U'$ and $t + t_i > 0$ for all $i = 1, 2, \cdots, m$. Since $z_0 \cdot t = \pi(x_i \cdot t) \in \pi(U') = \pi(V')$, $i = 1, 2, \cdots, m$, we can take a point $b \in V'$ such that $\pi(b) = z_0 \cdot t$. Then $b \cdot (-t) \in \pi^{-1}(z_0) \subseteq \bigcup_{i=1}^m V \cdot t_i$. Hence $b \cdot (-t) \in V \cdot t_i$, i.e., $b \in V \cdot (t + t_i) \cap V'$ for some $1 \le i \le m$. Now let $a = x_i \cdot t$. Then $\pi(a) = z_0 \cdot t$ and $a \in U \cdot (t + t_i) \cap U'$. It follows that $(a, b) \in ((U \times V) \cdot (t + t_i)) \cap (U' \times V') \cap R_\pi \subset (W \cdot (t + t_i)) \cap W'$. □

The following is a classical result of Auslander ([63]).

Proposition 3.4. *If $\pi : (X, \mathbb{T}) \to (Y, \mathbb{T})$ is an extension between minimal flows, then π is semi-open.*

It is well-known that every extension of minimal flows can be lifted to an open extension by almost 1-1 modifications. To be precise, let $\pi : (X, \mathbb{T}) \to (Y, \mathbb{T})$ be an extension between minimal flows, and let Y_0 be the set of continuity points of $\pi^{-1} : Y \to 2^X$: $y \mapsto \pi^{-1}(y)$. Recall that Y_0 is an invariant residual subset of Y.

Let $Y^* = \mathrm{cl}(\{\pi^{-1}(y) : y \in Y_0\})$ and $(2^X, \mathbb{T})$ be the flow on 2^X induced from $(X, \mathbb{T})$. Then Y^* is an invariant closed subset of 2^X. It is easy to see that for any $y^* \in Y^*$, $\pi(y^*)$ is a singleton. Define $\tau : Y^* \to Y$ as such that $\tau(y^*) = \pi(x)$, $x \in y^*$. Then $\tau : (Y^*, \mathbb{T}) \to (Y, \mathbb{T})$ is a flow extension.

Also let $X^* = \{(x, y^*) \in X \times Y^* : x \in Y^*\}$. Then X^* is a closed invariant subset of $(X \times Y^*, \mathbb{T})$. Denote $\tau' : X^* \to X$ and $\pi' : X^* \to Y^*$ as the natural projections.

Proposition 3.5. *The following holds.*

1. $(X^*, \mathbb{T})$ *is a minimal flow and the following diagram*

$$\begin{array}{ccc} (X, \mathbb{T}) & \xleftarrow{\;\tau'\;} & (X^*, \mathbb{T}) \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi'} \\ (Y, \mathbb{T}) & \xleftarrow{\;\tau\;} & (Y^*, \mathbb{T}) \end{array}$$

commutes.

2. τ, τ' *are almost* 1-1 *extensions.*
3. π' *is an open extension.*

Proof. See Theorem 3.1 in [63] or Lemma 14.41 in [2]. □

3.3. **Entropies.** Let $(X,\mathbb{T})$ be a compact flow and consider the time-1 map $T: X \to X$: $x \mapsto x\cdot 1$. We denote the discrete flow induced by T simply by (X,T).

Let $\mathcal{B}_X$ denote the collection of all Borel subsets of X. A *cover* of X is a finite family of Borel subsets of X whose union is X. A *partition* of X is a cover of X whose elements are pairwise disjoint. We denote the set of partitions of X by $\mathcal{P}_X$ and the set of open covers of X by $\mathcal{C}_X$. An open cover $\mathcal{U}$ is said to be *finer* than $\mathcal{V}$ (denoted by $\mathcal{U} \succeq \mathcal{V}$) if each element of $\mathcal{U}$ is contained in some element of $\mathcal{V}$. Let $\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$. Given non-negative integers M, N and $\mathcal{U} \in \mathcal{C}_X$ or $\mathcal{P}_X$, we let $\mathcal{U}_M^N = \bigvee_{n=M}^N T^{-n}\mathcal{U}$. Also, given $\mathcal{U} \in \mathcal{C}_X$, we let $N(\mathcal{U})$ be the minimal cardinality among all cardinalities of sub-open-covers of $\mathcal{U}$ and let

$$H(\mathcal{U}) = \log N(\mathcal{U}).$$

Clearly, if there is another open cover $\mathcal{V} \succeq \mathcal{U}$, then $H(\mathcal{V}) \geq H(\mathcal{U})$. In fact, for any two covers $\mathcal{U}$, $\mathcal{V} \in \mathcal{C}_X$ we have $H(\mathcal{U} \vee \mathcal{V}) \leq H(\mathcal{U}) + H(\mathcal{V})$. Consequently, for any open cover $\mathcal{U} \in \mathcal{C}_X$, $a_n = H(\mathcal{U}_0^{n-1})$ is a bounded sub-additive sequence, i.e., $a_{n+m} \leq a_n + a_m$, $n, m \in \mathbb{N}$. Hence $\lim_{n\to\infty} \frac{a_n}{n}$ exists and equals $\inf_{n\geq 1} \frac{a_n}{n}$. This limit, denoted by $h_{\text{top}}(T,\mathcal{U})$, is called the *entropy of* $\mathcal{U}$. The *topological entropy* $h_{top}(X,T)$ *of* (X,T) is simply defined as

$$h_{\text{top}}(X,T) = \sup_{\mathcal{U}\in\mathcal{C}_X} h_{\text{top}}(T,\mathcal{U}),$$

and, the *topological entropy* $h_{top}(X,\mathbb{T})$ *of* $(X,\mathbb{T})$ is simply defined as $h_{\text{top}}(X,T)$.

Let $\mathcal{M}(X)$, $\mathcal{M}(X,T)$, and $\mathcal{M}^e(X,T)$, respectively, be the set of Borel probability measures on X, the set of invariant Borel probability measures on X, and the set of invariant ergodic measures on X, respectively. For given $\alpha \in \mathcal{P}_X$ and $\mu \in \mathcal{M}(X)$, define

$$H_\mu(\alpha) = \sum_{A\in\alpha} -\mu(A)\log\mu(A).$$

Now let $\mu \in \mathcal{M}(X,T)$. Then for a given $\alpha \in \mathcal{P}_X$, $H_\mu(\alpha_0^{n-1})$ is a non-negative sub-additive sequence. The *measure-theoretic entropy of* μ *relative to* α is defined by

$$h_\mu(T,\alpha) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}) = \inf_{n\geq 1} \frac{1}{n} H_\mu(\alpha_0^{n-1}),$$

and the *measure-theoretic entropy of* μ is defined by

$$h_\mu(X,T) = \sup_{\alpha\in\mathcal{P}_X} h_\mu(T,\alpha).$$

The classical variational principle of entropy says that

$$h_{\text{top}}(X,T) = \sup_{\mu\in\mathcal{M}(X,T)} h_\mu(X,T)$$

and the supremum can be attained by an invariant ergodic measure. We refer the readers to [9, 50, 65] for more information on the classical theory of measure-theoretic and topological entropies.

Given partitions $\alpha, \beta \in \mathcal{P}_X$, $\mu \in \mathcal{M}(X)$ and σ-algebra $\mathcal{A} \subseteq \mathcal{B}_X$, define

$$\begin{aligned} &H_\mu(\alpha|\beta) = H_\mu(\alpha\vee\beta) - H_\mu(\beta), \\ &H_\mu(\alpha|\mathcal{A}) = \textstyle\sum_{A\in\alpha}\int_X -\mathbb{E}(1_A|\mathcal{A})\log\mathbb{E}(1_A|\mathcal{A})d\mu, \\ &H_\mu(\alpha|\beta\vee\mathcal{A}) = H_\mu(\alpha\vee\beta|\mathcal{A}) - H_\mu(\beta|\mathcal{A}), \end{aligned}$$

where $\mathbb{E}(1_A|\mathcal{A})$ is the expectation of 1_A with respect to $\mathcal{A}$. Then $H_\mu(\alpha|\beta)$ (resp. $H_\mu(\alpha|\mathcal{A})$) increases with respect to α and decreases with respect to β (resp. $\mathcal{A}$).

Let $\mu \in \mathcal{M}(X,T)$ and $\mathcal{A}$ be an invariant measurable σ-algebra of X. It is not hard to see that for a given $\alpha \in \mathcal{P}_X$, $H_\mu(\alpha_0^{n-1}|\mathcal{A})$ is a bounded sub-additive sequence. The *measure-theoretic conditional entropy of* α *with respect to* $\mathcal{A}$ is defined by

$$h_\mu(T,\alpha|\mathcal{A}) = \lim_{n\to\infty}\frac{1}{n}H_\mu(\alpha_0^{n-1}|\mathcal{A}) = \inf_{n\geq 1}\frac{1}{n}H_\mu(\alpha_0^{n-1}|\mathcal{A}),$$

and the *measure-theoretic conditional entropy of* (X,T,μ) *with respect to* $\mathcal{A}$ is defined by

$$h_\mu(X,T|\mathcal{A}) = \sup_{\alpha\in\mathcal{P}_X} h_\mu(T,\alpha|\mathcal{A}).$$

It is easy to see that $h_\mu(T,\alpha|\mathcal{A}) = H_\mu(\alpha|\bigvee_{i=1}^\infty(T^{-i}\alpha)\vee\mathcal{A})$.

Let $\pi:(X,T)\to(Y,S)$ be an extension between compact discrete flows. For each $\alpha\in\mathcal{P}_X$, the *measure-theoretic conditional entropy of* α *with respect to* (Y,S) is defined by

$$h_\mu(T,\alpha|Y) = h_\mu(T,\alpha|\pi^{-1}(\mathcal{B}_Y)) = \lim_{n\to\infty}\frac{1}{n}H_\mu(\alpha_0^{n-1}|\pi^{-1}(\mathcal{B}_Y)),$$

and the *measure-theoretic conditional entropy of* (X,T,μ) *with respect to* (Y,S) is defined by

$$h_\mu(X,T|Y) = \sup_{\alpha\in\mathcal{P}_X} h_\mu(T,\alpha|Y).$$

When (Y,S) is the trivial flow, the above coincides with the measure-theoretic entropy of (X,T) with respect to μ.

Let $\mu\in\mathcal{M}(X,T)$ and $\mathcal{A}$ be an invariant sub-σ-algebra of $\mathcal{B}_X$. The *relative Pinsker σ-algebra* $P_\mu(\mathcal{A})$ is defined as the smallest σ-algebra containing $\bigcup\{\xi\in\mathcal{P}_X : h_\mu(T,\xi|\mathcal{A})=0\}$. When $\mathcal{A}=\{\emptyset,X\}$, $P_\mu(\mathcal{A})$ coincides with P_μ – the classical Pinsker σ-algebra of the system. It is easy to see that $P_\mu(\mathcal{A})$ is invariant, $P_\mu(\mathcal{A})\supseteq P_\mu\vee\mathcal{A}$, and $P_\mu(\mathcal{A},k)=P_\mu(\mathcal{A})$ for any $k\in\mathbb{Z}\setminus\{0\}$, where $P_\mu(\mathcal{A},k)$ denotes the smallest σ-algebra containing $\bigcup\{\xi\in\mathcal{P}_X : h_\mu(T^k,\xi|\mathcal{A})=0\}$. Given $\alpha\in\mathcal{P}_X$, we let $\alpha^-=\bigvee_{n=1}^\infty T^{-i}\alpha$ and $\alpha^T=\bigvee_{n=-\infty}^{+\infty}T^{-i}\alpha$. Then a relative version of the classical Pinsker formula (see [17, 20, 50]) says that if $\alpha,\beta\in\mathcal{P}_X$, then

$$h_\mu(T,\alpha\vee\beta|\mathcal{A}) = h_\mu(T,\beta|\mathcal{A}) + H_\mu(\alpha|\beta^T\vee\alpha^-\vee\mathcal{A}). \tag{3.3}$$

In particular, when $\mathcal{A}$ is trivial, $h_\mu(T,\alpha\vee\beta)=h_\mu(T,\beta)+H_\mu(\alpha|\beta^T\vee\alpha^-)$.

Proposition 3.6. *Let μ and $\mathcal{A}$ be given as above. Then for each $\xi\in\mathcal{P}_X$,*

$$h_\mu(T,\xi|P_\mu(\mathcal{A})) = H_\mu(\xi|\xi^-\vee P_\mu(\mathcal{A})) = H_\mu(\xi|\xi^-\vee\mathcal{A}) = h_\mu(T,\xi|\mathcal{A}).$$

Proof. For any $\alpha\in P_X$ and any invariant sub-σ-algebra $\mathcal{C}$ of $\mathcal{B}_X$, it is easy to see that $h_\mu(\alpha,T|\mathcal{C}) = H_\mu(\alpha|\alpha^-\vee\mathcal{C})$. Hence for each $\xi\in\mathcal{P}_X$, we have

$$h_\mu(T,\xi|P_\mu(\mathcal{A})) = H_\mu(\xi|\xi^-\vee P_\mu(\mathcal{A})) \text{ and } H_\mu(\xi|\xi^-\vee\mathcal{A}) = h_\mu(T,\xi|\mathcal{A}).$$

Now we fix $\xi\in\mathcal{P}_X$ and let $\eta\subset P_\mu(\mathcal{A})$ be any finite measurable partition. It follows from (3.3) that

$$\begin{aligned}
&H_\mu(\xi|\xi^-\vee\mathcal{A}) + H_\mu(\eta|\eta^-\vee\xi^T\vee\mathcal{A})\\
&= h_\mu(T,\xi|\mathcal{A}) + H_\mu(\eta|\xi^T\vee\eta^-\vee\mathcal{A})\\
&= h_\mu(T,\xi\vee\eta|\mathcal{A}) = H_\mu(\xi\vee\eta|\xi^-\vee\eta^-\vee\mathcal{A})\\
&= H_\mu(\eta|\xi^-\vee\eta^-\vee\mathcal{A}\vee\xi) + H_\mu(\xi|\xi^-\vee\eta^-\vee\mathcal{A}).
\end{aligned} \tag{3.4}$$

Since $\eta\subset P_\mu(\mathcal{A})$, we have $H_\mu(\eta|\eta^-\vee\mathcal{A}) = h_\mu(T,\eta|\mathcal{A})=0$,

$$H_\mu(\eta|\eta^-\vee\xi^T\vee\mathcal{A})=0, \quad \text{and } H_\mu(\eta|\eta^-\vee\xi^-\vee\mathcal{A}\vee\xi)=0. \tag{3.5}$$

Combining (3.4) and (3.5), we have

$$H_\mu(\xi|\xi^-\vee\eta^-\vee\mathcal{A}) = H_\mu(\xi|\xi^-\vee\mathcal{A}). \tag{3.6}$$

Let $\eta_n\subset P_\mu(\mathcal{A})$ be an increasing sequence of finite measurable partitions of X such that $\bigvee_{n=1}^\infty\eta_n = P_\mu(\mathcal{A})$ (mod μ). It follows from (3.6) that $H_\mu(\xi|\xi^-\vee P_\mu(\mathcal{A})) = H_\mu(\xi|\xi^-\vee\mathcal{A})$. □

3.4. **Measure-theoretic extensions.** Let $(X,\mathcal{B},\mu)$ be a standard Borel space, μ be a regular probability measure on X, and $T : X \to X$ be a measurable transformation. The quadruple $(X,\mathcal{B},\mu,T)$ is said to be a *metric dynamical system* (MDS for short) if T is measure preserving, i.e., $\mu(B) = \mu(T^{-1}B)$ for all $B \in \mathcal{B}$. If, in addition, T is bijective and T^{-1} is also measure-preserving, then $(X,\mathcal{B},\mu,T)$ is said to be *invertible*. In the following, a MDS is always assumed to be invertible. A MDS $(X,\mathcal{B},\mu)$ is said to be *ergodic* if whenever $A \in \mathcal{B}$ is such that $\mu(A\Delta T^{-1}A) = 0$ then either $\mu(A) = 0$ or $\mu(A) = 1$.

Let $(X,\mathcal{B},\mu,T)$ be a MDS and $(Y,\mathcal{C},\nu,S)$ be a *measure-theoretic factor* of $(X,\mathcal{B},\mu,T)$, i.e., there exists a measure-preserving map $\pi : X \to Y$, called a measure-theoretic *factor map* or *extension*, such that $\pi\circ T = S\circ\pi$ μ-a.e.. It is well-known that μ admits a ν-a.s. unique disintegeration $\mu = \int_Y \mu_y d\nu(y)$ over $(Y,\mathcal{C},\nu,S)$ ([15, Proposition 5.9]), where μ_y, $y \in Y$, are Borel probability measures on X satisfying

$$\mu_{Sy} = T\mu_y,\ \nu\text{-a.e. } y \in Y. \tag{3.7}$$

For each $i = 1, 2, \cdots, n$, let $\pi_i : (X_i,\mathcal{B}_i,\mu_i,T_i) \to (Y,\mathcal{C},\nu,S)$ be a factor map between MDSs and $\mu_i = \int_Y \mu_{i,y} d\nu(y)$ be the disintegeration of μ_i over $(Y,\mathcal{C},\nu,S)$. Define

$$\mu_1 \times_Y \mu_2 \times_Y \cdots \times_Y \mu_n = \int_Y \mu_{1,y} \times \mu_{2,y} \times \cdots \times \mu_{n,y}\, d\nu(y). \tag{3.8}$$

Then by (3.7), $T_1 \times T_2 \times \cdots \times T_n$ preserves the measure $\mu_1 \times_Y \mu_2 \times_Y \cdots \times_Y \mu_n$. The MDS $(X_1 \times X_2 \times \cdots \times X_n, \mathcal{B}_1 \times \mathcal{B}_2 \times \cdots \times \mathcal{B}_n, \mu_1 \times_Y \mu_2 \times_Y \cdots \times_Y \mu_n, T_1 \times T_2 \times \cdots \times T_n)$ is called the *product* of $(X_i,\mathcal{B}_i,\mu_i,T_i)$, $i = 1, 2, \cdots, n$, relative to $(Y,\mathcal{C},\nu,S)$.

Let $\pi : (X,\mathcal{B},\mu,T) \to (Y,\mathcal{C},\nu,S)$ be a factor map between ergodic MDSs and let $\mu = \int_Y \mu_y d\nu(y)$ be the disintegeration of μ over $(Y,\mathcal{C},\nu,S)$. π is said to be *relatively weakly mixing* if the MDS $(X \times X, \mathcal{B}\times\mathcal{B}, \mu\times_Y\mu, T\times T)$ is ergodic, and is said to be *compact* if there exists a dense set $\mathcal{F}$ of functions in $L^2(X,\mathcal{B},\mu)$ with the following properties: for any $f \in \mathcal{F}$ and $\delta > 0$, there exists a finite set of functions $g_1, g_2, \cdots, g_k \in L^2(X,\mathcal{B},\mu)$ such that $\min_{1\le i\le k} ||T^n f - g_j||_{L^2(\mu_y)} < \delta$, ν-a.e. $y \in Y$, for all $n \in \mathbb{Z}$.

Now consider a compact discrete flow (X,T). For $\mu \in \mathcal{M}(X,T)$, we let $P_\mu(\mathcal{A})$ be the relative Pinsker σ-algebra of invariant σ-algebra $\mathcal{A}$ of $\mathcal{B}_X$ and denote the completion of Borel σ-algebra $\mathcal{B}_X$ of X under μ by $\mathcal{B}_\mu$. Then $(X,\mathcal{B}_\mu,\mu,T)$ is a Lebesgue system. Let $(Z,\mathcal{Z},\eta,R)$ be the *Pinsker factor* of $(X,\mathcal{B}_\mu,\mu,T)$ and $\pi : (X,\mathcal{B}_\mu,\mu,T) \to (Z,\mathcal{Z},\eta,R)$ be the *measure-theoretic Pinsker factor map* with respect to $\mathcal{A}$, i.e., $\pi : X \to Z$ is measure-preserving, $\pi\circ T = R\circ\pi$ μ-a.e., and $\pi^{-1}\mathcal{Z} = P_\mu(\mathcal{A})$ (mod μ).

Let $\mu = \int_Z \mu_z d\eta(z)$ be the disintegration of μ over $(Z,\mathcal{Z},\eta,R)$. Then for each integer $n \ge 2$, $\lambda_n^{\mathcal{A}}(\mu) = \int_Z \mu_z^{(n)} d\eta(z)$ is a $T^{(n)}$-invariant measure on $X^{(n)}$, where

$$\mu_z^{(n)} = \underbrace{\mu_z \times \mu_z \times \cdots \times \mu_z}_{n},\ X^{(n)} = \underbrace{X \times X \times \cdots \times X}_{n},\ T^{(n)} = \underbrace{T \times T \times \cdots \times T}_{n}.$$

Moreover, it follows from basic properties of disintegration ([15, 54]) that for any $A_1, A_2, \cdots, A_n \in \mathcal{B}_\mu$ and $\alpha \in \mathcal{P}_X$,

$$\lambda_n^{\mathcal{A}}(\mu)(\Pi_{i=1}^n A_i) = \int_X \Pi_{i=1}^n \mathbb{E}(1_A|P_\mu(\mathcal{A}))(x) d\mu(x) \tag{3.9}$$

and

$$\begin{aligned} H_\mu(\alpha|P_\mu(\mathcal{A})) &= \textstyle\int_X \sum_{A\in\alpha} -\mathbb{E}(1_A|P_\mu(\mathcal{A})(x) \log \mathbb{E}(1_A|P_\mu(\mathcal{A})(x) d\mu(x) \\ &= \textstyle\int_X \sum_{A\in\alpha} -\mu_{\pi(x)}(A) \log \mu_{\pi(x)}(A) d\mu(x) \\ &= \textstyle\int_Z (\int_X \sum_{A\in\alpha} -\mu_{\pi(x)}(A) \log \mu_{\pi(x)}(A) d\mu_z(x))\, d\eta(z) \\ &= \textstyle\int_Z H_{\mu_z}(\alpha) d\eta(z). \end{aligned} \tag{3.10}$$

The following result should be well-known. As we are not aware of a suitable reference for it, a proof of the result is given for the sake of completeness.

Theorem 3.3. *Let (X,T) be a compact discrete flow, $\mu\in\mathcal{M}^e(X,T)$, and $\mathcal{A}$ be a T-invariant σ-algebra of $\mathcal{B}_X$. Let $\pi:(X,\mathcal{B}_\mu,\mu,T)\to(Z,\mathcal{Z},\eta,R)$ be the measure-theoretic Pinsker factor map with respect to $\mathcal{A}$ and $\mu=\int_Z\mu_z d\eta(z)$ be the disintegration of μ over $(Z,\mathcal{Z},\eta,R)$. If $h_\mu(X,T|\mathcal{A})>0$, then*

(1) *μ_z is non-atomic for η-a.e $z\in Z$;*
(2) *$\lambda_n^{\mathcal{A}}(\mu)$ is a $T^{(n)}$-invariant ergodic measure on $X^{(n)}$.*

Proof. (1) If (1) is not true, then the ergodicity of μ implies that there exists a positive integer k such that μ_z is purely atomic with exactly k points in its support for η-a.e. $z\in Z$. Hence for each $\beta\in\mathcal{P}_X$ and η-a.e. $z\in Z$,

$$H_{\mu_z}(\beta)\le\log k. \tag{3.11}$$

Now for any $\alpha\in P_X$, we have by Proposition 3.6, (3.10), and (3.11) that

$$\begin{aligned} h_\mu(T,\alpha|\mathcal{A}) &= h_\mu(T,\alpha|P_\mu(\mathcal{A}))=\lim_{n\to\infty}\tfrac{1}{n}H_\mu(\textstyle\bigvee_{i=0}^{n-1}T^{-i}\alpha|P_\mu(\mathcal{A}))\\ &=\lim_{n\to\infty}\tfrac{1}{n}\textstyle\int_Z H_{\mu_z}(\bigvee_{i=0}^{n-1}T^{-i}\alpha)d\eta(z)\le\lim_{n\to\infty}\tfrac{1}{n}\int_Z\log k\,d\eta(z)=0.\end{aligned}$$

Since α is arbitrary, $h_\mu(X,T|\mathcal{A})=0$, a contradiction.

(2) We first claim that $\pi:(X,\mathcal{B}_\mu,\mu,T)\to(Z,\mathcal{Z},\eta,T)$ is a relatively weakly mixing extension. If not, then by a classical result of Furstenberg and Zimmer ([15, 68, 69]) there exist $(Y,\mathcal{C},\nu,S)$ and factor maps $\pi_1:X\to Y$, $\pi_2:Y\to Z$ such that $\pi=\pi_2\pi_1$ and π_2 is a non-trivial compact extension. We note that $\pi_1^{-1}\pi_2^{-1}\mathcal{Z}=P_\mu(\mathcal{A})$. Now, for any $A\in\mathcal{C}$, we have by Proposition 3.6 that

$$\begin{aligned} &h_\mu(T,\{\pi_1^{-1}A,\pi_1^{-1}(Y\setminus A)\}|\mathcal{A})\\ &=h_\mu(T,\{\pi_1^{-1}A,\pi_1^{-1}(Y\setminus A)\}|P_\mu(\mathcal{A}))\\ &=h_\mu(T,\{\pi_1^{-1}A,\pi_1^{-1}(Y\setminus A)\}|\pi_1^{-1}\pi_2^{-1}\mathcal{Z})\\ &=h_\nu(S,\{A,Y\setminus A\}|\pi_2^{-1}\mathcal{Z}).\end{aligned}$$

Since π_2 is a compact extension, the conditional sequential entropy characterization of compact extensions ([25]) implies that $h_\mu(T,\{\pi_1^{-1}A,\pi_1^{-1}(Y\setminus A)\}|\mathcal{A})=0$. This shows that $\pi_1^{-1}A\in P_\mu(\mathcal{A})$. Since A is arbitrary, $\pi_1^{-1}\mathcal{C}\subseteq P_\mu(\mathcal{A})$ (modμ), and moreover, $P_\mu(\mathcal{A})=\pi_1^{-1}\pi_2^{-1}\mathcal{Z}\subseteq\pi_1^{-1}\mathcal{C}\subseteq P_\mu(\mathcal{A})$ $(\mathrm{mod}\,\mu)$. Hence $\pi_1^{-1}\pi_2^{-1}\mathcal{Z}=\pi_1^{-1}\mathcal{C}=P_\mu(\mathcal{A})$ $(\mathrm{mod}\,\mu)$, i.e., $\pi_2^{-1}\mathcal{Z}=\mathcal{C}$ $(\mathrm{mod}\,\nu)$. This shows that π_2 is an isomorphism, a contradiction to the fact that π_2 is a non-trivial compact extension. Hence $\pi:(X,\mathcal{B}_\mu,\mu,T)\to(Z,\mathcal{Z},\eta,T)$ is a relatively weakly mixing extension.

Since $\lambda_n^{\mathcal{A}}(\mu)=\underbrace{\mu\times_Z\mu\times_Z\cdots\times_Z\mu}_{n}$ and μ is ergodic, we have by Proposition 6.3 in [15] that $(X^{(n)},\mathcal{B}^{(n)},\mu^{(n)},T^{(n)})$ is ergodic, where $\mathcal{B}^{(n)}=\underbrace{\mathcal{B}\times\mathcal{B}\times\cdots\times\mathcal{B}}_{n}$. □

4. Entropy and ordering

The main purpose of this section is to prove Theorem 1. While Theorem 1 should more or less follow from an entropy inequality in [8], we will prove a general entropy preservation result (Theorem 4.1 below) under an ordering condition which not only implies Theorem 1 but also has the advantage of treating other zero entropy problems. For instance, applying our entropy preservation result, one can similarly show that all almost automorphic minimal sets obtained in [60, 61] for an almost periodically forced monotone system admit zero topological entropy.

4.1. **Entropy preservation.** Given a compact discrete flow (X,T), a finite subset A of X is called a *full scrambled set* if for each map $f:A\to A$ there exists an infinite sequence $\{n_i\}\subset\mathbb{N}$ such that $\lim_{i\to\infty}T^{n_i}x=f(x)$ for any $x\in A$.

Lemma 4.1. *Let $\pi:(X,T)\to(Y,S)$ be an extension between compact discrete flows and μ be an ergodic measure of (X,T). If $h_\mu(X,T|Y)>0$, then for each integer $n\ge 2$ there exist $y\in Y$ and $x_1,x_2,\cdots,x_n\in\pi^{-1}(y)$ such that $x_1,x_2,\cdots,x_n$ are pairwise different and $\{x_1,x_2,\cdots,x_n\}$ is a full scrambled set of (X,T).*

Proof. We follow the arguments in the proof of Theorem 2.1 in [7].

Let $(Z, \mathcal{Z}, \eta, R)$ be the Pinsker factor of $(X, \mathcal{B}_\mu, \mu, T)$ with respect to $\pi^{-1}\mathcal{B}_Y$ and $\mu = \int_Z \mu_z d\eta(z)$ be the disintegration of μ over $(Z, \mathcal{Z}, \eta, R)$. For each integer $n \geq 2$, let $\lambda_n(\mu) = \int_Z \mu_z^{(n)} d\eta(z)$ and $W^n = \text{supp}(\lambda_n(\mu))$. Since μ is ergodic and $h_\mu(X, T|Y) > 0$, we have by Proposition 3.3 that μ_z is non-atomic for η-a.e. $z \in Z$ and $\lambda_n(\mu)$ is an ergodic measure. Hence $(W^n, T^{(n)})$ is transitive, i.e., it contains a transitive point – point whose orbit is dense. Let W^n_{trans} denote the set of all transitive points of $(W^n, T^{(n)})$ and G_n be the set of generic points in W^n with respect to $\lambda_n(\mu)$, i.e., $G_n = \{w \in W^n : \frac{1}{N}\sum_{i=0}^{N-1} \delta_{T^i \times T^i \cdots \times T^i(w)} \to \lambda_n(\mu) \text{ as } N \to \infty\}$. Since $\lambda_n(\mu)$ is ergodic, we have $G_n \subset W^n_{\text{Trans}}$. Then by Birkhoff ergodic theorem,

$$1 = \lambda_n(\mu)(G_n) = \int_Z \mu_z^{(n)}(G_n) d\eta(z).$$

It follows that there exists a subset Z_n of Z with $\eta(Z_n) = 1$ such that $\mu_z^{(n)}(G_n) = 1$ and μ_z is non-atomic for all $z \in Z_n$. For each $z \in Z_n$, let $S_z = \text{supp}(\mu_z)$. Then S_z is a closed subset of X without isolated points and

$$G_n \cap S_z^{(n)} \subset W^n_{\text{trans}} \cap S_z^{(n)} =: L_z^n, \text{ where } S_z^{(n)} = \underbrace{S_z \times S_z \times \cdots \times S_z}_{n}.$$

Since $\mu_z^{(n)}(G_n \cap S_z^{(n)}) = 1$, $G_n \cap S_z^{(n)}$ is a dense subset of $S_z^{(n)}$. This shows that for each $z \in Z_n$,

$$S_z^{(n)} = \text{cl}(L_z^n) \subseteq W^n. \tag{4.1}$$

Now, fix $z \in Z_n$ and take $(x_1, x_2, \cdots, x_n) \in L_z^n \subset W^n_{\text{trans}}$. By (4.1) and the fact that S_z is not a singleton, $x_1, x_2, \cdots, x_n$ are pairwise different. Let $A = \{x_1, x_2, \cdots, x_n\}$ and $f : A \to A$ be any map. We have by (4.1) that $(f(x_1), f(x_2), \cdots, f(x_n)) \in S_z^{(n)} \subset W^n$. Since $(x_1, x_2, \cdots, x_n) \in W^n_{\text{trans}}$, there exists an infinite sequence $\{n_i\} \subset \mathbb{N}$ such that

$$\lim_{i\to\infty} (T^{n_i}x_1, T^{n_i}x_2, \cdots, T^{n_i}x_n) = (f(x_1), f(x_2) \cdots, f(x_n)),$$

i.e., $\lim_{i\to\infty} T^{n_i}x = f(x)$ for all $x \in A$. This shows that A is a full scrambled set of (X, T).

It remains to show that there exists $y \in Y$ such that $\{x_1, x_2, \cdots, x_n\} \subseteq \pi^{-1}(y)$. If this is not true, then there exist two disjoint nonempty open subsets U_1 and U_2 of Y such that $\{\pi(x_i)\}_{i=1}^n \subset U_1 \cup U_2$ and $\{\pi(x_i)\}_{i=1}^n \cap U_j \neq \emptyset, j = 1, 2$. For each $i = 1, 2, \cdots, n$, take $s(i) \in \{1, 2\}$ such that $x_i \in \pi^{-1}U_{s(i)}$. Since $(x_1, x_2, \cdots, x_n) \in \text{supp}(\lambda_n(\mu))$ and $\Pi_{i=1}^n \pi^{-1}U_{s(i)}$ is an open neighborhood of $(x_1, x_2, \cdots, x_n)$, we have

$$\lambda_n(\mu)(\Pi_{i=1}^n \pi^{-1}U_{s(i)}) > 0.$$

But by (3.9) and the facts that $\pi^{-1}U_{s(i)} \in P_\mu(\pi^{-1}\mathcal{B}_Y)$, $\pi^{-1}U_1 \cap \pi^{-1}U_2 = \emptyset$, and $\{s(1), \cdots, s(n)\} = \{1, 2\}$, we also have

$$\begin{aligned}\lambda_n(\mu)(\Pi_{i=1}^n \pi^{-1}U_{s(i)}) &= \textstyle\int_X \Pi_{i=1}^n \mathbb{E}(1_{\pi^{-1}U_{s(i)}} | P_\mu(\pi^{-1}\mathcal{B}_Y))(x) d\mu(x) \\ &= \textstyle\int_X \Pi_{i=1}^n 1_{\pi^{-1}U_{s(i)}}(x) d\mu(x) = 0,\end{aligned}$$

which is a contradiction. □

Lemma 4.2. *Let $\pi : (X, T) \to (Y, S)$ be an extension between compact discrete flows. If $h_{top}(X, T) > h_{top}(Y, S)$, then for each integer $n \geq 2$ there exist $y \in Y$ and $x_1, x_2, \cdots, x_n \in \pi^{-1}(y)$ such that $x_1, x_2, \cdots, x_n$ are pairwise different and $\{x_1, x_2, \cdots, x_n\}$ is a full scrambled set of (X, T).*

Proof. Since $h_{\text{top}}(X, T) > h_{\text{top}}(Y, S)$, $h_{\text{top}}(Y, S) < +\infty$. By the variational principle of entropy there exists an ergodic measure μ of (X, T) with $h_\mu(X, T) > h_{\text{top}}(Y, S)$. Hence $\nu = \phi\mu \in \mathcal{M}^e(Y, S)$ and $h_\mu(X, T) > h_\nu(Y, S)$.

Let $\{\alpha_i\}_{i=1}^\infty \subset \mathcal{P}_X$, $\{\beta_j\}_{j=1}^\infty \subset \mathcal{P}_Y$ be such that $\alpha_1 \preceq \alpha_2 \preceq \cdots$, $\bigvee_{i=1}^\infty \alpha_i = \mathcal{B}_X \pmod{\mu}$, $\beta_1 \preceq \beta_2 \preceq \cdots$, and $\bigvee_{j=1}^\infty \beta_j = \mathcal{B}_Y \pmod{\nu}$. We have by (3.3) that

$$h_\mu(T, \alpha_i \vee \pi^{-1}\beta_j) = h_\mu(T, \pi^{-1}\beta_j) + H_\mu(\alpha_i | (\pi^{-1}\beta_j)^T \vee (\alpha_i)^-). \tag{4.2}$$

Since $h_\mu(T,\pi^{-1}\beta_j)=h_\nu(S,\beta_j)$, (4.2) yields that

$$H_\mu(\alpha_i|(\pi^{-1}\beta_j)^T\vee(\alpha_i)^-)\geq h_\mu(T,\alpha_i)-h_\nu(\beta_j,S)\geq h_\mu(T,\alpha_i)-h_\nu(Y,S). \tag{4.3}$$

Note that $(\pi^{-1}\beta_j)^T\vee(\alpha_i)^-\nearrow\pi^{-1}\mathcal{B}_Y\vee(\alpha_i)^-$ as $j\to\infty$. Taking $j\to\infty$ in (4.3), we have by Matingale theorem that $H_\mu(\alpha_i|\pi^{-1}\mathcal{B}_Y\vee(\alpha_i)^-)\geq h_\mu(T,\alpha_i)-h_\nu(Y,S)$. Hence

$$h_\mu(X,T|Y)\geq h_\mu(\alpha_i,T|Y)=H_\mu(\alpha_i|\pi^{-1}\mathcal{B}_Y\vee(\alpha_i)^-)\geq h_\mu(T,\alpha_i)-h_\nu(Y,S). \tag{4.4}$$

Taking $i\to\infty$ in (4.4), we have by Kolmogorov-Sinai theorem that

$$h_\mu(X,T|Y)\geq h_\mu(X,T)-h_\nu(Y,S)>0.$$

The lemma now follows from Lemma 4.1. □

When $h_\nu(Y,S)<+\infty$, it can be shown that $h_\mu(X,T|Y)=h_\mu(X,T)-h_\nu(Y,S)$. When $h_\nu(Y,S)=+\infty$, we note that $h_\mu(X,T)=h_\nu(Y,S)=+\infty$. But in this case, it can also happen that $h_\mu(X,T|Y)>0$. Therefore the condition $h_\mu(X,T|Y)>0$ in Lemma 4.1 is more general than the condition $h_\mu(X,T)>h_\nu(Y,S)$ in Lemma 4.2.

Given an integer $n\geq 2$, we denote by $Per_n(X)$ the set of all coordinate permutations on $X^{(n)}$. An *n-partial order relation* R on X is a subset of $X^{(n)}$ such that

a) $\tau_0(R)\cap R=\emptyset$ for some $\tau_0\in Per_n(X)$;
b) R is *essentially closed*, i.e., for any $\{w_k\}_{k=1}^\infty\subset R$ and $w=(x_1,x_2,\cdots,x_n)\in X^{(n)}$, if $\lim_{k\to\infty}w_k=w$ and $x_i\neq x_j$ for $1\leq i\neq j\leq n$, then $w\in R$.

We say that a compact flow $(X,\mathbb{T})=(X,\{\Pi_t\}_{t\in\mathbb{T}})$ *preserves an n-partial order* R if

$$\underbrace{\Pi_t\times\Pi_t\times\cdots\times\Pi_t}_{n}(R)\subseteq R$$

for all $t>0$. Also, we refer the relation $PR_e^n(X,\mathbb{T})=\{(x_1,x_2,\cdots,x_n):x_i\neq x_j,\ 1\leq i<j\leq n,\text{ and }\liminf_{t\to+\infty}\operatorname{diam}(\{x_1\cdot t,x_2\cdot t,\cdots,x_n\cdot t\})=0\}$ as *the proper n-proximal relation of* $(X,\mathbb{T})$.

Theorem 4.1. *Let $\pi:(X,\mathbb{T})\to(Y,\mathbb{T})$ be an extension between compact flows. If for some integer $n\geq 2$, the flow $(X,\mathbb{T})$ preserves an n-partial order relation R on X and $PR_e^n(X,\mathbb{T})\cap R_\pi^n\subseteq\bigcup_{\tau\in Per_n(X)}\tau(R)$, where $R_\pi^n=\{(x_1,x_2,\cdots,x_n)\in X^{(n)}:\pi(x_1)=\cdots=\pi(x_n)\}$, then $h_{top}(X,\mathbb{T})=h_{top}(Y,\mathbb{T})$.*

Proof. Let T and S be the time-1 maps of $(X,\mathbb{T})$ and $(Y,\mathbb{T})$ respectively. If $h_{\rm top}(X,T)\neq h_{\rm top}(Y,S)$, then $h_{\rm top}(X,T)>h_{\rm top}(Y,S)$. It follows from Lemma 4.2 that there exist $y\in Y$ and $x_1,x_2,\cdots,x_n\in\pi^{-1}(y)$ such that $x_i\neq x_j$, $1\leq i\neq j\leq n$, and $\{x_1,x_2,\cdots,x_n\}$ is a full scrambled set of (X,T). Clearly, $(x_1,x_2,\cdots,x_n)\in PR_n^e(X,\mathbb{T})\cap R_\pi\subseteq\bigcup_{\tau\in Per_n(X)}\tau(R)$. Without loss of generality, we assume that $(x_1,x_2,\cdots,x_n)\in R$. Since R is an n-partial relation, there exists $\tau_0\in Per_n(X)$ such that $\tau_0(R)\cap R=\emptyset$. Denote $\tau_0(x_1,x_2,\cdots,x_n)=(x_1',x_2',\cdots,x_n')$. Then $x_i'\neq x_j'$ for all $1\leq i\neq j\leq n$. Since $\{x_1,x_2,\cdots,x_n\}$ is a full scrambled set of (X,T), there exists a sequence $\{n_i\}\subset\mathbb{N}$ such that

$$\lim_{i\to\infty}(T^{n_i}x_1,T^{n_i}x_2,\cdots,T^{n_i}x_n)=(x_1',x_2',\cdots,x_n').$$

Note that $(T^{n_i}x_1,T^{n_i}x_2,\cdots,T^{n_i}x_n)\in R$ and $x_i'\neq x_j'$ for all $1\leq i\neq j\leq n$. We have by the essential closeness of R that $(x_1',x_2',\cdots,x_n')\in R$. Now $(x_1',x_2',\cdots,x_n')\in\tau_0(R)$, a contradiction to the fact that $\tau_0(R)\cap R=\emptyset$. Hence $h_{\rm top}(X,T)=h_{\rm top}(Y,S)$. □

Corollary 4.1. *Let $(X,\mathbb{T})$ be a compact flow which preserves an n-partial order relation R on X for some integer $n\geq 2$. If $PR_e^n(X,\mathbb{T})\subseteq\bigcup_{\tau\in Per_n(X)}\tau(R)$, then $h_{top}(X,\mathbb{T})=0$.*

Proof. It follows from Theorem 4.1 by taking $(Y,\mathbb{T})$ as the trivial flow. □

4.2. **Zero entropy of APCFs.** The follows theorem immediately implies Theorem 1.

Theorem 4.2. *For a SPCF* $(S^1 \times Y, \mathbb{T})$, $h_{top}(S^1 \times Y, \mathbb{T}) = h_{top}(Y, \mathbb{T})$.

Proof. Let $\pi : S^1 \times Y \to Y$ be the natural projection. Clearly, $\pi : (S^1 \times Y, \mathbb{T}) \to (Y, \mathbb{T})$ is a flow extension. Consider

$$R = \{((e^{2\pi\phi_1 i}, y), (e^{2\pi\phi_2 i}, y), (e^{2\pi\phi_3 i}, y)) : y \in Y \text{ and } \phi_1 < \phi_2 < \phi_3 < 1 + \phi_1\}.$$

It is clear that R is a 3-partial relation on $X = S^1 \times Y$ which is preserved by $(X, \mathbb{T})$ and $PR_e^3(X, \mathbb{T}) \cap R_\pi^3 \subseteq \bigcup_{\tau \in Per_3(X)} \tau(R)$. It follows from Theorem 4.1 that $h_{\rm top}(S^1 \times Y, \mathbb{T}) = h_{\rm top}(Y, \mathbb{T})$. □

5. Li-Yorke chaos and proximality

5.1. **General conditions on the existence of Li-Yorke chaos.** The following lemmas will be needed in the proof of our general result on Li-Yorke chaos of a proximal extension.

Lemma 5.1. *Let* X *and* Y *be compact metric spaces,* $\pi : X \to Y$ *be a semi-open, surjective, continuous map, and* K *be a residual subset of* X*. Then*

$$A = A_K = \{y \in Y : K \cap \pi^{-1}(y) \text{ is a residual subset of } \pi^{-1}(y)\}$$

is a residual subset of Y*.*

Proof. See Proposition 3.1 in [63]. □

Let X be a compact metric space. A subset $K \subseteq X$ is called a *Mycielski set* if it is a countable union of Cantor sets. The following result is a special case of Mycielski theorem ([48]).

Lemma 5.2. *Let* X *be a compact metric space with no isolated point. If* R *is a residual subset of* $X \times X$*, then there exists a Mycielski set* K *of* X *which is dense in* X *such that for any two distinct points* x, y *in* K*,* $(x, y) \in R$*.*

Proof. See Theorem 1 in [48] or Lemma 2.6 in [7]. □

The following result is more or less known for maps ([19]) but the proof does not automatically carry over to the case of $\mathbb{R}$-flows.

Theorem 5.1. *Let* $\pi : (X, \mathbb{T}) \to (Y, \mathbb{T})$ *be a proximal extension of minimal flows which is not almost* 1-1*. Then there exists a residual subset* Y_c *of* Y *such that each fiber* $\pi^{-1}(y)$*,* $y \in Y_c$*, admits an uncountable scrambled set. In particular,* $(X, \mathbb{T})$ *is Li-Yorke chaotic.*

Proof. Let Y_0, Y^*, X^* and π', τ, τ' be as in Proposition 3.5. Recall that π' is an open extension. Let $y^* \in Y^*$ and $x, x' \in y^*$.

Since $\pi(x) = \pi(x') = \tau(y^*)$ and π is a proximal extension, x, x' are proximal, so are $(x, y^*), (x', y^*)$. This shows that π' is a proximal extension as well. It follows from Proposition 3.3 that $(R_{\pi'}, \mathbb{T})$ is positively transitive. Thus $Tran^+(R_{\pi'})$ is a residual subset of $R_{\pi'}$. Since $\rho : R_{\pi'} \to Y^*$: $((x, y^*), (x', y^*)) \mapsto y^*$ is an open map, it follows from Lemma 5.1 that there is a residual subset $Y_c^* \subseteq Y^*$ such that for every point $y^* \in Y_c^*$, $Tran^+(R_{\pi'}) \cap ((\pi')^{-1}(y^*) \times (\pi')^{-1}(y^*))$ is a residual subset of $(\pi')^{-1}(y^*) \times (\pi')^{-1}(y^*)$. Let $Y_c = \tau(Y_c^*)$. Since by Proposition 3.4 τ is semi-open, Lemma 5.1 implies that Y_c is a residual set.

Fix $y_0 \in Y_c$ and let $y_0^* \in Y_c^*$ be such that $\tau(y_0^*) = y_0$, i.e., $y_0^* \subset \pi^{-1}(y_0)$. We first claim that the closed subset $(\pi')^{-1}(y_0^*) = \{(x, y_0^*) : x \in y_0^*\}$ has no isolated point. Suppose for contradiction that there exists $x \in y_0^*$ such that (x, y_0^*) is an isolated point of $(\pi')^{-1}(y_0^*)$. Obviously, $\{((x, y_0^*), (x, y_0^*))\}$ is a relatively open subset of $(\pi')^{-1}(y_0^*) \times (\pi')^{-1}(y_0^*)$, hence $((x, y_0^*), (x, y_0^*)) \in Trans^+(R_{\pi'})$ as $Trans^+(R_{\pi'}) \cap ((\pi')^{-1}(y_0^*) \times (\pi')^{-1}(y_0^*))$ is a residual subset of $(\pi')^{-1}(y_0^*) \times (\pi')^{-1}(y_0^*)$. It follows that $R_{\pi'} = \{(x^*, x^*) : x^* \in X^*\}$, i.e., π' is an isomorphism, and hence $\pi^{-1}(y^*)$ is a singleton for each $y^* \in Y^*$. In particular, $\pi^{-1}(y)$ is a singleton for each $y \in Y_0$, i.e., π is an almost 1-1 extension, a contradiction.

Next, by applying Lemma 5.2 with $R = Tran^+(R_{\pi'}) \cap ((\pi')^{-1}(y_0^*) \times (\pi')^{-1}(y_0^*))$, we obtain a Mycielski set $K^*_{y_0} \subset (\pi')^{-1}(y_0^*)$ such that for any two distinct points x^*, x_1^* in $K^*_{y_0}$, $(x^*, x_1^*) \in Tran^+(R_{\pi'})$.

Now let $K_{y_0} = \{x \in X : (x, y_0^*) \in K^*_{y_0}\}$. Since K_{y_0} and $K^*_{y_0}$ are homeomorphic, $K_{y_0} \subseteq \pi^{-1}(y_0)$ is also a Mycieski set hence it is uncountable.

It remains to show that K_{y_0} is a scrambled subset of $(X, \mathbb{T})$. Let x, x_1 be any two distinct points in K_{y_0}. We note that $((x, y_0^*), (x_1, y_0^*)) \in Tran^+(R_{\pi'})$. Since both $((x, y_0^*), (x, y_0^*))$ and $((x, y_0^*), (x_1, y_0^*))$ are in $R_{\pi'}$, there are positive increasing sequences $t_i \to +\infty$, $s_j \to +\infty$ such that

$$\lim_{i\to\infty}((x, y_0^*), (x_1, y_0^*)) \cdot t_i = ((x, y_0^*), (x_1, y_0^*)),$$
$$\lim_{j\to\infty}((x, y_0^*), (x_1, y_0^*)) \cdot s_j = ((x, y_0^*), (x, y_0^*)).$$

This implies that $\lim_{i\to\infty}(x, x_1) \cdot t_i = (x, x_1)$ and $\lim_{j\to\infty}(x, x_1) \cdot s_j = (x, x)$, i.e., $\{x, x_1\}$ is a Li-Yorke pair of $(X, \mathbb{T})$. Hence K_{y_0} is an uncountable scrambled set of X. This completes the proof. □

An extension $\pi : (X, \mathbb{T}) \to (Y, \mathbb{T})$ between compact flows is said to be *positively asymptotic* if for each $y \in Y$, any two points $x, x' \in \pi^{-1}(y)$ are positively asymptotic, i.e., $\lim_{t\to+\infty} d(x \cdot t, x' \cdot t) = 0$, where d is a compatible metric on X.

Corollary 5.1. *Let $\pi : (X, \mathbb{T}) \to (Y, \mathbb{T})$ be a positively asymptotic extension between minimal flows. Then π is an almost* 1-1 *extension.*

Proof. Obviously, π is a proximal extension. If π is not almost 1-1, then by Theorem 5.1 there exists an uncountable scrambled set $K_y \subset \pi^{-1}(y)$ for some $y \in Y$. In particular, there exist $x_1, x_2 \in K_y \subset \pi^{-1}(y)$ which are not positively asymptotic, a contradiction. □

5.2. **A strict dynamical dichotomy of minimal sets.** We now consider a SPCF $(S^1 \times Y, \mathbb{T}) = (S^1 \times Y, \{\Lambda_t\}_{t\in\mathbb{T}})$ in the form (1.1), i.e.,

$$\Lambda_t(s_0, y_0) = (\psi(s_0, y_0, t), y_0 \cdot t), \quad t \in \mathbb{T}.$$

We denote d_Y as a compatible metric on Y and $\pi : S^1 \times Y \to Y$ as the natural projection.

For $s_1 \neq s_2 \in S^1$, we denote $[s_1, s_2]$ as the closed arc from s_1 to s_2 oriented counter-clockwise, and let $(s_1, s_2) = [s_1, s_2] \setminus \{s_1, s_2\}$. We also denote $[s, s] = \{s\}$ for any $s \in S^1$. For a fixed $y_0 \in Y$, consider the family of maps $f_t : S^1 \to S^1$: $s \mapsto \psi(s, y_0, t)$, $t \in \mathbb{T}$. Then each f_t is an orientation preserving homeomorphism of S^1.

Lemma 5.3. *Let $s_1 \neq s_2 \in S^1$ and $\{t_j\} \subset \mathbb{T}$ be such that $\lim_{j\to\infty} f_{t_j}(s_i) = s_i'$ for some $s_i' \in S^1$, $i = 1, 2$. Then the following holds.*

1. *If $s_1' \neq s_2'$, then $\lim_{j\to\infty} f_{t_j}([s_1, s_2]) = [s_1', s_2']$ under the Hausdorff metric on 2^{S^1}.*
2. *If $s_1' = s_2' = s'$, then either $\lim_{j\to\infty} f_{t_j}([s_1, s_2]) = \{s'\}$ or $\lim_{j\to\infty} f_{t_j}([s_2, s_1]) = \{s'\}$ by taking subsequences if necessary.*
3. *If $s_1' = s_2' = s'$ and $\limsup_{j\to\infty} f_{t_j}([s_1, s_2]) \neq S^1$, then $\lim_{i\to\infty} f_{t_j}([s_1, s_2]) = \{s'\}$.*
4. *If $\lim_{t\to+\infty} |f_t(s_1) - f_t(s_2)| = 0$, then there exists $t_0 > 0$ such that, as $t \geq t_0$, $|f_t(s_1^*) - f_t(s_2^*)| \leq |f_t(s_1) - f_t(s_2)|$ for either all $s_1^*, s_2^* \in [s_1, s_2]$ or all $s_1^*, s_2^* \in [s_2, s_1]$. In particular, $\lim_{t\to+\infty} |f_t(s_1^*) - f_t(s_2^*)| = 0$ for either all $s_1^*, s_2^* \in [s_1, s_2]$ or all $s_1^*, s_2^* \in [s_2, s_1]$.*

Proof. (1), (2) and (3) are obvious.

(4) Denote $A_1 = [s_1, s_2]$ and $A_2 = [s_2, s_1]$. We let $0 < \epsilon_0 < \frac{\mathrm{diam}(S^1)}{3}$ be such that if $|s - s'| \leq \epsilon_0$ then $|\psi(s, y, r) - \psi(s', y, r)| \leq \frac{\mathrm{diam}(S^1)}{3}$ for all $y \in Y$ and $r \in [0, 1]$. We also let $t_0 > 0$ be such that $|f_t(s_1) - f_t(s_2)| \leq \epsilon_0$ for all $t \geq t_0$. Then for any $t \geq t_0$, there exists $i(t) = 1$ or 2 such that $\mathrm{diam}(f_t(A_{i(t)})) \leq |f_t(s_1) - f_t(s_2)| \leq \epsilon_0$.

Since $\mathrm{diam}(A_1 \cup A_2) = \mathrm{diam}(S^1)$, we have

$$\mathrm{diam}(f_t(S^1 \setminus A_{i(t)})) \geq \mathrm{diam}(S^1) - \epsilon_0 > \frac{2}{3}\mathrm{diam}(S^1)$$

for all $t \geq t_0$. In the following, we show that $i(t)$ equals a constant, say $i_0 = 1$ or 2, on $[t_0, +\infty) \cap \mathbb{T}$. If this is not true, then there exist $t_1 \in [t_0, +\infty) \cap \mathbb{T}$ and $r \in (0, 1] \cap \mathbb{T}$ such that $i(t_1) \neq i(t_1 + r)$. On one hand, since $i(t_1) \neq i(t_1 + r)$, we have

$$\mathrm{diam}(f_{t_1+r}(A_{i(t_1)})) \geq \mathrm{diam}(f_{t_1+r}(S^1 \setminus A_{i(t_1+r)})) \geq \mathrm{diam}(S^1) - \epsilon_0 > \frac{2}{3}\mathrm{diam}(S^1).$$

But on the other hand, since $\mathrm{diam}(f_{t_1}(A_{i(t_1)})) \leq \epsilon_0$, we have

$$\mathrm{diam}(f_{t_1+r}(A_{i(t_1)})) = \mathrm{diam}(\{\psi(s, y \cdot t_1, r) : s \in f_{t_1}(A_{i(t_1)})\}) \leq \frac{\mathrm{diam}(S^1)}{3}.$$

This is a contradiction.

Now for any $s_1^*, s_2^* \in A_{i_0}$ and $t \geq t_0$, we have $|f_t(s_1^*) - f_t(s_2^*)| \leq \mathrm{diam}(f_t(A_{i(t)})) \leq |f_t(s_1) - f_t(s_2)|$. □

We now assume that base flow $(Y, \mathbb{T})$ is minimal in the SPCF $(S^1 \times Y, \mathbb{T})$. Let X be a minimal set of $(S^1 \times Y, \mathbb{T})$, Y_0 be the set of continuity points of $\pi^{-1} : Y \to 2^X$: $y \mapsto \pi^{-1}(y)$, and $(X^*, \mathbb{T}), (Y^*, \mathbb{T}), \tau, \tau', \pi'$ be defined as in Proposition 3.5 with respect to the extension $\pi : (X, \mathbb{T}) \to (Y, \mathbb{T})$. Recall that Y_0 is an invariant residual subset of Y, $Y^* = \mathrm{cl}\{\pi^{-1}(y) : y \in Y_0\}$, $X^* = \{(x, y^*) \in X \times Y^* : x \in y^*\}$, $(X^*, \mathbb{T})$ is a minimal flow, $\tau : (Y^*, \mathbb{T}) \to (Y, \mathbb{T})$ and $\tau' : (X^*, \mathbb{T}) \to (X, \mathbb{T})$ are almost 1-1 extensions (hence $(Y^*, \mathbb{T})$ is point-distal if $(Y, \mathbb{T})$ is), and $\pi' : (X^*, \mathbb{T}) \to (Y^*, \mathbb{T})$ is an open extension.

Let $Z = S^1 \times Y^*$ and define the skew-product flow $(Z, \mathbb{T}) = (Z, \{\Pi_t^Z\}_{t \in \mathbb{T}})$ by

$$\Pi_t^Z(s, y^*) = (\psi(s, \tau(y^*), t), y^* \cdot t).$$

Consider $\rho : X^* \to Z$: $((s, \tau(y^*)), y^*) \mapsto (s, y^*)$ and let $Z^* = \rho(Z)$ endow with the metric

$$d_{Z^*}((s_1, y_1^*), (s_2, y_2^*)) = |s_1 - s_2| + d_{Y^*}(y_1^*, y_2^*), \ \ (s_1, y_1^*), (s_2, y_2^*) \in Z^*,$$

where d_{Y^*} is a compatible metric on Y^*. Since for any $((s, \tau(y^*)), y^*) \in X^*$ and $t \in \mathbb{T}$,

$$\begin{aligned} \rho(((s, \tau(y^*)), y^*) \cdot t) &= \rho(\Lambda_t(s, \tau(y^*)), y^* \cdot t) = \rho((\psi(s, \tau(y^*), t), \tau(y^*) \cdot t), y^* \cdot t) \\ &= (\psi(s, \tau(y^*), t), y^* \cdot t) = \Pi_t^Z(s, y^*) = \Pi_t^Z(\rho((s, \tau(y^*)), y^*)), \end{aligned}$$

we see that $\rho : (X^*, \mathbb{T}) \to (Z^*, \mathbb{T})$ is a flow isomorphism. Hence $(Z^*, \mathbb{T})$ is a minimal flow. Let $\pi^* : Z \to Y^*$ be the natural projection and denote $\pi_1 = \pi^*|_{Z^*}$.

Lemma 5.4. *The following diagram*

$$\begin{array}{ccccc} (X, \mathbb{T}) & \xleftarrow{\tau'} & (X^*, \mathbb{T}) & \xrightarrow{\rho} & (Z^*, \mathbb{T}) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi_1} \\ (Y, \mathbb{T}) & \xleftarrow{\tau} & (Y^*, \mathbb{T}) & = & (Y^*, \mathbb{T}) \end{array}$$

commutes, where τ, τ' are almost 1-1, π', π_1 are open, and ρ is 1-1.

Proof. Since π' is open, so is $\pi_1 = \pi' \circ \rho^{-1}$.

With Proposition 3.5, we only need to check the commutativity of the right half of the diagram. Let $((s, \tau(y^*)), y^*) \in X^*$. Then

$$\pi_1(\rho((s, \tau(y^*)), y^*)) = \pi_1(s, y^*) = y^* = \pi'((s, \tau(y^*)), y^*).$$

□

Proposition 5.1. *Suppose that the base flow $(Y, \mathbb{T})$ of the SPCF $(S^1 \times Y, \mathbb{T})$ is point-distal. If there exists a second category subset Y_u^* of Y^* such that for each $y^* \in Y_u^*$ there exists no uncountable scrambled set in $\pi_1^{-1}(y^*)$, then $(Z^*, \mathbb{T})$ is point-distal.*

Proof. Let $\mathcal{A}$ denote the collection of all proper, closed, sub-arcs of S^1 with end points being roots of unity and consider the set

$$\mathcal{D} = \{(I_1, I_2) : I_1, I_2 \in \mathcal{A} \text{ and } I_2 \subset \text{int}(I_1)\}.$$

It is clear that $\mathcal{D}$ is countable.

Since $(Y^*, \mathbb{T})$ is point-distal, the set Y_d^* of distal points of Y^* is a residual subset. Clearly, $Y_w^* = Y_u^* \cap Y_d^*$ is a second category subset of Y^*.

For each $y^* \in Y^*$, we let $S(y^*) = \{s \in S^1 : (s, y^*) \in Z^*\}$. Then $S(y^*)$ is a closed subset of S^1. Since π_1 is open, the map $\theta : Y^* \to 2^{S^1}$: $y^* \mapsto S(y^*)$ is continuous. Define

$$Y_i^* = \{y^* \in Y_w^* : S(y^*) \text{ contains an isolated point}\} \text{ and } Y_p^* = Y_w^* \setminus Y_i^*.$$

Since Y_w^* is a second category subset of Y^*, either Y_i^* or Y_p^* is a second category subset of Y^*.

There are two cases to consider.

Case 1. Y_i^* is a second category subset of Y^*.

We note that for each $y^* \in Y_i^*$, there exists $(I_1^{y^*}, I_2^{y^*}) \in \mathcal{D}$ such that $S(y^*) \cap I_1^{y^*} = S(y^*) \cap I_2^{y^*}$ is a singleton. Thus the map $\Phi : Y_i^* \to \mathcal{D}$: $y^* \mapsto (I_1^{y^*}, I_2^{y^*})$ is well-defined.

Since $Y_i^* = \bigcup_{(I_1,I_2)\in\mathcal{D}} \Phi^{-1}(I_1, I_2)$, $\mathcal{D}$ is countable, and Y_i^* is a second category subset of Y^*, there exist $(I_1^0, I_2^0) \in \mathcal{D}$ and a nonempty open subset U of Y^* such that $\overline{\Phi^{-1}(I_1^0, I_2^0)} \supseteq U$. Using the continuity of the map $\theta : Y^* \to 2^{S^1}$: $y^* \mapsto S(y^*)$, we have that for each $y^* \in \overline{\Phi^{-1}(I_1^0, I_2^0)}$, $S(y^*) \cap \text{int}(I_1^0) = S(y^*) \cap I_2^0$ is a singleton. In particular, for each $y^* \in U$, $S(y^*) \cap \text{int}(I_1^0) = S(y^*) \cap I_2^0$ is a singleton. Let $W = \text{int}(I_1^0)$. Then $W \cap S(y^*)$ is a singleton for each $y^* \in U$.

Fix points $y^* \in Y^*$ and $s \in S(y^*)$. Then $(s, y^*) \in Z^*$. Since $(W \times U) \cap Z^*$ is a nonempty open subset of Z^* and $(Z^*, \mathbb{T})$ is a minimal flow, there exists $t_0 \in \mathbb{T}$ such that $\Pi_{t_0}^{Z^*}(s, y^*) \in W \times U$. Hence there exists an open neighborhood V of s in S^1 such that $\Pi_{t_0}^{Z^*}((V \times \{y^*\}) \cap Z^*) \subset (W \times U) \cap Z^*$. Since $\Pi_{t_0}^{Z^*}((V \times \{y^*\}) \cap Z^*) \subset S(y^* \cdot t_0) \times \{y^* \cdot t_0\}$ and $y^* \cdot t_0 \in U$, $\Pi_{t_0}^{Z^*}((V \times \{y^*\}) \cap Z^*) \subseteq (S(y^* \cdot t_0) \cap W) \times \{y^* \cdot t_0\}$ is a singleton, it follows that $(V \times \{y^*\}) \cap Z^*$ is a singleton, i.e., s is an isolated point of $S(y^*)$.

Thus, for each $y^* \in Y^*$, $S(y^*)$ is a discrete closed subset of S^1, hence a finite subset of S^1. This shows that $\pi_1 : (Z^*, \mathbb{T}) \to (Y^*, \mathbb{T})$ is a finite to one and open extension of the point-distal flow $(Y^*, \mathbb{T})$. By Proposition 3.2 2), $(Z^*, \mathbb{T})$ is point-distal.

Case 2. Y_p^* is a second category subset of Y^*.

Let $y_0^* \in Y_p^*$ be fixed such that $S(y_0^*)$ contains no isolated point. Then $S(y_0^*)$ is a perfect subset of S^1. It is not hard to see that there exists $s_0 \in S(y_0^*)$ such that for any $\epsilon > 0$ sufficiently small, $[s_0, s_0 e^{2\pi\epsilon i}] \cap S(y_0^*)$ and $[s_0 e^{-2\pi\epsilon i}, s_0] \cap S(y_0^*)$ are two uncountable sets. Clearly, $z_0^* = (s_0, y_0^*) \in Z^*$. Let $y_0 = \tau(y_0^*)$ and $f_t : S^1 \to S^1$: $s \mapsto \psi(s, y_0, t)$.

Suppose for contradiction that $(Z^*, \mathbb{T})$ is not point-distal.

Claim 1. There exists $\epsilon_0 > 0$ such that if (s_1, y^*), $(s_2, y^*) \in Z^*$ are distal, then

$$\sup_{t\geq 0}|\psi(s_1, \tau(y^*), t) - \psi(s_2, \tau(y^*), t)| > \epsilon_0.$$

Since $(Z^*, \mathbb{T})$ is not point-distal, we have by Proposition 3.1 that there exists $z_1^* = (s', y_1^*) \in Z^* \setminus \{z_0^*\}$ such that z_1^*, z_0^* are positively proximal. In particular, y_1^*, y_0^* are positively proximal. Since $y_0^* \in Y_d^*$, $y_1^* = y_0^*$. Hence $s' \neq s_0$ and there exists a sequence $t_i \to +\infty$ and a point $s'' \in S^1$ such that $\lim_{i\to\infty} f_{t_i}(s') = \lim_{i\to\infty} f_{t_i}(s_0) = s''$. It follows from Lemma 5.3 (2) that there exists a subsequence $\{i_k\} \subset \{i\}$ such that either $\lim_{k\to\infty} f_{t_{i_k}}([s', s_0]) = \{s''\}$ or $\lim_{k\to\infty} f_{t_{i_k}}([s_0, s']) = \{s''\}$ under Hausdroff metric. We let $B = [s', s_0]$ or $[s_0, s']$ be such that for any $s_1, s_2 \in B \cap S(y_0^*)$,

$$\liminf_{t\to+\infty} d_{Z^*}((s_1, y_0^*) \cdot t, (s_2, y_0^*) \cdot t) = \lim_{k\to\infty} |f_{t_{i_k}}(s_1) - f_{t_{i_k}}(s_2)| = 0.$$

By the choice of s_0, $B \cap S(y_0^*)$ is an uncountable set.

For each $(s,y^*)\in Z^*$, we consider the set

$$AR_+(s,y^*)=\{s_1\in S(y^*):\lim_{t\to+\infty} d_{Z^*}((s_1,y^*)\cdot t,(s,y^*)\cdot t)=0\}.$$

Obviously, if $s\in S(y_0^*)$, then $AR_+(s,y_0^*)=\{s_1\in S(y_0^*):\lim_{t\to+\infty}|f_t(s_1)-f_t(s)|=0\}$. Hence for any $s_1,s_2\in S(y_0^*)$, either $AR_+(s_1,y_0^*)=AR_+(s_2,y_0^*)$ or $AR_+(s_1,y_0^*)\cap AR_+(s_2,y_0^*)=\emptyset$. It follows that there exists a set $I\subset B\cap S(y_0^*)$ such that

(1) for any $s\in I$, $B\cap AR_+(s,y_0^*)\neq\emptyset$;
(2) for any $s_1\neq s_2\in I$, $AR_+(s_1,y_0^*)\cap AR_+(s_2,y_0^*)=\emptyset$;
(3) $B\cap S(y_0^*)=\bigcup_{s\in I}(B\cap AR_+(s,y_0^*))$.

By (2) above, if $s_1\neq s_2\in I$, then $\limsup_{t\to+\infty} d_{Z^*}((s_1,y_0^*)\cdot t,(s_2,y_0^*)\cdot t)>0$, hence (s_1,y_0^*) and (s_2,y_0^*) form a Li-Yorke pair. This shows that $I\times\{y_0^*\}\subseteq\pi_1^{-1}(y_0^*)$ is a scrambled set of $(Z^*,\mathbb{T})$. Since $\pi_1^{-1}(y_0^*)$ contains no uncountable scrambled set, I must be countable. Note that $B\cap S(y_0^*)$ is uncountable. There must be a point $s^0\in I$ such that $B\cap AR_+(s^0,y_0^*)$ is uncountable, in particular, $AR_+(s^0,y_0^*)$ is uncountable. Let $\epsilon>0$ be given. Since $AR_+(s^0,y_0^*)$ is uncountable, there exist $s_1^1,s_2^1\in AR_+(s^0,y_0^*)$ such that $[s_1^1,s_2^1]\cap AR_+(s^0,y_0^*)$ and $[s_2^1,s_1^1]\cap AR_+(s^0,y_0^*)$ are both uncountable. Using Lemma 5.3 (4) and the fact that $\lim_{t\to+\infty} d_{Z^*}((s_1^1,y_0^*)\cdot t,(s_2^1,y_0^*)\cdot t)=0$, we have that there exists $t_0\geq 0$ such that, as $t\geq t_0$,

$$d_{Z^*}((s_1',y_0^*)\cdot t,(s_2',y_0^*)\cdot t)\leq d_{Z^*}((s_1^1,y_0^*)\cdot t,(s_2^1,y_0^*)\cdot t)\leq\epsilon$$

for all $s_1',s_2'\in[s_1^1,s_2^1]$ or $s_1',s_2'\in[s_2^1,s_1^1]$. Let $[s_1^0,s_2^0]\subseteq[s_1^1,s_2^1]$ or $[s_2^1,s_1^1]$ be such that $[s_1^0,s_2^0]\cap S(y_0^*)$ is an uncountable set, $[s_1^0,s_2^0]\cap S(y_0^*)\subseteq AR_+(s_0,y^*)$, and $\sup_{t\geq0} d_{Z^*}((s_1',y^*)\cdot t,(s_2',y^*)\cdot t)\leq\epsilon$ for all $s_1',s_2'\in[s_1^0,s_2^0]\cap S(y_0^*)$. Also let $s_3^0\in(s_1^0,s_2^0)\cap S(y_0^*)$ and $\epsilon_0>0$ be such that $\{s\in S^1:|s-s_3^0|\leq 2\epsilon_0\}\subset(s_1^0,s_3^0)$.

Let $(s_1,y^*),(s_2,y^*)\in Z^*$ be distal. Since $(Z^*,\mathbb{T})$ is minimal, there exists a positive sequence $t_n\to+\infty$ such that $\Pi_{t_n}^{Z^*}(s_1,y^*)\to(s_3^0,y_0^*)$ and $\Pi_{t_n}^{Z^*}(s_2,y^*)\to(s_4^0,y_0^*)$ for some $s_4^0\in S(y_0^*)$. Since (s_3^0,y_0^*), $(s_4^0,y_0^*)\in Z^*$ are also distal, $s_4^0\notin(s_1^0,s_2^0)$. Hence $|s_3^0-s_4^0|\geq 2\epsilon_0$. It follows that $\sup_{t\geq0}|\psi(s_1,\tau(y^*),t)-\psi(s_2,\tau(y^*),t)|\geq|s_3^0-s_4^0|>\epsilon_0$. This proves Claim 1.

Claim 2. For each $\epsilon>0$, there exist a nonempty open set $U_\epsilon\subset Y^*$ and a point $(I_1^\epsilon,I_2^\epsilon)\in\mathcal{D}$ such that if $y^*\in U_\epsilon$, then $S(y^*)\cap I_2^\epsilon\neq\emptyset$ and $\sup_{t\geq0} d_{Z^*}((s_1',y^*)\cdot t,(s_2',y^*)\cdot t)\leq\epsilon$ for all $s_1',s_2'\in\mathrm{int}(I_1^\epsilon)\cap S(y^*)$.

Let $y^*\in Y_p^*$. Then $S(y^*)$ contains no isolated point and no uncountable scrambled set. Similar to the proof of Claim 1 there are $s_1\neq s_2\in S(y^*)$ such that $[s_1,s_2]\cap S(y^*)$ is an uncountable set, $[s_1,s_2]\cap S(y^*)\subseteq AR_+(s,y^*)$ for some $s\in S(y^*)$, and $\sup_{t\geq0} d_{Z^*}((s_1',y^*)\cdot t,(s_2',y^*)\cdot t)\leq\epsilon$ for all $s_1',s_2'\in[s_1,s_2]\cap S(y^*)$. Hence there exists $(I_1^{y^*},I_2^{y^*})\in\mathcal{D}$ such that $I_2^{y^*}\cap S(y^*)\neq\emptyset$ and $\sup_{t\geq0} d_{Z^*}((s_1',y^*)\cdot t,(s_2',y^*)\cdot t)\leq\epsilon$ for any $s_1',s_2'\in I_1^{y^*}\cap S(y^*)$. We define the map $\Phi_\epsilon:Y_p^*\to\mathcal{D}$ as $\Phi_\epsilon(y^*)=(I_1^{y^*},I_2^{y^*})$, $y^*\in Y_p^*$.

Since $Y_p^*=\bigcup_{(I_1,I_2)\in\mathcal{D}}\Phi_\epsilon^{-1}(I_1,I_2)$, $\mathcal{D}$ is countable, and Y_p^* is a second category subset of Y^*, we have that there exists $(I_1^\epsilon,I_2^\epsilon)\in\mathcal{D}$ and a nonempty open subset U_ϵ of Y^* such that $\overline{\Phi_\epsilon^{-1}(I_1^\epsilon,I_2^\epsilon)}\supseteq U_\epsilon$.

We note that for any $y^*\in\Phi_\epsilon^{-1}(I_1^\epsilon,I_2^\epsilon)$, $I_2^\epsilon\cap S(y^*)\neq\emptyset$, $\sup_{t\geq0} d_{Z^*}((s_1',y^*)\cdot t,(s_2',y^*)\cdot t)\leq\epsilon$ for any $s_1',s_2'\in I_1^\epsilon\cap S(y^*)$. It follows from the continuity of the map $\theta:Y^*\to 2^{S^1}$: $y^*\mapsto S(y^*)$ that for each $y^*\in\overline{\Phi_\epsilon^{-1}(I_1^\epsilon,I_2^\epsilon)}$, $S(y^*)\cap I_2^\epsilon\neq\emptyset$, and $\sup_{t\geq0} d_{Z^*}((s_1',y^*)\cdot t,(s_2',y^*)\cdot t)\leq\epsilon$ for any $s_1',s_2'\in\mathrm{int}(I_1^\epsilon)\cap S(y^*)$. In particular, for each $y^*\in U_\epsilon$, we have $S(y^*)\cap I_2^\epsilon\neq\emptyset$ and $\sup_{t\geq0} d_{Z^*}((s_1',y^*)\cdot t,(s_2',y^*)\cdot t)\leq\epsilon$ for any $s_1',s_2'\in\mathrm{int}(I_1^\epsilon)\cap S(y^*)$. This proves Claim 2.

Claim 3. If $(s_1,y^*),(s_2,y^*)\in Z^*$ are proximal, then $s_1\in AR_+(s_2,y^*)$. Moreover, for each $(s,y^*)\in Z^*$, $AR_+(s,y^*)$ is an open subset of $S(y^*)$.

Let $(s_1,y^*),(s_2,y^*)\in Z^*$ be proximal. For any $\epsilon>0$, we let $U_\epsilon\subset Y^*$ and $(I_1^\epsilon,I_2^\epsilon)\in\mathcal{D}$ be as in Claim 2. Then for each $y_1^*\in U_\epsilon$, we have $S(y_1^*)\cap I_2^\epsilon\neq\emptyset$ and $\sup_{t\geq0} d_{Z^*}((s_1',y_1^*)\cdot t,(s_2',y_1^*)\cdot t)\leq\epsilon$

for all $s_1', s_2' \in \mathrm{int}(I_1^\epsilon) \cap S(y_1^*)$. Since $(Z^*, \mathbb{T})$ is minimal and $(\mathrm{int}(I_1^\epsilon) \times U_\epsilon) \cap Z^*$ is an open subset of Z^*, there exists $t_1 \in \mathbb{T}$ such that $(s_1, y^*) \cdot t_1, (s_2, y^*) \cdot t_1 \in (\mathrm{int}(I_1^\epsilon) \times U_\epsilon) \cap Z^*$. Hence

$$\sup_{t \geq 0} d_{Z^*}((s_1, y^*) \cdot (t_1 + t), (s_2, y^*) \cdot (t_1 + t)) \leq \epsilon.$$

Since $\epsilon > 0$ is arbitrary, $s_1 \in AR_+(s_2, y^*)$.

For a fixed $\epsilon \in (0, \epsilon_0)$, we let $y_1^* \in Y^*$ and $s_1', s_2' \in \mathrm{int}(I_1^\epsilon) \cap S(y_1^*)$. We have by Claim 1 and Claim 2 that $(s_1', y_1^*), (s_2', y_1^*)$ are proximal. Repeat the above arguments for $(s_1', y_1^*), (s_2', y_1^*)$ in place of $(s_1, y^*), (s_2, y^*)$ respectively, we conclude that $s_1' \in AR_+(s_2', y_1^*)$. Hence $\mathrm{int}(I_1^\epsilon) \cap S(y_1^*) \subset AR_+(s, y_1^*)$ for each $s \in \mathrm{int}(I_1^\epsilon) \cap S(y_1^*)$.

Let $(s, y^*) \in Z^*$ and $t \in \mathbb{T}$ be such that $(s, y^*) \cdot t \in (\mathrm{int}(I_1^\epsilon) \times U_\epsilon) \cap Z^*$, i.e., $\psi(s, \tau(y^*), t) \in \mathrm{int}(I_1^\epsilon)$ and $y^* \cdot t \in U_\epsilon$. Clearly, there exists an open neighborhood V of s in S^1 such that

$$\{\psi(s', \tau(y^*), t) : s' \in V \cap S(y^*)\} \subset \mathrm{int}(I_1^\epsilon) \cap S(y^* \cdot t) \subset AR_+((s, y^*) \cdot t).$$

This implies that $V \cap S(y^*) \subset AR_+(s, y^*)$, i.e., $AR_+(s, y^*)$ is an open subset of $S(y^*)$.

Claim 4. $\pi_1 : (Z^*, \mathbb{T}) \to (Y^*, \mathbb{T})$ is a finite to one extension.

Since for any $s_1, s_2 \in S(y_0^*)$, either $AR_+(s_1, y_0^*) = AR_+(s_2, y_0^*)$ or $AR_+(s_1, y_0^*) \cap AR_+(s_2, y_0^*) = \emptyset$, there exists $J \subset S(y_0^*)$ such that $\{AR_+(s, y_0^*)\}_{s \in J}$ is a partition of $S(y_0^*)$. By Claim 3, for each $s \in J$, $AR_+(s, y_0^*)$ is an open subset of $S(y_0^*)$. So $\{AR_+(s, y_0)\}_{s \in J}$ is an open cover and also a partition of $S(y_0^*)$. Hence J must be a finite set, say, $J = \{s_1, s_2, \cdots, s_n\}$. For each $i = 1, 2, \cdots, n$, since $AR_+(s_i, y_0^*) = S(y_0^*) \setminus \bigcup_{j \neq i} AR_+(s_j, y_0^*)$, we see that $AR_+(s_i, y_0^*)$ is also a closed subset of $S(y_0^*)$.

If $n = 1$, then for any $s_1', s_2' \in S(y_0^*)$, $(s_1', y_0^*), (s_2', y_0^*)$ are proximal. This implies that π_1 is a proximal extension. Since for each $y^* \in Y_u^*$ there exists no uncountable scrambled set in $\pi_1^{-1}(y^*)$, we have by Theorem 5.1 that π_1 is an almost 1-1 extension. Hence there exists y_1^* such that $|S(y_1^*)| = |\pi_1^{-1}(y_1^*)| = 1$. Moreover, since $\theta : y^* \to S(y^*)$ is continuous, $|\pi_1^{-1}(y^*)| = |S(y^*)| = 1$ for any $y^* \in Y^*$, i.e., π_1 is a flow isomorphism.

If $n \geq 2$, then there exists $s' \in S(y_0^*) \setminus AR_+(s_1, y_0^*)$. Since $AR_+(s_1, y_0^*)$ is closed, we can find points $a_1, b_1 \in AR_+(s_1, y_0^*)$ such that $AR_+(s_1, y_0^*) \subseteq [a_1, b_1]$ (if $a_1 = b_1$, then $[a_1, b_1] = \{a_1\}$) and $s' \in S^1 \setminus [a_1, b_1]$. By Lemma 5.3 (4) and the fact that $s' \notin AR_+(s_1, y_0^*)$, we have that $[a_1, b_1] \cap S(y_0^*) \subseteq AR_+(s_1, y_0^*)$ and $\mathrm{diam}(f_t([a_1, b_1])) \leq |f_t(a_1) - f_t(b_1)|$ as t sufficiently large, where $f_t(s) = \psi(s, \tau(y_0^*), t)$, $t \in \mathbb{T}$. Hence $AR_+(s_1, y_0^*) = [a_1, b_1] \cap S(y_0^*)$. Similarly, for each $i = 2, 3, \cdots, n$ there exist points $a_i, b_i \in AR_+(s_i, y_0^*)$ such that $AR_+(s_i, y_0^*) = [a_i, b_i] \cap S(y_0^*)$ and $\mathrm{diam}(f_t([a_i, b_i])) \leq |f_t(a_i) - f_t(b_i)|$ as t sufficiently large.

Let $t_k \to +\infty$ be such that $y_0^* \cdot t_k \to y_0^*$ and $\lim_{k \to \infty} f_{t_k}([a_i, b_i]) = \{c_i\}$ for some $c_i \in S(y_0^*)$. Using the continuity of $\theta : y^* \to S(y^*)$ and the fact that

$$S(y_0^* \cdot t) = f_t(S(y_0^*)) = \bigcup_{i=1}^n f_t(AR_+(s_i, y_0^*)) = \bigcup_{i=1}^n f_t([a_i, b_i] \cap S(y_0^*)), \quad t \in \mathbb{T},$$

we have

$$S(y_0^*) = \lim_{k \to \infty} S(y_0^* \cdot t_k) = \lim_{k \to \infty} \bigcup_{i=1}^n f_{t_k}([a_i, b_i] \cap S(y_0^*)) = \{c_1, c_2, \cdots, c_n\}.$$

By the continuity of θ again, we have $|\pi_1^{-1}(y^*)| = |S(y^*)| = |S(y_0^*)|$ for any $y^* \in Y^*$. This shows that π_1 is a finite to one extension. The proof of Claim 4 is now complete.

Now, since $(Y^*, \mathbb{T})$ is point-distal and π_1 is finite to one and open, we have by Proposition 3.2 2) that $(Z^*, \mathbb{T})$ is point-distal, a contradiction. □

Theorem 5.2. *Let X be a minimal set of a SPCF $(S^1 \times Y, \mathbb{T})$ with point-distal base flow $(Y, \mathbb{T})$. Then X is either point-distal or residually Li-Yorke chaotic.*

Proof. Let $X^*, Y^*, Z^*, \tau, \tau', \rho, \pi, \pi_1, \pi'$ be as in Lemma 5.4 for the present minimal set X.

We first consider the case that there exists a second category subset Y_u^* of Y^* such that for each $y^* \in Y_u^*$ there exists no uncountable scrambled set in $\pi_1^{-1}(y^*)$. By Proposition 5.1, $(Z^*, \mathbb{T})$ is point-distal. Since $\rho : (X^*, \mathbb{T}) \to (Z^*, \mathbb{T})$ is a flow isomorphism, $(X^*, \mathbb{T})$ is also point-distal. Note that $\tau' : (X^*, \mathbb{T}) \to (X, \mathbb{T})$ is almost 1-1. We conclude that $(X, \mathbb{T})$ is point-distal in this case.

Next, we consider the case that there exists a residual subset Y_c^* of Y^* such that for each $y^* \in Y_c^*$ there exists an uncountable scrambled set in $\pi_1^{-1}(y^*)$. Let $y_0^* \in Y_c^*$ and F be an uncountable scrambled set in $\pi_1^{-1}(y_0^*)$. Then there exists an uncountable subset S of S^1 such that $F = \{(s, y_0^*) : s \in S\}$. Let $y_0 = \tau(y_0^*)$ and $F' = \{(s, y_0) : s \in S\}$. Clearly, F' is an uncountable set. Since $\pi_2 = \tau' \circ \rho^{-1} : (Z^*, \mathbb{T}) \to (X, \mathbb{T})$ is a flow extension and $\pi_2(F) = F'$, we have $F' \subset \pi^{-1}(y_0)$. Let $s_1 \neq s_2 \in S$. Then $(s_1, y_0^*), (s_2, y_0^*)$ form a Li-Yorke pair. Hence

$$\liminf_{t\to+\infty} |\psi(s_1, y_0, t) - \psi(s_2, y_0, t)| = 0 \text{ and } \limsup_{t\to+\infty} |\psi(s_1, y_0, t) - \psi(s_2, y_0, t)| > 0,$$

i.e., $\{(s_1, y_0), (s_2, y_0)\}$ is also a Li-Yorke pair. Therefore, $F' \subseteq \pi^{-1}(y_0)$ is an uncountable scrambled set of $(X, \mathbb{T})$.

Let $Y_c = \tau(Y_c^*)$. Since Y_c^* is a residual subset of Y^* and τ is an almost 1-1 extension, Y_c is a residual subset of Y and there exists an uncountable scrambled set in $\pi^{-1}(y)$ for each $y \in Y_c$. Hence $(X, \mathbb{T})$ is residually Li-Yorke chaotic. □

A point-distal or even an almost automorphic minimal set can also be Li-Yorke chaotic. But our next theorem says that it cannot be residually Li-Yorke chaotic.

Lemma 5.5. *Let X and Y be compact metric spaces, and $\pi : X \to Y$ be a semi-open, surjective, continuous map. Then*

$$X_0 = \{x \in X : \text{for any open neighborhood } U \text{ of } x,\ \pi(U) \text{ is a neighborhood of } \pi(x)\}$$

is a residual subset of X.

Proof. It follows from arguments of Lemma 3.1 in [63]. □

Theorem 5.3. *Consider a SPCF $(S^1 \times Y, \mathbb{T})$ with point-distal base flow $(Y, \mathbb{T})$. If a minimal set is point-distal, then it is not residually Li-Yorke chaotic.*

Proof. We use the explicit expression (1.1) for a SPCF $(S^1 \times Y, \mathbb{T}) = (S^1 \times Y, \{\Lambda_t\}_{t\in\mathbb{T}})$. Suppose for contradiction that the SPCF has a point-distal minimal set M which is also residually Li-Yorke chaotic. Then the set M_d of distal points of M is a residual subset. It follows from Lemma 5.5 that

$$Y_d = \{y \in Y : M_d \cap \pi^{-1}(y) \text{ is a residual subset of } \pi^{-1}(y) \cap M\}$$

is a residual subset of Y. Since M is residually Li-Yorke chaotic, there exists a residual subset Y_c of Y such that each fiber $\pi^{-1}(y)$, $y \in Y_c$, admits an uncountable scrambled set W_y.

Fix $y \in Y_d \cap Y_c$ and let $E_y = \{s \in S^1 : (s, y) \in W_y\}$. Then E_y is an uncountable subset of S^1 and it is not hard to see that there exists $s_0 \in E_y$ such that for any $\epsilon > 0$ sufficiently small, $[s_0, s_0 e^{2\pi\epsilon i}] \cap E_y$ and $[s_0 e^{-2\pi\epsilon i}, s_0] \cap E_y$ are two uncountable sets.

Consider the family of maps $f_t : S^1 \to S^1$: $s \mapsto \psi(s, y, t)$, $t \in \mathbb{T}$. Take $s_1 \in E_y \setminus \{s_0\}$. Then $(s_0, y), (s_1, y)$ are positively proximal. Hence there exists a sequence $t_i \to +\infty$ and a point $s'' \in S^1$ such that $\lim_{i\to\infty} f_{t_i}(s_1) = \lim_{i\to\infty} f_{t_i}(s_0) = s''$. It follows from Lemma 5.3 (2) that there exists a subsequence $\{i_k\} \subset \{i\}$ such that either $\lim_{k\to\infty} f_{t_{i_k}}([s_1, s_0]) = \{s''\}$ or $\lim_{k\to\infty} f_{t_{i_k}}([s_0, s_1]) = \{s''\}$ under Hausdroff metric. We let $B = [s_1, s_0]$ or $[s_0, s_1]$ be such that for any $s_2 \in B$,

$$\liminf_{t\to+\infty} d((s_0, y) \cdot t, (s_2, y) \cdot t) = \lim_{k\to\infty} |f_{t_{i_k}}(s_0) - f_{t_{i_k}}(s_2)| = 0.$$

According to the choice of s_0, $B \cap E_y$ is an uncountable set. Now since $\pi^{-1}(y) \cap M_d$ is a residual subset of $\pi^{-1}(y) \cap M$, there exists $s_2^* \in \mathrm{int}(B) \cap E_y$ such that $(s_2^*, y) \in M_d$. This is impossible since $\liminf_{t\to+\infty} d((s_0, y) \cdot t, (s_2^*, y) \cdot t) = 0$. □

Now Theorem 2 immediately follows from Theorems 5.2, 5.3 above.

6. A general topological classification of Minimal sets

In this section, we consider a SPCF $(S^1 \times Y, \mathbb{T}) = (S^1 \times Y, \{\Lambda_t\}_{t\in\mathbb{T}})$ with minimal base flow $(Y, \mathbb{T})$. We adopt the explicit form (1.1), i.e.,

$$\Lambda_t(s_0, y_0) = (\psi(s_0, y_0, t), y_0 \cdot t), \ \ t \in \mathbb{T},$$

and denote d_Y as a compatible metric on Y and $\pi : S^1 \times Y \to Y$ as the natural projection.

Let M be a minimal set of the SPCF $(S^1 \times Y, \mathbb{T})$. For each $y \in Y$, we denote $M_y = \{s \in S^1 : (s, y) \in M\}$. Since M_y is a closed subset of S^1, each connected component of M_y is either the whole circle or a closed interval in the circle (which can be degenerate). Consider the function $\zeta_M : Y \to R^1$:

$$\zeta_M(y) = \sup\{|B| : B \text{ is a connected component of } M_y\},$$

where $|B|$ denotes the length of B. For each $y \in Y$, it is clear that there exists a component B in M_y such that $\zeta_M(y) = |B|$, i.e.,

$$\zeta_M(y) = \max\{|B| : B \text{ is a connected component of } M_y\}.$$

Lemma 6.1. *The function ζ_M is non-negative and upper semi-continuous, i.e., $\zeta_M(y) \geq 0$ and $\limsup_{y_n\to y} \zeta_M(y_n) \leq \zeta_M(y)$ for each $y \in Y$.*

Proof. The lemma is clear because M is compact. □

Let $Y_0(\zeta_M) = \{y \in Y : \zeta_M(y) = 0\}$.

Lemma 6.2. *Either $\inf_{y\in Y} \zeta_M(y) > 0$ or $Y_0(\zeta_M)$ is a residual subset of Y.*

Proof. By Lemma 6.1, $Y_0(\zeta_M)$ is a G_δ-set. Hence it is sufficient to show that if $\inf_{y\in Y} \zeta_M(y) = 0$, then $Y_0(\zeta_M)$ is a dense subset of Y.

Assume that $\inf_{y\in Y} \zeta_M(y) = 0$ and let $Y_c(\zeta_M)$ be the set of points of continuity of ζ_M. Since ζ_M is upper semi-continuous, $Y_c(\zeta_M)$ is a residual set.

If $\zeta_M(y_0) > 0$ for some $y_0 \in Y_c(\zeta_M)$, then there exist open neighborhood U of y_0 and $c > 0$ such that $\zeta_M(y) \geq c$ for all $y \in U$. By the minimality of Y, we let $t_1 < t_2 < \cdots < t_n$ be such that $\bigcup_{i=1}^n U \cdot t_i = Y$. Since $\inf_{y\in U} \zeta_M(y) > 0$, we have that $c_i =: \inf_{y\in U} \zeta_M(y \cdot t_i) > 0$, for all $i = 1, 2, \cdots, n$. Hence $\inf_{y\in Y} \zeta_M(y) \geq \min\{c_i : i = 1, 2, \cdots, n\} > 0$, a contradiction to the fact that $\inf_{y\in Y} \zeta_M(y) = 0$. This shows that for any $y_0 \in Y_c(\zeta(M))$, $\zeta_M(y_0) = 0$, i.e., $Y_c(\zeta_M) \subseteq Y_0(\zeta_M)$. Hence $Y_0(\zeta_M)$ is a residual subset of Y. □

Recall that a minimal set M of a SPCF $(S^1 \times Y, \mathbb{T})$ is a *Cantorian* if there exists a residual subset Y_0 of Y such that for each $y \in Y_0$, M_y is a Cantor set.

The following theorem immediately yields Theorem 3.

Theorem 6.1. *Let M be a minimal set of a SPCF $(S^1 \times Y, \mathbb{T})$ with minimal base flow $(Y, \mathbb{T})$. Then precisely one of the following holds:*

a) *M is an almost N-1 extension of Y for some positive integer N;*
b) *$M = S^1 \times Y$;*
c) *M is a Cantorian.*

Proof. By Lemma 6.2, there are two cases:

a) $\inf_{y\in Y} \zeta_M(y) > 0$;
b) $Y_0(\zeta_M)$ is a residual subset of Y.

We first consider the case a). Let

$$\mathcal{A} = \{[e^{2\pi ir}, e^{2\pi il}] : r < l < 1 + r, \ r, l \in \mathbb{Q}\}.$$

Since $\mathcal{A}$ is countable, we can rewrite it as $\mathcal{A} = \{[a_i, b_i]\}_{i\in\mathbb{N}}$. Since $\inf_{y\in Y} \zeta_M(y) > 0$, we have that for each $y \in Y$, there exists $i(y) \in \mathbb{N}$ such that $[a_{i(y)}, b_{i(y)}] \subseteq M_y$. In particular, $[a_{i(y)}, b_{i(y)}] \times \{y\} \subseteq M$.

Denote $Y_i = \{y \in Y : i(y) = i\}$ for $i \in \mathbb{N}$. Then $\bigcup_{i\in\mathbb{N}} Y_i = Y$. Hence there exists $i_* \in \mathbb{N}$ such that $W =: \mathrm{int}(\overline{Y}_{i_*}) \neq \emptyset$. Since $[a_{i_*}, b_{i_*}] \times Y_{i_*} \subseteq M$ and M is closed, we have that $[a_{i_*}, b_{i_*}] \times \overline{Y}_{i_*} \subseteq M$, which implies that $(a_{i_*}, b_{i_*}) \times W \subseteq M$.

Let $y \in Y$ and denote $S(y) = \{s \in S^1 : (s, y) \in M_y\}$. For any $s \in S(y)$, since M is minimal, there exists $t \in \mathbb{T}$ such that $(s, y) \cdot t \in (a_{i_*}, b_{i_*}) \times W$, which implies that $s \in \mathrm{int}_{S^1}(S(y))$. It follows that $S(y)$ is an open subset of S^1. Since $S(y)$ is also closed, $S(y) = S^1$. Since y is arbitrary, $M = \bigcup_{y\in Y} S(y) \times \{y\} = S^1 \times Y$.

We now consider the case b). Let

$$\begin{aligned} Y^c(M) &= \{y \in Y : M_y \text{ is a Cantor set}\}, \\ Y^i(M) &= \{y \in Y : M_y \text{ has an isolated point}\}. \end{aligned}$$

Since it is clear that M_y is a Cantor set for any $y \in Y_0(\zeta_M) \setminus Y^i(M)$, we have that $Y^c(M) \supseteq Y_0(\zeta_M) \setminus Y^i(M)$.

If $Y^c(M)$ is a residual subset of Y, then by definition M is a Cantorian.

If $Y^c(M)$ is not a residual subset of Y, then $Y^i(M)$ is of second category, or $Y^c(M) \supseteq Y_0(\zeta_M) \setminus Y^i(M)$ is a residual subset of Y since $Y_0(\zeta_M)$ is a residual subset of Y.

Consider the countable set

$$\mathcal{D} = \{(I_1, I_2) : I_1, I_2 \in \mathcal{A} \text{ and } I_2 \subset \mathrm{int}(I_1)\}.$$

We note that for each $y \in Y^i(M)$, there exists $(I_1^y, I_2^y) \in \mathcal{D}$ such that $S(y) \cap I_1^y = S(y) \cap I_2^y$ is a singleton. Thus the map $\Phi : Y^i(M) \to \mathcal{D}$: $y \mapsto (I_1^y, I_2^y)$ is well-defined.

Since $Y^i(M) = \bigcup_{(I_1,I_2)\in\mathcal{D}} \Phi^{-1}(I_1, I_2)$, $\mathcal{D}$ is countable, and $Y^i(M)$ is of second category, there exist $(I_1^0, I_2^0) \in \mathcal{D}$ and a nonempty open subset U of Y such that $\overline{\Phi^{-1}(I_1^0, I_2^0)} \supseteq U$. Since $\theta : y \to S(y)$ is upper semi-continuous, the set Y_0 of all continuity points of θ is an invariant residual subset of Y. Let $W = \mathrm{int}(I_1^0)$. For each $y \in U \cap Y_0 \subset \overline{\Phi^{-1}(I_1^0, I_2^0)} \cap Y_0$, since $S(y) \cap \mathrm{int}(I_1^0) = S(y) \cap I_2^0$ is a singleton, $W \cap S(y)$ is also a singleton.

Fix $y \in Y_0$ and $s \in S(y)$. Then $(s, y) \in M$. Since $(W \times U) \cap M$ is a nonempty open subset of M and M is a minimal set, there exists $t_0 \in \mathbb{T}$ such that $(s, y) \cdot t_0 \in W \times U$. Hence there exists an open neighborhood V of s in S^1 such that $(V \times \{y\}) \cdot t_0 \cap M \subset (W \times U) \cap M$. Since $(V \times \{y\}) \cdot t_0 \cap M \subset S(y \cdot t_0) \times \{y \cdot t_0\}$ and $y \cdot t_0 \in U \cap Y_0$, we have that $(V \times \{y\}) \cdot t_0 \cap M \subseteq (S(y \cdot t_0) \cap W) \times \{y \cdot t_0\}$ is a singleton. It follows that $(V \times \{y\}) \cap M$ is a singleton, i.e., s is an isolated point of $S(y)$.

Thus, for each $y \in Y_0$, $S(y)$ is a discrete closed subset of S^1, hence it is a finite subset of S^1. This shows that $\pi : M \to Y$ is an almost finite to one extension. It follows from Proposition 3.2 1) that $\pi : M \to Y$ is an almost N-1 extension for some positive integer N.

Finally, the above is a strict trichotomy because in both cases a) and b), M cannot be a Cantorian. □

7. Finite To One Extensions And Almost Automorphic Dynamics

7.1. **Local connectivity and almost automorpy.** Let X be a complete metric space. Recall that $x \in X$ is a *locally connected point* if for any open neighborhood U of x there is a connected closed neighborhood V of x such that $V \subseteq U$. We denote by X_{lc} the set of locally connected points in X.

Lemma 7.1. *Suppose that $X_{lc} \neq \emptyset$ and $(X, \mathbb{T})$ is minimal. Then X_{lc} is an invariant residual subset of X.*

Proof. The invariance of X_{lc} is clear. For each $k \in \mathbb{N}$, we consider the open set

$$X_{lc}^k = \{x \in X : \text{ there exists a connected closed neighborhood } V \text{ of } x \text{ such that } V \subseteq B(x, \frac{1}{k})\},$$

where $B(x, \frac{1}{k}) = \{z \in X : d(x, z) < \frac{1}{k}\}$. Then $X_{lc} = \bigcap_{k=1}^{\infty} X_{lc}^k$, i.e., X_{lc} is a G_δ subset of X. It follows from the minimality of X that X_{lc} is also dense. □

The following result is known as the Ramsey theorem ([53]).

Lemma 7.2. *If the set $C = \{(i,j) \in \mathbb{N}\times\mathbb{N} : 1 \le i < j < \infty\}$ is divided into finite sets $C_1, C_2, \cdots, C_\ell$, then there is a sequence $\{i_n\}$ of natural numbers for which all pairs $(i_m, i_n), m < n$, are in C_j for some $j \in \{1,2,\cdots,\ell\}$.*

Our main result Theorem 5 is a direct consequence of the following theorem.

Theorem 7.1. *Consider an almost n-1 extension $\pi : (X,\mathbb{T}) \to (Y,\mathbb{T})$ between minimal flows in which $(Y,\mathbb{T})$ is almost periodic minimal. If $X_{lc} \neq \emptyset$, then the following holds.*

1) *$(X,\mathbb{T})$ is almost automorphic;*
2) *For each $y \in Y$, the fiber $\pi^{-1}(y)$ has precisely n connected components.*

Proof. 1) Let

$$X_0 = \{x \in X : \text{for any open neighborhood } U \text{ of } x,\ \pi(U) \text{ is a neighborhood of } \pi(x)\}.$$

By Lemma 5.5, X_0 is a residual subset of X. Since by Lemma 7.1 X_{lc} is residual, $X_0 \cap X_{lc}$ is also a residual subset of X. Hence by Lemma 5.1,

$$Y_1 = \{y \in Y : \pi^{-1}(y) \cap X_0 \cap X_{lc} \text{ is a residual subset of } \pi^{-1}(y)\}$$

is a residual subset of Y. Let Y_0 be the set of continuity points of the map $\Phi : Y \to 2^X$: $y \mapsto \pi^{-1}(y)$. Since Y_0 is a residual subset of Y, so is $Y_0 \cap Y_1$.

Let $y_0 \in Y_0 \cap Y_1$. Then $|\pi^{-1}(y_0)| = n$, say, $\pi^{-1}(y_0) = \{x_1, x_2, \cdots, x_n\}$. Since $\pi^{-1}(y_0) \cap X_0$ is a residual subset of $\pi^{-1}(y_0)$, we have that $x_i \in X_0$ for all $i = 1,2,\cdots,n$. By Theorem 3.1, we want to show that x_1 is a Δ^*-recurrent point, i.e., for any open neighborhood U of x_1, the recurrent time set $N(x_1, U) = \{t \in \mathbb{T} : x_1 \cdot t \in U\}$ is a Δ^*-set. More precisely, let $\{s_i\}_{i=1}^\infty$ be a sequence in $\mathbb{T}$. We need to show that $N(x_1,U) \cap \{s_k - s_{k'} : k > k'\} \neq \emptyset$.

Let U_i be open neighborhoods of x_i, for $i = 1,2,\cdots,n$ respectively, such that $U_1 \subseteq U$ and $\mathrm{cl}(U_i) \cap \mathrm{cl}(U_j) = \emptyset$ for any $1 \le i \neq j \le n$. Since $y_0 \in Y_0$, there exists an open neighborhood W of y_0 such that for each $y \in W$, $\pi^{-1}(y) \cap U_i \neq \emptyset$, $i = 1,2,\cdots,n$, and $\pi^{-1}(y) \subset \bigcup_{i=1}^n U_i$.

Since $(Y,\mathbb{T})$ is almost periodic, there exists an invariant compatible metric d_Y on Y, i.e., $d_Y(y_1 \cdot t, y_2 \cdot t) = d_Y(y_1, y_2)$ for any $y_1, y_2 \in Y$ and $t \in \mathbb{T}$. Let $\delta > 0$ be such that the open ball $B_\delta(y_0)$ centered at y_0 with radius δ is contained in W. Let $m = n!$. Then there exist $r \in \{0,1,\cdots,m-1\}$, a subsequence $\{i_k\} \subset \mathbb{N}$, and a homeomorphism $g : Y \to Y$ such that $\frac{s_{i_k}-r}{m} \in \mathbb{T}$ and

$$\sup_{y\in Y} d_Y(y \cdot \frac{s_{i_k}-r}{m}, g(y)) \le \frac{\delta}{2^k} \tag{7.1}$$

for all $k = 1,2,\cdots$. Then for any $u, v \in \mathbb{N}$ with $u \neq v$, it follows from (7.1) that

$$\begin{aligned}\sup_{y\in Y} d_Y(y\cdot(\frac{s_{i_u}-s_{i_v}}{m}), y) &= \sup_{y\in Y} d_Y(y\cdot\frac{s_{i_u}-r}{m}, y\cdot\frac{s_{i_v}-r}{m})\\ &\le \sup_{y\in Y} d_Y(y\cdot\frac{s_{i_u}-r}{m}, g(y)) + \sup_{y\in Y} d_Y(y\cdot\frac{s_{i_v}-r}{m}, g(y)) \le \frac{\delta}{2^u} + \frac{\delta}{2^v}.\end{aligned}$$

Denote $r_k = \frac{s_{i_k}-r}{m}$, $k = 1,2,\cdots$, $R = \{r_i - r_j : i > j\}$, and $Per(n)$ as the set of all permutations of $\{1,2,\cdots,n\}$. Let $t \in R$. Since $y_0 \cdot t \in W$, $\pi^{-1}(y_0 \cdot t) \cap U_i \neq \emptyset$ for all $i = 1,2,\cdots,n$. Hence there exists a unique $P_t \in Per(n)$ such that $x_j \cdot t \in U_{P_t(j)}$ for all $j = 1,2,\cdots,n$.

For each $P \in Per(n)$, we let $R_P = \{t \in R : P_t = P\}$ and $C_P = \{(i,j) \in \mathbb{N}\times\mathbb{N} : r_j - r_i \in P\}$. Since $R = \cup_{P\in Per(n)} R_P = \{r_i - r_j : i > j\}$, we have $C = \cup_{P\in Per(n)} C_P$. Applying Lemma 7.2, one finds a subsequence $\{l_j\} \subset \mathbb{N}$ and $Q \in Per(n)$ such that $R_Q \supseteq \{r_{l_i} - r_{l_j} : i > j\}$. Let $u_i = r_{l_i}$, $i \in \mathbb{N}$. It is clear that

a) $\{u_i - u_j : i > j\} \subseteq R_Q$;
b) $\{m(u_i - u_j) : i > j\} \subseteq \{s_k - s_{k'} : k > k'\}$;
c) $\sup_{y\in Y} d(y \cdot (u_i - u_j), y) < \frac{\delta}{2^i} + \frac{\delta}{2^j}$ for any $i > j$.

Since $Q \in Per(n)$, $Q^m(j) = j$ for each $j = 1, 2, \cdots, n$. In particular, $Q^m(1) = 1$. Let $W_m = U_1$. Since $\lim_{i>j\to\infty} \sup_{y\in Y} d(y \cdot (u_i - u_j), y) = 0$, there exists a positive integer N_m and an open neighborhood $V_m \subseteq W$ of y_0 such that $V_m \cdot (u_i - u_j) \subseteq W$ for all $i > j \geq N_m$.

Since X is local connected, there exists a connected closed neighborhood W_{m-1} of $x_{Q^{m-1}(1)}$ such that $W_{m-1} \subseteq U_{Q^{m-1}(1)} \cap \pi^{-1}(V_m)$. Now for any $i > j \geq N_m$, since $\pi^{-1}(V_m \cdot (u_i - u_j)) \subseteq \pi^{-1}(W) \subseteq \bigcup_{k=1}^n U_k$, we have $W_{m-1} \cdot (u_i - u_j) \subseteq \bigcup_{k=1}^n U_k$. Note that $W_{m-1} \cdot (u_i - u_j)$ is both connected and closed, $x_{Q^{m-1}(1)} \cdot (u_i - u_j) \in W_m = U_1$, and $\mathrm{cl}(U_k) \cap \mathrm{cl}(U_{k'}) = \emptyset$ for $1 \leq k < k' \leq n$. It follows that $W_{m-1} \cdot (u_i - u_j) \subseteq W_m$, i.e., we find a connected closed neighborhood W_{m-1} of $x_{Q^{m-1}(1)}$ and a positive integer N_m such that $W_{m-1} \cdot (u_i - u_j) \subseteq W_m$ for all $i > j \geq N_m$.

By repeating the above process and using induction, we find that, for each $v = m-1, m-2, \cdots, 1, 0$, there is a connected closed neighborhood W_v of $x_{Q^v(1)}$ and a positive integer N_{v+1} such that $W_v \cdot (u_i - u_j) \subseteq W_{v+1}$ for all $i > j \geq N_{v+1}$. Let $N = \max\{N_{v+1} : v = m-1, m-2, \cdots, 1, 0\}$. Then for any $i > j \geq N$, we have

$$\begin{aligned} W_0 \cdot (m(u_i - u_j)) &= (W_0 \cdot (u_i - u_j)) \cdot ((m-1)(u_i - u_j)) \subseteq W_1 \cdot ((m-1)(u_i - u_j)) \subseteq \cdots \\ &\subseteq W_v \cdot ((m-v)(u_i - u_j)) \subseteq \cdots \subseteq W_m = U_1 \subseteq U. \end{aligned}$$

Since $x_1 \in W_0$, $x_1 \cdot (m(u_i - u_j)) \in U$. This together with b) above implies that $m(u_i - u_j) \in N(x_1, U) \cap \{s_k - s_{k'} : k > k'\} \neq \emptyset$. Since $\{s_i\}_{i=1}^\infty$ is arbitrary, $N(x, U)$ is a Δ^*-set for any neighborhood U of x_1. Hence x_1 is a Δ^*-recurrent point, i.e., $(X, \mathbb{T})$ is almost automorphic.

2) Let d_Y be an invariant compatible metric on Y, d be the metric on X, and Y_0, Y_1 be the residual sets defined in 1).

First, we show that for each $y \in Y$, the fiber $\pi^{-1}(y)$ has at least n connected components. Suppose this is not true. Then there exists $y_1 \in Y$ such that $\pi^{-1}(y_1)$ has m-connected components $\{B_r\}_{r=1}^m$ for some $m \leq n-1$. For a fixed $y_0 \in Y_0 \cap Y_1$, we let $\pi^{-1}(y_0) = \{x_1, x_2, \cdots, x_n\}$. Also let U_i be open neighborhoods of x_i, for $i = 1, 2, \cdots, n$ respectively, such that $U_1 \subseteq U$ and $\mathrm{cl}(U_i) \cap \mathrm{cl}(U_j) = \emptyset$ for any $1 \leq i \neq j \leq n$. Since $y_0 \in Y_0$, there exists an open neighborhood W of y_0 such that for each $y \in W$, $\pi^{-1}(y) \cap U_i \neq \emptyset$, $i = 1, 2, \cdots, n$, and $\pi^{-1}(y) \subset \bigcup_{i=1}^n U_i$. Let $t \in \mathbb{T}$ be such that $y_1 \cdot t \in W$. Then $\pi^{-1}(y_1 \cdot t) = \bigcup_{r=1}^m B_r \cdot t \subseteq \bigcup_{i=1}^n U_i$ and $\pi^{-1}(y_1 \cdot t) \cap U_i \neq \emptyset$, $i = 1, 2, \cdots, n$. For each $r = 1, 2, \cdots, m$, since $B_r \cdot t$ is a closed connected subset of $\pi^{-1}(y_1 \cdot t)$ and $\mathrm{cl}(U_i) \cap \mathrm{cl}(U_j) = \emptyset$, there exists $i(r) \in \{1, 2, \cdots, n\}$ such that $B_r \cdot t \subseteq U_{i(r)}$. Hence $\{1, 2, \cdots, n\} \setminus \{i(1), i(2), \cdots, i(m)\} \neq \emptyset$. Let $i_0 \in \{1, 2, \cdots, n\} \setminus \{i(1), i(2), \cdots, i(m)\}$. Then $U_{i_0} \cap \pi^{-1}(y_1 \cdot t) \subseteq \bigcup_{r=1}^m U_{i_0} \cap U_{i(r)} = \emptyset$, a contradiction.

Next, suppose for contradiction that there exists $y \in Y$ such that $\pi^{-1}(y)$ has at least $(n+1)$-connected components $\{A_j\}_{j=1}^{n+1}$. For a fixed $y_0 \in Y_0 \cap Y_1$, we let $\pi^{-1}(y_0) = \{x_1, x_2, \cdots, x_n\}$ and $\epsilon = \min\{d(x_i, x_j) : 1 \leq i < j \leq n\}$. Also let N be a natural number such that $\epsilon > \frac{2}{N}$. For a given integer $m \geq N$, we consider the sets $U_i^m = \pi^{-1}(B(y_0, \frac{1}{m})) \cap B(x_i, \frac{1}{m})$, $i = 1, 2, \cdots, n$, where $B(y_0, r) = \{z \in Y : d_Y(z, y_0) < r\}$ and $B(x_i, r) = \{x \in X : d(x_i, x) < r\}$, $i = 1, 2, \cdots, n$. For each $i = 1, 2, \cdots, n$, since $x_i \in X_{lc}$, there exists a connected closed neighborhood V_i^m of x_i such that $V_i^m \subseteq U_i^m$. Let $V^m = \bigcup_{i=1}^n V_i^m$. Then V^m is a closed neighborhood of $\pi^{-1}(y_0)$. Since π^{-1} is continuous at y_0, there exists a neighborhood W_m of y_0 such that $\pi^{-1}(W_m) \subset V^m$.

Let $t_m \in \mathbb{T}$ be such that $y \cdot t_m \in W_m$. Then $A_j \cdot t_m \subseteq V^m$ for all $j = 1, 2, \cdots, n+1$. Since $A_j \cdot t_m$ is connected and $V_i^m \cap V_j^m = \emptyset$ for all $1 \leq i < j \leq n$, there exists a unique integer $n(j, m) \in \{1, 2, \cdots, n\}$ such that $A_j \cdot t_m \subseteq V_{n(j,m)}^m$. Hence there have to be integers $j_1(m), j_2(m)$ with $1 \leq j_1(m) < j_2(m) \leq n+1$ such that $n(j_1(m), m) = n(j_2(m), m)$, which we denote as $n(m)$.

Let $E_m = V_{n(m)}^m \cdot (-t_m)$. It is clear that E_m is a connected closed set, $A_{j_1(m)} \cup A_{j_2(m)} \subseteq E_m$, and $\pi(E_m) = \pi(V_{n(m)}^m) \cdot (-t_m) \subseteq B(y_0, \frac{1}{m}) \cdot (-t_m) = B(y_0 \cdot (-t_m), \frac{1}{m})$. Since $y \in \pi(E_m)$, we have $\pi(E_m) \subseteq B(y, \frac{2}{m})$.

Now we take a sequence $N \leq m_1 < m_2 < \cdots$ such that

(i) $j_1(m_1) = j_1(m_2) = \cdots$, denoted by j_1;
(ii) $j_2(m_1) = j_2(m_2) = \cdots$, denoted by j_2;
(iii) $\lim_{\ell\to\infty} E_{m_\ell} = E$ for some $E \in 2^X$ under the Hausdorff metric on 2^X.

It is clear that $1 \leq j_1 < j_2 \leq n+1$, $A_{j_1} \cup A_{j_2} \subseteq E$, and E is a connected closed set of X. Since $\pi(E_m) \subseteq B(y, \frac{2}{m})$, $E \subseteq \pi^{-1}(y)$. Note that E is connected, $A_{j_1} \cup A_{j_2} \subseteq E$, and A_{j_1}, A_{j_2} are two connected components of $\pi^{-1}(y)$. We must have $E = A_{j_1} = A_{j_2}$, which is a contradiction to the fact that $A_{j_1} \cap A_{j_2} = \emptyset$. □

7.2. **SPCF with at least two minimal sets.** We consider a SPCF $(S^1 \times Y, \mathbb{T}) = (S^1 \times Y, \{\Lambda_t\}_{t\in\mathbb{T}})$ in the form (1.1), i.e.,

$$\Lambda_t(s_0, y_0) = (\psi(s_0, y_0, t), y_0 \cdot t), \quad t \in \mathbb{T}.$$

We denote d_Y as a compatible metric on Y and $\pi : S^1 \times Y \to Y$ as the natural projection.

The following result may be regarded as a topological counterpart to the Furstenberg measure-theoretic characterization ([14]) for SPCFs.

Theorem 7.2. *Consider a SPCF $(S^1 \times Y, \mathbb{T})$ with minimal base flow $(Y, \mathbb{T})$ and assume that it has at least two minimal sets. Then the the following holds.*

a) *There is a positive integer n such that each minimal set is an almost n-1 extension of Y.*
b) *If the SPCF becomes APCF and one of its minimal set is almost automorphic, then so are others.*

Proof. a) Let M, M_0 be two minimal sets of $(S^1 \times Y, \mathbb{T})$. For each $y \in Y$, we consider the sets $S(y) = \{s \in S^1 : (s, y) \in M\}$ and $S_0(y) = \{s \in S^1 : (s, y) \in M_0\}$. Clearly, for each $y \in Y$, $S(y)$ and $S_0(y)$ are closed subsets of S^1, $S(y) \cap S_0(y) = \emptyset$, and the maps $\rho, \rho_0 : Y \to 2^{S^1 \times Y}$ defined by $\rho(y) = S(y)$, $\rho_0(y) = S_0(y)$, $y \in Y$, are upper semi-continuous. We denote by Y^c and Y_0^c, respectively, as the sets of continuity points of ρ, ρ_0, respectively. Then both Y^c and Y_0^c are residual subsets of Y.

Fix a point $y_0 \in Y^c \cap Y_0^c$. Since $S^1 \setminus S_0(y_0)$ is an open subset of S^1, $S^1 \setminus S_0(y)$ is a countable union of proper, open sub-arcs of S^1, i.e., $S^1 \setminus S_0(y_0) = \bigcup_{i=1}^{I} A_i$, where $1 \leq I \leq +\infty$ and each A_i is a proper, open sub-arc of S^1. Since $\bigcup_{i=1}^{I} A_i \supseteq S(y_0)$, there exists a positive integer $N(y_0)$ such that $\bigcup_{i=1}^{N(y_0)} A_i \supset S(y_0)$. Without loss of generality, we assume that $A_i \cap S(y_0) \neq \emptyset$ for all $i = 1, 2, \cdots, N(y_0)$.

Claim: For each $i = 1, 2, \cdots, N(y_0)$, $A_i \cap S(y_0)$ is a singleton. In particular, $|S(y_0)| = N(y_0) < \infty$.

Suppose for contradiction that the Claim is not true. Then there exists some $1 \leq i \leq N(y_0)$ such that $A_i \cap S(y_0)$ is not a singleton. We denote $A_i = (c, d)$ and let $B_i = [a, b]$ be a closed sub-arc of A_i such that $A_i \cap S(y_0) = B_i \cap S(y_0)$. It is clear that $a, b \in S(y_0)$ and $c, d \in S_0(y_0)$. Using minimality of M, we let $t_j \to +\infty$ be a sequence such that $\lim_{j\to\infty}(b, y_0) \cdot t_j = (a, y_0)$. By taking subsequences if necessary, we assume that $\lim_{j\to\infty}(a, y_0) \cdot t_j = (a', y_0)$, $\lim_{j\to\infty}(c, y_0) \cdot t_j = (c', y_0)$, and $\lim_{j\to\infty}(d, y_0) \cdot t_j = (d', y_0)$, for some $a' \in S(y_0)$ and $c', d' \in S_0(y_0)$. Let f_t be as in Lemma 5.3. Then $f_{t_j}(b) \to a$, $f_{t_j}(a) \to a'$, $f_{t_j}(c) \to c'$, and $f_{t_j}(d) \to d'$, as $j \to \infty$.

We first show that $c' = c$, $d' = d$, and $a' = a$. Suppose $c' \neq c$. Since $c' \neq a$, we have by Lemma 5.3 (1) that $\lim_{j\to\infty} f_{t_j}([c, b]) = [c', a] \supsetneq [c, a]$. Hence there is a sufficiently small open neighborhood V of c in S^1 such that $\overline{V} \subset (c', a)$. Since $y_0 \in Y_0^c$ and $c \in S_0(y_0)$, there exists an open neighborhood U of y_0 in Y such that $S_0(y) \cap V \neq \emptyset$ for all $y \in U$. Note that $f_{t_j}((c, b)) = (f_{t_j}(c), f_{t_j}(b))$, $\overline{V} \subset (c', a)$, and $y_0 \cdot t_j \to y_0$. There exists a positive integer J such that $f_{t_j}((c, b)) \supset V$ and $y \cdot t_j \in U$ as $j \geq J$. Moreover, for fixed $j \geq J$, there exists $s \in (c, b)$ such that $f_{t_j}(s) \in S_0(y_0 \cdot t_j) \cap V$. This implies that $(f_{t_j}(s), y_0 \cdot t_j) \in M_0$, i.e., $(s, y_0) \cdot t_j \in M_0$. Hence $(s, y_0) \in M_0$, i.e., $s \in S_0(y_0)$, which contradicts to the fact that $(c, b) \cap S_0(y_0) = \emptyset$. Therefore, $c' = c$. Similarly, $d' = d$. Since $a \in [c, b]$, $a' = \lim_{j\to\infty} f_{t_j}(a) \in [c, a]$. Hence $a' \in [c, a] \cap S(y_0) = \{a\}$, i.e., $a' = a$.

Next, we show that

$$S(y_0 \cdot t_j) \not\to S(y_0). \tag{7.2}$$

Since $f_{t_j}([a, b]) \subseteq f_{t_j}([c, b])$, $\limsup_{j\to\infty} f_{t_j}([a, b]) \subseteq [c, a] \neq S^1$. Also note that $f_{t_j}(a) \to a$ and $f_{t_j}(b) \to a$. We have by Lemma 5.3 (3) that $\lim_{j\to\infty} f_{t_j}([a, b]) = \{a\}$. If $c = d$, then

$S(y_0) \subseteq [a,b]$. Hence $S(y_0 \cdot t_j) = f_{t_j}(S(y_0)) = f_{t_j}(S(y_0) \cap [a,b]) \to \{a\} \neq S(y_0)$. Now suppose that $c \neq d$. Since $\lim_{j\to\infty} f_{t_j}(c) = c$ and $\lim_{j\to\infty} f_{t_j}(d) = d$, we have by Lemma 5.3 (1) that $\lim_{j\to\infty} f_{t_j}([d,c]) = [d,c]$. Note that $S(y_0 \cdot t_j) = f_{t_j}(S(y_0)) = f_{t_j}(S(y_0) \cap [a,b]) \cup f_{t_j}(S(y_0) \cap [d,c])$. It follows that $\limsup_{j\to\infty} S(y_0 \cdot t_j) \subseteq (\{a\} \cup [d,c])$. Since $b \notin \{a\} \cup [d,c]$, (7.2) holds.

Now, since $y_0 \in Y^c$ and $y_0 \cdot t_j \to y_0$, we have that $S(y_0 \cdot t_j) \to S(y_0)$, which is a contradiction (7.2). This proves the Claim.

It follows from the Claim that $S(y)$ is a finite set for any $y \in Y^c \cap Y_0^c$. Hence M is an almost finite to one extension of Y. It follows from Proposition 3.2 1) that M is an almost n-1 extension of Y for some positive integer $n = n(M)$. Similarly, M_0 is an almost n_0-1 extension for some positive integer $n_0 = n(M_0)$. In fact, from the proof of Proposition 3.2 1), we also see that $|S(y)| = n$ for any $y \in Y^c$ and $|S(y)| = n_0$ for any $y \in Y_0^c$. For a fixed $y \in Y^c \cap Y_0^c$, since $S^1 \setminus S_0(y)$ has precisely n_0 connected components, $N(y) \leq n_0$. Using the Claim, we also have $n = |S(y)| = N(y)$. Hence $n \leq n_0$. Similarly, $n_0 \leq n$. This shows that $n = n_0$.

b) Let M_0, M be two minimal sets of $(S^1 \times Y, \mathbb{T})$ among which M_0 is almost automorphic. We consider the set A_0 of almost automorphic points of M_0. Since A_0 is a residual subset of M_0, it follows from Lemma 5.5 that

$$Y_0^* = \{y \in Y : A_0 \cap \pi^{-1}(y) \text{ is a residual subset of } \pi^{-1}(y) \cap M_0\}$$

is a residual subset of Y.

Let $Y_0^c, Y^c, S_0(y), S(y)$ be as in a) for the minimal sets M_0, M and take any point $y_0 \in Y^c \cap Y_0^c \cap Y_0^*$. By a), $n = |S_0(y_0)| = |S(y_0)|$. Since $A_0 \cap \pi^{-1}(y_0) = \pi^{-1}(y_0) \cap M_0$, we see that for each $s \in S_0(y_0)$, $(s, y_0) \in A_0$. If $n = 1$, then M is an almost 1-1 extension of Y, hence it is almost automorphic by Theorem 3.2.

We now assume $n \geq 2$. Since $S^1 \setminus S_0(y_0)$ has precisely n connected components and each connected component contains precisely one point in $S(y_0)$, there exist $a_1 \neq a_2 \in S_0(y_0)$ and $b \in S(y_0)$ such that $\{b\} = [a_1, a_2] \cap S(y_0)$. We want to show that (b, y_0) is an almost automorphic point of M.

Let $\{t_i'\}$ be any sequence in $\mathbb{T}$. Since (a_1, y_0) and (a_2, y_0) are almost automorphic points of M_0, there exist a subsequence $\{t_i\} \subseteq \{t_i'\}$ and $a_1', a_2' \in S_0(y)$ for some $y \in Y$ such that $\lim_{i\to\infty}(a_j, y_0) \cdot t_i = (a_j', y)$ and $\lim_{i\to\infty}(a_j', y) \cdot (-t_i) = (a_j, y_0)$, for $j = 1, 2$. Taking a subsequence of $\{t_i\}$ if necessary, we may assume that there exist $b' \in S(y)$ and $b'' \in S(y_0)$ such that $\lim_{i\to\infty}(b, y_0) \cdot t_i = (b', y)$ and $\lim_{i\to\infty}(b', y) \cdot (-t_i) = (b'', y_0)$.

Since $\lim_{i\to\infty} \lim_{m\to\infty}(a_j, y_0) \cdot (t_m - t_i) = (a_j, y_0)$, $j = 1, 2$, and $\lim_{i\to\infty} \lim_{m\to\infty}(b, y_0) \cdot (t_m - t_i) = (b'', y_0)$, there exist sequences $\{m_k\}$ and $\{i_k\}$ such that if $r_k = t_{m_k} - t_{i_k}$ for all k, then $\lim_{k\to\infty}(a_j, y_0) \cdot r_k = (a_j, y_0)$, $j = 1, 2$, and $\lim_{k\to\infty}(b, y_0) \cdot r_k = (b'', y_0)$. Again, let f_t be as in Lemma 5.3. Since $\lim_{k\to\infty} f_{r_k}(a_j) = a_j$, we have by Lemma 5.3 (1) that $\lim_{k\to\infty} f_{r_k}([a_1, a_2]) = [a_1, a_2]$. Now, $b'' = \lim_{k\to\infty} f_{r_k}(b) \in \lim_{k\to\infty} f_{r_k}([a_1, a_2]) = [a_1, a_2]$. Hence $b'' \in [a_1, a_2] \cap S(y_0) = \{b\}$, i.e., $b'' = b$. This shows that (b, y_0) is an almost automorphic point of M, implying that M is almost automorphic. □

Theorem 7.3. *Consider an APCF $(S^1 \times Y, \mathbb{T})$ in which Y is locally connected. If there are at least two minimal sets, then each minimal set is almost automorphic.*

Proof. Let M, M_0 be two minimal sets of $(S^1 \times Y, \mathbb{T})$ and let $S(y), S_0(y)$, ρ, ρ_0, Y^c, Y_0^c be defined as in the proof of Theorem 7.2 for the present M, M_0.

Fix $y_0 \in Y^c \cap Y_0^c$. It follows from the proof of Theorem 7.2 that $n =: |S_0(y_0)| = |S(y_0)| < +\infty$. Again, if $n = 1$, then M is an almost 1-1 extension of Y, hence by Theorem 3.2 it is almost automorphic. We now assume that $n \geq 2$. Since $S^1 \setminus S_0(y_0)$ has precisely n connected components and each of them has precisely one point in $S(y_0)$, there exist points $0 \leq t_1 < r_1 < t_2 < r_2 < \cdots < t_n < r_n < 1 + t_1$ such that $S(y_0) = \{a_1, a_2, \cdots, a_n\}$ and $S_0(y_0) = \{b_1, b_2, \cdots, b_n\}$, where $a_j = e^{2\pi i t_j}$, $b_j = e^{2\pi i r_j}$, $j = 1, 2, \cdots, n$.

We want to show that (a_1, y_0) is an almost automorphic point of M. By Theorem 3.1, it is sufficient to show that for any open neighborhood U of (a_1, y_0) in $S^1 \times Y$, the recurrent time set $N((a_1, y_0), U)$ is a Δ^*-set.

Let U be an open neighborhood of (a_1, y_0) in $S^1 \times Y$. It is clear that there exist open neighborhoods W_1 of y_0 in Y and E of a_1 in S^1 such that $E \times W_1 \subseteq U$.

Let $\delta > 0$ be sufficiently small and

$$a_j^+ =: e^{2\pi i(t_j+\delta)},\ a_j^- =: e^{2\pi i(t_j-\delta)},\ A_j = [a_j^-, a_j^+],$$
$$b_j^+ =: e^{2\pi i(r_j+\delta)},\ b_j^- =: e^{2\pi i(r_j-\delta)},\ B_j = [b_j^-, b_j^+],$$

$j = 1, 2, \cdots, n$, be such that $A_1 \subset E$ and $A_1, B_1, A_2, B_2, \cdots, A_n, B_n$ are pairwise disjoint, i.e.,

$$t_1 - \delta < t_1 + \delta < r_1 - \delta < r_1 + \delta < t_2 - \delta < \cdots < t_n + \delta < r_n - \delta < r_n + \delta < 1 + t_1 - \delta.$$

Since $y_0 \in Y^c \cap Y_0^c$, there exists an open neighborhood W of y_0 with $W \subseteq W_1$ such that for all $y \in W$, $S(y) \subseteq \bigcup_{j=1}^n \mathrm{int}(A_j)$, $S_0(y) \subseteq \bigcup_{j=1}^n \mathrm{int}(B_j)$, $S(y) \cap \mathrm{int}(A_j) \neq \emptyset$, and $S_0(y) \cap \mathrm{int}(B_j) \neq \emptyset$ for all $j = 1, 2, \cdots, n$. Thus for each $y \in W$ and $j = 1, 2, \cdots, n$ there exist $t_{j,y}^{\pm}, r_{j,y}^{\pm}$ with $t_j - \delta \leq t_{j,y}^- \leq t_{j,y}^+ \leq t_j + \delta$ and $r_j - \delta \leq r_{j,y}^- \leq r_{j,y}^+ \leq r_j + \delta$ such that if

$$a_{j,y}^+ =: e^{2\pi i t_{j,y}^+},\ a_j^- =: e^{2\pi i t_{j,y}^-} \in S(y),$$
$$b_{j,y}^+ =: e^{2\pi i r_{j,y}^+},\ b_j^- =: e^{2\pi i r_{j,y}^-} \in S_0(y),$$

then $S(y) \cap A_j = S(y) \cap A_{j,y} \subseteq \mathrm{int}(A_j)$ and $S_0(y) \cap B_j = S_0(y) \cap B_{j,y} \subseteq \mathrm{int}(B_j)$, where $A_{j,y} = [a_{j,y}^-, a_{j,y}^+]$ and $B_{j,y} = [b_{j,y}^-, b_{j,y}^+]$. It is clear that $A_{1,y}, B_{1,y}, A_{2,y}, B_{2,y}, \cdots, A_{n,y}, B_{n,y}$ are pairwise disjoint for each $y \in W$, and, for each $j = 1, 2, \cdots, n$, the map $E_j^- : W \to [t_j - \delta, t_j + \delta]$: $y \mapsto t_{j,y}^-$ is lower semi-continuous and the map $E_j^+ : W \to [t_j - \delta, t_j + \delta]$: $y \mapsto t_{j,y}^+$ is upper semi-continuous. It follows that for each $j = 1, 2, \cdots, n$, both maps $E_j : W \to 2^{[t_j-\delta, t_j+\delta]}$: $y \mapsto [t_{j,y}^-.t_{j,y}^+]$ and $\phi_j : W \to 2^{S^1}$: $y \mapsto A_{j,y} = [a_{j,y}^-, a_{j,y}^+] = \{e^{2\pi t} : t \in [t_{j,y}^-, t_{j,y}^+]\}$ are upper semi-continuous.

Claim 1: Given $j \in \{1, 2, \cdots, n\}$, $y \in W$, and $t \in \mathbb{T}$ such that $y \cdot t \in W$, there exists a unique $L_j^t(y) \in \{1, 2, \cdots, n\}$ such that $(A_{j,y} \times \{y\}) \cdot t = A_{L_j^t(y), y\cdot t} \times \{y \cdot t\}$. Moreover, for fixed y, t as above, the map $L_{\{\cdot\}}^t(y) : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}$ is a permutation of $\{1, 2, \cdots, n\}$.

Let j, y, t be given as above. We consider the orientation preserving homeomorphism $h : S^1 \to S^1$: $s \mapsto \psi(s, y, t)$. It is clear that $(A_{j,y} \times \{y\}) \cdot t = h(A_{j,y}) \times \{y \cdot t\}$, $h(A_{j,y}) = [h(a_{j,y}^-), h(a_{j,y}^+)]$, and $h(a_{j,y}^-), h(a_{j,y}^+) \in S(y \cdot t)$. If there exists $b \in h(A_{j,y}) \cap S_0(y \cdot t)$, i.e., $(b, y \cdot t) \in (A_{j,y} \times \{y\}) \cdot t$ and $(b, y \cdot t) \in M_0$, then there exists $b' \in A_{j,y}$ such that $(b', y) \cdot t = (b, y \cdot t) \in M_0$. It follows that $(b', y) \in M_0$, i.e., $b' \in A_{j,y} \cap S_0(y) \subseteq A_j \cap (\bigcup_{i=1}^n B_i) = \emptyset$, which is impossible. Hence

$$h(A_{j,y}) \cap S_0(y \cdot t) = \emptyset. \tag{7.3}$$

Since $y \cdot t \in W$ and $h(a_{j,y}^-), h(a_{j,y}^+) \in S(y \cdot t)$, there exist $i_1, i_2 \in \{1, 2, \cdots, n\}$ such that $h(a_{j,y}^-) \in A_{i_1, y\cdot t} \subseteq A_{i_1}$ and $h(a_{j,y}^+) \in A_{i_2, y\cdot t} \subseteq A_{i_2}$. Since $[h(a_{j,y}^-), h(a_{j,y}^+)] \cap S_0(y \cdot t) = \emptyset$, we must have $i_1 = i_2$. For otherwise, $i_1 \neq i_2$, and the arc $[h(a_{j,y}^-), h(a_{j,y}^+)]$ intersects both A_{i_1} and A_{i_2}. It follows that $B_{i_1} \subseteq [h(a_{j,y}^-), h(a_{j,y}^+)]$, and hence $b_{i_1, y\cdot t}^- \in B_{i_1} \cap S_0(y \cdot t) \subseteq [h(a_{j,y}^-), h(a_{j,y}^+)] \cap S_0(y \cdot t)$, which is impossible by (7.3).

Now let $L_j^t(y) = i_1$. Then $h(a_{j,y}^-), h(a_{j,y}^+) \in A_{L_j^t(y), y\cdot t}$. Since $A_{L_j^t(y), y\cdot t}$ is a sub-arc of S^1, either (i) $[h(a_{j,y}^-), h(a_{j,y}^+)] \subseteq A_{L_j^t(y), y\cdot t}$ or (ii) $[h(a_{j,y}^-), h(a_{j,y}^+)] \supseteq S^1 \setminus A_{L_j^t(y), y\cdot t}$. But the case (ii) is impossible because $S_0(y \cdot t) \subseteq S^1 \setminus A_{L_j^t(y), y\cdot t}$ and $[h(a_{j,y}^-), h(a_{j,y}^+)] \cap S_0(y \cdot t) = \emptyset$. It now follows from (i) that

$$(A_{j,y} \times \{y\}) \cdot t \subseteq A_{L_j^t(y), y\cdot t} \times \{y \cdot t\}. \tag{7.4}$$

Such $L_j^t(y)$ is unique because $A_{1,y\cdot t}, A_{2,y\cdot t}, \cdots, A_{n,y\cdot t}$ are pairwise disjoint. Let $y' = y \cdot t$ and $t' = -t$. Then $y', y' \cdot t' \in W$. From the above, for each $i = 1, 2, \cdots, n$, there exists a unique $L_i^{t'}(y') \in \{1, 2, \cdots, n\}$ such that

$$(A_{i,y'} \times \{y'\}) \cdot t' \subseteq A_{L_i^{t'}(y'), y'\cdot t'} \times \{y' \cdot t'\}. \tag{7.5}$$

For each $j = 1, 2, \cdots, n$, we let $i(j) = L_j^t(y)$. By (7.4) and (7.5), we have

$$A_{j,y} \times \{y\} \subseteq (A_{i(j),y'} \times \{y'\}) \cdot t' \subseteq A_{L_{i(j)}^{t'}(y'), y} \times \{y\}.$$

Since $A_{1,y\cdot t}, A_{2,y\cdot t}, \cdots, A_{n,y\cdot t}$ are pairwise disjoint, $L_{i(j)}^{t'}(y') = j$ and $A_{j,y} \times \{y\} = (A_{i(j),y'} \times \{y'\}) \cdot t'$. This implies that

$$(A_{j,y} \times \{y\}) \cdot t = ((A_{i(j),y'} \times \{y'\}) \cdot t') \cdot t = A_{i(j),y'} \times \{y'\} = A_{L_j^t(y), y\cdot t} \times \{y \cdot t\},$$

i.e., $(A_{j,y} \times \{y\}) \cdot t = A_{L_j^t(y), y\cdot t} \times \{y \cdot t\}$.

Note that $L_j^t(y) \in \{1, 2, \cdots, n\}$, $(A_{j,y} \times \{y\}) \cdot t = A_{L_j^t(y), y\cdot t} \times \{y \cdot t\}$ for each $j = 1, 2, \cdots, n$, and $A_{1,y}, A_{2,y}, \cdots, A_{n,y}$ are pairwise disjoint. We have that $L_{\{\cdot\}}^t(y) : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}$ is one to one, i.e., a permutation of $\{1, 2, \cdots, n\}$. This proves Claim 1.

Claim 2. Let V be a nonempty, connected, closed subset of W and $t \in \mathbb{T}$ be such that $V \cdot t \subset W$. Then there exists a permutation P of $\{1, 2, \cdots, n\}$ such that for each $j = 1, 2, \cdots, n$ and $y \in V$, $(A_{j,y} \times \{y\}) \cdot t = A_{P(j), y\cdot t} \times \{y \cdot t\}$.

Let $j \in \{1, 2, \cdots, n\}$ and $y \in V$ be given. By Claim 1, there exists a unique $L_j^t(y) \in \{1, 2, \cdots, n\}$ such that $(A_{j,y} \times \{y\}) \cdot t = A_{L_j^t(y), y\cdot t} \times \{y \cdot t\}$.

We first show that the map $L_j^t(\cdot) : V \to \{1, 2, \cdots, n\}$ is continuous. Let $\{y_k\}_{k=1}^{\infty} \subset V$ converges to some $y \in V$ and denote $i = L_j^t(y)$. If $L_j^t(y_k) \not\to L_j^t(y)$, then by Claim 1 there exists a subsequence $\{k_1 < k_2 < \cdots\}$ of $\{k\}$ and $r \in \{1, 2, \cdots, n\} \setminus \{i\}$ such that $L_j^t(y_{k_\ell}) = r$ for each $\ell \in \mathbb{N}$. Take a sequence of points $z_\ell \in A_{j, y_{k_\ell}}$, $\ell \in \mathbb{N}$. We assume without loss of generality that $\lim_{\ell\to\infty} z_\ell = z$ for some $z \in S^1$. By the upper semi-continuity of the map $\phi_j : W \to 2^{S^1}$: $y \mapsto A_{j,y}$, we have that $z \in A_{j,y}$. Since $(z_\ell, y_{k_\ell}) \cdot t \in A_{r, y_{k_\ell}} \times \{y_{k_\ell} \cdot t\}$, it again follows from the upper semi-continuity of ϕ_j that $(z, y) \cdot t = \lim_{\ell\to\infty} (z_\ell, y_{k_\ell}) \cdot t \in A_{r, y\cdot t} \times \{y \cdot t\}$. But since $(z, y) \cdot t \in (A_{j,y} \times \{y\}) \cdot t \in A_{i, y\cdot t} \times \{y \cdot t\}$, $(z, y) \cdot t \in A_{i, y\cdot t} \times \{y \cdot t\} \cap A_{r, y\cdot t} \times \{y \cdot t\}$. Hence $A_{i, y\cdot t} \cap A_{r, y\cdot t} \neq \emptyset$, a contradiction to the fact that $i \neq r$. This shows the continuity of $L_j^t(\cdot) : V \to \{1, 2, \cdots, n\}$.

Now, for each $j = 1, 2, \cdots, n$, $L_j^t(\cdot) : V \to \{1, 2, \cdots, n\}$ must be a constant map since its domain is connected and its range is discrete. Let $y \in V$ and $P(j) = L_j^t(y)$, $j = 1, 2, \cdots, n$. Then

$$(A_{j,y} \times \{y\}) \cdot t = A_{P(j), y\cdot t} \times \{y \cdot t\}.$$

It follows from Claim 1 that $P : \{1, 2 \cdots, n\} \to \{1, 2, \cdots, n\}$ is a permutation of $\{1, 2, \cdots, n\}$.

Claim 3. For any sequence $\{s_i\}$ in $\mathbb{T}$, $N((a_1, y_0), U) \cap \{s_k - s_{k'} : k > k'\} \neq \emptyset$, i.e., $N((a_1, y_0), U)$ is a Δ^*-set.

Let $m = n!$ and d be an invariant compatible metric on Y, i.e., $d(y_1 \cdot t, y_2 \cdot t) = d(y_1, y_2)$, $y_1, y_2 \in Y$, $t \in \mathbb{T}$. Since $(Y, \mathbb{T})$ is almost periodic, there exists an increasing subsequence $\{i_k\} \subset \mathbb{N}$, an integer $r \in \{0, 1, \cdots, m-1\}$, and a homeomorphism $g : Y \to Y$ such that $\frac{s_{i_k} - r}{m} \in \mathbb{T}$ and

$$\sup_{y\in Y} d(y \cdot \frac{s_{i_k} - r}{m}, g(y)) \leq \frac{1}{2^k} \tag{7.6}$$

for all $k = 1, 2, \cdots$. Since Y is locally connected, there exists a connected closed neighborhood V of y_0 such that $B_\delta(V) =: \{y \in Y : d(y, V) < \delta\} \subseteq W$ and $B(y_0, \delta) \subseteq V$ for some $\delta > 0$. Let K be a natural number such that $\frac{1}{2^K} < \frac{\delta}{2m}$. Then for any $u, v \in \mathbb{N}$ with $u > v \geq K$, we have by (7.6) that

$$\begin{aligned}\sup_{y\in Y} d(y\cdot(\frac{s_{i_u}-s_{i_v}}{m}), y) &= \sup_{y\in Y} d(y\cdot\frac{s_{i_u}-r}{m}, y\cdot\frac{s_{i_v}-r}{m}) \\ &\leq \sup_{y\in Y} d(y\cdot\frac{s_{i_u}-r}{m}, g(y)) + \sup_{y\in Y} d(y\cdot\frac{s_{i_v}-r}{m}, g(y)) \\ &\leq \frac{1}{2^u} + \frac{1}{2^v} < \frac{1}{2^{K-1}} < \frac{\delta}{m}.\end{aligned}$$

It follows that $d(V\cdot(\frac{s_{i_u}-s_{i_v}}{m}), V) < \frac{\delta}{m}$, and, for each $\ell = 1, 2, \cdots, n$,

$$\begin{aligned}d\left(y_0\cdot(\ell\frac{s_{i_u}-s_{i_v}}{m}), y_0\right) &\leq \sum_{j=1}^{\ell} d\left(y_0\cdot(j\frac{s_{i_u}-s_{i_v}}{n}), y_0\cdot((j-1)\frac{s_{i_u}-s_{i_v}}{n})\right) \\ &= \sum_{j=1}^{\ell} d\left(y_0\cdot(\frac{s_{i_u}-s_{i_v}}{m}), y_0\right) < \frac{\ell}{m}\delta \leq \delta,\end{aligned}$$

i.e., $y_0\cdot(\ell\frac{s_{i_u}-s_{i_v}}{m}) \subseteq V$. Let $t = \frac{s_{i_u}-s_{i_v}}{m}$. Since $V\cdot t \subseteq W$, we have by Claim 2 that there exists a permutation P of $\{1, 2, \cdots, n\}$ such that for each $j = 1, 2, \cdots, n$ and $y \in V$, $(A_{j,y}\times\{y\})\cdot t = A_{P(j),y\cdot t}\times\{y\cdot t\}$. Clearly, $P^m(j) = j$ for each $j = 1, 2, \cdots, n$. Thus, for any $w = 0, 1, 2, \cdots, m-1$, since $y_0\cdot(wt) \in V$, we have

$$(A_{P^w(1),y_0\cdot(wt)}\times\{y_0\cdot(wt)\})\cdot t = A_{P^{w+1}(1),y_0\cdot((w+1)t)}\times\{y\cdot((w+1)t)\},$$

where $P^0(1) = 1$. By induction, we further have

$$(A_{1,y_0}\times\{y_0\})\cdot(mt) = A_{1,y_0\cdot(mt)}\times y_0\cdot(mt) \subseteq A_1\times V \subseteq U.$$

Since $A_{1,y_0} = \{(a_1, y_0)\}$ and $mt = s_{i_u} - s_{i_v}$, we have $(a_1, y_0)\cdot(s_{i_u}-s_{i_v}) \in U$. Hence $s_{i_u}-s_{i_v} \in N((a_1,y_0),U)\cap\{s_k - s_{k'} : k > k'\}$, i.e., $N((a_1,y_0),U)\cap\{s_k - s_{k'} : k > k'\} \neq \emptyset$. Since $\{s_i\}$ is arbitrary, $N((a_1,y_0),U)$ is a Δ^*-set.

This completes the proof. □

Now, parts 1) and 2) of Theorem 4 are the respective parts in Theorem 7.2 above, and, part 3) of Theorem 4 is just Theorem 7.3 above.

8. Mean motion, Transitivity, and connectivity

In this section, we consider a SPCF $(S^1\times Y, \mathbb{T}) = (S^1\times Y, \{\hat{\Lambda}_t\}_{t\in\mathbb{T}})$ in the angular form (1.3), i.e.,

$$\hat{\Lambda}_t(\phi_0, y_0) = (\phi(\phi_0, y_0, t), y_0\cdot t),\quad t\in\mathbb{T}, \tag{8.1}$$

where $\phi, \phi_0 \in R^1 \pmod 1$, $y_0 \in Y$. Let $\tilde{\phi}(\tilde{\phi}_0, y_0, t)$ be the lift of $\phi(\phi_0, y_0, t)$ in R^1 satisfying $\tilde{\phi}(\tilde{\phi}_0+1, y_0, t) \equiv \tilde{\phi}(\tilde{\phi}_0, y_0, t)+1$. Then it is clear that $\hat{\Lambda}_t$ is generated from the flow $\tilde{\Lambda}_t : R^1\times Y \to R^1\times Y$:

$$\tilde{\Lambda}_t(\tilde{\phi}_0, y_0) = (\tilde{\phi}(\tilde{\phi}_0, y_0, t), y_0\cdot t),\quad t\in\mathbb{T}$$

when $\tilde{\phi}_0, \tilde{\phi}$ are identified modulo 1.

Through the section, for simplicity, we will often use the same symbol ϕ_0 to denote a point $\phi_0 \in S^1$ and its lift $\tilde{\phi}_0 \in R^1$.

8.1. **Rotation number and mean motion.** It is more or less known that a SPCF (8.1) with uniquely ergodic base flow $(Y, \mathbb{T})$ admits a well-defined rotation number (see [24] for the discrete case and [37] for certain almost periodic continuous case). Below, for the sake of completeness, we give a unified proof of this result for both discrete and continuous cases.

The following result is known as the Oxtoby ergodic theorem (see [14]).

Lemma 8.1. *Let $(X, \mathbb{T})$ be uniquely ergodic and $f \in C(X, R^1)$. Then for any $x \in X$,*

$$\lim_{T\to+\infty} \frac{1}{\lambda_{\mathbb{T}}([0,T)\cap\mathbb{T})} \int_{[0,T)\cap\mathbb{T}} f(x\cdot t)\, d\lambda_{\mathbb{T}}(t) = \int_X f(z) d\mu(z),$$
$$\lim_{T\to+\infty} \frac{1}{\lambda_{\mathbb{T}}((-T,0]\cap\mathbb{T})} \int_{(-T,0]\cap\mathbb{T}} f(x\cdot t)\, d\lambda_{\mathbb{T}}(t) = \int_X f(z) d\mu(z),$$

where $\lambda_{\mathbb{T}}$ is the Haar measure on $\mathbb{T}$ with $\lambda_{\mathbb{T}}([0,1)\cap\mathbb{T}) = 1$ and μ is the unique $\mathbb{T}$-invariant probability measure on $(X, \mathbb{T})$.

Theorem 8.1. *Consider the SPCF (8.1) with uniquely ergodic base flow $(Y, \mathbb{T})$. Then for any $\phi_0 \in R^1$, $y_0 \in Y$, the limit*

$$\rho = \lim_{t\to\infty} \frac{\tilde{\phi}(\phi_0, y_0, t)}{t}$$

exists and is independent of choice of $(\phi_0, y_0) \in R^1 \times Y$.

Proof. For simplicity, we only consider the limit as $t \to +\infty$.

First, we observe from the periodicity of $\tilde{\phi}$ in the first argument that for any $t \in \mathbb{T}$, $y \in Y$, if $\phi_1^*, \phi_2^* \in R^1$ are such that $|\phi_1^* - \phi_2^*| < l$ for some positive integer l, then also

$$|\tilde{\phi}(\phi_1^*, y, t) - \tilde{\phi}(\phi_2^*, y, t)| < l. \tag{8.2}$$

Next, for any $\phi_0 \in R^1$, $y \in Y$, $t, s \in \mathbb{T}$, we let $0 \le \phi_1, \phi_2 < 1$ be such that

$$\phi_1 \equiv \phi_0,\ \ \phi_2 \equiv \tilde{\phi}(\phi_0, y, t),\ \ (\mathrm{mod}\, 1).$$

Then

$$\begin{aligned} \tilde{\phi}(\phi_0, y\cdot t, s) - \phi_0 &= \tilde{\phi}(\phi_1, y\cdot t, s) - \phi_1, \\ \tilde{\phi}(\phi_0, y, t+s) - \tilde{\phi}(\phi_0, y, t) &= \tilde{\phi}(\phi_2, y\cdot t, s) - \phi_2. \end{aligned}$$

It follows from (8.2) that

$$\begin{aligned} &|\tilde{\phi}(\phi_0, y, t+s) - \tilde{\phi}(\phi_0, y, t) - \tilde{\phi}(\phi_0, y\cdot t, s) + \phi_0| \\ &= |\tilde{\phi}(\phi_2, y\cdot t, s) - \tilde{\phi}(\phi_1, y\cdot t, s) + \phi_2 - \phi_1| \\ &\le |\tilde{\phi}(\phi_2, y\cdot t, s) - \tilde{\phi}(\phi_1, y\cdot t, s)| + |\phi_2 - \phi_1| \le 4, \end{aligned}$$

i.e.,

$$-4 \ \le\ \tilde{\phi}(\phi_0, y, t+s) - \tilde{\phi}(\phi_0, y, t) - \tilde{\phi}(\phi_0, y\cdot t, s) + \phi_0 \le 4. \tag{8.3}$$

Integrating (8.3) with respect to t from 0 to a positive number $T \in \mathbb{T}$ yields that

$$-4 \le \frac{1}{T}\{\int_{[0,T)\cap\mathbb{T}} (\tilde{\phi}(\phi_0, y, t+s) - \tilde{\phi}(\phi_0, y, t))\, d\lambda_{\mathbb{T}}(t) - \int_{[0,T)\cap\mathbb{T}} (\tilde{\phi}(\phi_0, y\cdot t, s) - \phi_0)\, d\lambda_{\mathbb{T}}(t)\} \le 4. \tag{8.4}$$

For any positive number $s \in \mathbb{T}$, we let $M_s = \sup\{|\tilde{\phi}(\phi_0, z, \tau) - \phi_0| : \phi_0 \in R^1, z \in Y, |\tau| \le s\}$. It is clear that $0 \le M_s < +\infty$ and

$$\begin{aligned}
&|\int_{[0,T)\cap\mathbb{T}} (\tilde{\phi}(\phi_0, y, t+s) - \tilde{\phi}(\phi_0, y, t))\, d\lambda_{\mathbb{T}}(t) - s\tilde{\phi}(\phi_0, y, T)| \\
&= |\int_{[0,s)\cap\mathbb{T}} (\tilde{\phi}(\phi_0, y, T+t) - \tilde{\phi}(\phi_0, y, t))\, d\lambda_{\mathbb{T}}(t) - s\tilde{\phi}(\phi_0, y, T)| \\
&= |\int_{[0,s)\cap\mathbb{T}} (\tilde{\phi}(\tilde{\phi}(\phi_0, y, T), y\cdot T, t) - \tilde{\phi}(\phi_0, y, T))\, d\lambda_{\mathbb{T}}(t) - \int_{[0,s)\cap\mathbb{T}} (\tilde{\phi}(\phi_0, y, t) - \phi_0)\, d\lambda_{\mathbb{T}}(t) - s\phi_0| \\
&\le s(|\phi_0| + 2M_s) =: s\Phi_s.
\end{aligned}$$

Combining this with (8.4) yields that

$$-\frac{4T + s\Phi_s}{sT} \le \frac{1}{T}\tilde{\phi}(\phi_0, y, T) - \frac{1}{sT}\int_{[0,T)\cap\mathbb{T}} (\tilde{\phi}(\phi_0, y\cdot t, s) - \phi_0) d\lambda_{\mathbb{T}}(t) \le \frac{4T + s\Phi_s}{sT}. \tag{8.5}$$

Now consider functions $\rho^*(y) = \limsup_{t\to+\infty} \dfrac{\tilde{\phi}(\phi_0, y, t)}{t}$ and $\rho_*(y) = \liminf_{t\to+\infty} \dfrac{\tilde{\phi}(\phi_0, y, t)}{t}$. We note by (8.2) that both $\rho^*(y)$ and $\rho_*(y)$ are independent of ϕ_0. Since $(Y, \mathbb{T})$ is uniquely ergodic, we have by Lemma 8.1 that

$$\lim_{T\to+\infty} \frac{1}{T}\int_{[0,T)\cap\mathbb{T}} (\tilde{\phi}(\phi_0, y\cdot t, s) - \phi_0) d\lambda_{\mathbb{T}}(t) = \int_Y (\tilde{\phi}(\phi_0, z, s) - \phi_0) d\mu(z), \tag{8.6}$$

where μ is the unique $\mathbb{T}$-invariant Borel probability measure on $(Y, \mathbb{T})$. By letting $T \to +\infty$ in (8.5) and applying (8.5) and (8.6), we have that

$$-\frac{4}{s} + \frac{1}{s}\int_Y (\tilde{\phi}(\phi_0, z, s) - \phi_0) d\mu(z) \le \rho_*(y) \le \rho^*(y) \le \frac{4}{s} + \frac{1}{s}\int_Y (\tilde{\phi}(\phi_0, z, s) - \phi_0) d\mu(z).$$

Now, taking limit $s \to +\infty$ in the above, we see that $\rho_*(y) = \rho^*(y)$ and equals

$$\lim_{s\to+\infty} \frac{1}{s}\int_Y (\tilde{\phi}(\phi_0, z, s) - \phi_0) d\mu(z),$$

which is a constant, denoted by ρ. □

The limit in the theorem is referred to as the *rotation number* associated with the SPCF (8.1).

Recall that the SPCF (8.1) is said to admit *mean motion* if

$$\sup_{t\in\mathbb{T}} |\tilde{\phi}(\phi_0, y_0, t) - \phi_0 - \rho t| < \infty$$

for all $(\phi_0, y_0) \in R^1 \times Y$.

Theorem 8.2. *Consider the SPCF* (8.1) *with strictly ergodic base flow* $(Y, \mathbb{T})$. *Then the followings are equivalent:*

a) (8.1) *admits mean motion;*
b) $\sup_{t\ge0} |\tilde{\phi}(\phi_0, y_0, t) - \phi_0 - \rho t| < +\infty$ *for some* $(\phi_0, y_0) \in R^1 \times Y$*;*
c) $\sup_{t\le0} |\tilde{\phi}(\phi_0, y_0, t) - \phi_0 - \rho t| < +\infty$ *for some* $(\phi_0, y_0) \in R^1 \times Y$*;*
d) $\sup_{t\ge0} (\tilde{\phi}(\phi_0, y_0, t) - \phi_0 - \rho t) < +\infty$ *for all* $(\phi_0, y_0) \in R^1 \times Y$*;*
e) $\sup_{t\le0} (\tilde{\phi}(\phi_0, y_0, t) - \phi_0 - \rho t) < +\infty$ *for all* $(\phi_0, y_0) \in R^1 \times Y$*;*
f) $\inf_{t\ge0} (\tilde{\phi}(\phi_0, y_0, t) - \phi_0 - \rho t) > -\infty$ *for all* $(\phi_0, y_0) \in R^1 \times Y$*;*
g) $\inf_{t\le0} (\tilde{\phi}(\phi_0, y_0, t) - \phi_0 - \rho t) > -\infty$ *for all* $(\phi_0, y_0) \in R^1 \times Y$.

Proof. It is clear that a) implies b)-g).

Suppose b) holds and let

$$M = \sup_{t\ge0} |\tilde{\phi}(\phi_0, y_0, t) - \phi_0 - \rho t|.$$

Then by the flow property,

$$|\tilde\phi(\tilde\phi(\phi_0, y_0, s), y_0 \cdot s, t) - \tilde\phi(\phi_0, y_0, s) - \rho t) + (\tilde\phi(\phi_0, y_0, s) - \phi_0 - \rho s| \le M \tag{8.7}$$

for all $s + t \ge 0$. Consider the ω-limit set $\omega(\phi_0, y_0)$ of (ϕ_0, y_0) with respect to the flow (8.1). For simplicity, we view $\omega(\phi_0, y_0)$ as a subset of $[0, 1] \times Y$. Let $(\phi_*, y_*) \in \omega(\phi_0, y_0)$ and $s_n \to +\infty$ be a sequence such that $\tilde\Lambda_{s_n}(\phi_0, y_0) \to (\phi_*, y_*)$. By taking a subsequence if necessary, we let $r(\phi_*, y_*) = \lim_{n\to\infty} |\tilde\phi(\phi_0, y_0, s_n) - \phi_0 - \rho s_n|$. It follows from (8.7) that

$$|\tilde\phi(\phi_*, y_*, t) - \phi_* - \rho t| \le M - r(\phi_*, y_*)$$

for all $t \in \mathbb{T}$. Hence by (8.2),

$$\sup_{t\in\mathbb{T}} |\tilde\phi(\phi^0, y^0, t) - \phi^0 - \rho t| < \infty$$

for all $(\phi^0, y^0) \in R^1 \times Y$, i.e., a) holds. This shows that b) implies a).

Similarly, c) implies a).

Now let d) hold. Suppose for contradiction that a) fails. It follows from the flow property and the equivalence between a) and b) that

$$\sup_{n\in\mathbb{N}} |\tilde\phi(\phi^*, y^*, n) - \phi^* - \rho n| = +\infty \tag{8.8}$$

for all $(\phi^*, y^*) \in R^1 \times Y$. Let E be a minimal set of the time-1 map $\hat\Lambda_1$ and consider the function $u : R^1 \times Y \to R^1$: $u(\phi, y) = \tilde\phi(\phi, y, 1) - \phi - \rho$. Since $u(\phi + 1, y) \equiv u(\phi, y)$, u can be viewed as a continuous function on $S^1 \times Y$. Using flow property and induction, it is easy to see that

$$\sum_{i=0}^{n-1} u(\tilde\Lambda_i(\phi, y)) = \tilde\phi(\phi, y, n) - \phi - \rho n \tag{8.9}$$

for all $n \in \mathbb{N}$ and $(\phi, y) \in R^1 \times Y$. Hence by (8.8), $|\sum_{i=0}^{n-1} u(\tilde\Lambda_i(\phi^*, y^*))|$ is unbounded on $\mathbb{N}$ for any $(\phi^*, y^*) \in E$. Since $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} u(\tilde\Lambda_i(\phi^*, y^*)) = 0$ for any $(\phi^*, y^*) \in E$, there is a residual subset E_* of E such that for any $(\phi_*, y_*) \in E_*$ the function $\sum_{i=0}^{n-1} u(\tilde\Lambda_i(\phi_*, y_*))$ oscillates from $-\infty$ to $+\infty$ as $n \to +\infty$ (see e.g., [31, 38]). In particular,

$$\sup_{n\in\mathbb{N}} (\tilde\phi(\phi_*, y_*, n) - \phi_* - \rho n) = \sup_{n\in\mathbb{N}} \sum_{i=0}^{n-1} u(\tilde\Lambda_i(\phi_*, y_*)) = +\infty \tag{8.10}$$

for all $(\phi_*, y_*) \in E_*$. This is a contradiction to the condition in d). Hence d) implies a).

Similarly, either e) or f) or g) implies a). □

8.2. **APCF with mean motion.** The aim of this subsection is to prove Theorem 6. We will need the following

Lemma 8.2. *Let M be a minimal set of an almost periodically forced skew-product flow $(R^1 \times Y, \mathbb{T}) = (R^1 \times Y, \{\Pi_t\}_{t\in\mathbb{T}})$:*

$$\Pi_t(x_0, y_0) = (x(x_0, y_0, t), y \cdot t), \quad t \in \mathbb{T},$$

where $(Y, \mathbb{T})$ is an almost periodic minimal flow. Then M is an almost 1-1 *extension of Y hence is almost automorphic.*

Proof. In the case $\mathbb{T} = \mathbb{R}$, the lemma is a special case of the main result in [60] concerning totally monotone skew-product semiflows. The proof for the discrete case follows from that of the continuous case (see also [67]) almost word by word. □

The following theorem is just our main result Theorem 6.

Theorem 8.3. *Suppose that* (8.1) *is an APCF which admits mean motion. Then the following holds.*

1) *Each minimal set of* (8.1) *is almost automorphic whose frequency module is generated by the rotation number and the forcing frequencies.*
2) *If a minimal set of* (8.1) *is an almost* N*-1 extension of* Y *for some positive integer* N*, then* N *is the smallest positive integer whose multiplication to the rotation number is contained in the frequency module of the forcing.*

Proof. 1) Let E be a minimal set of (8.1). It is sufficient to only consider the case $\mathbb{T} = \mathbb{Z}$, because, if $\mathbb{T} = \mathbb{R}$, then a point of E is almost automorphic iff it is almost automorphic for the time-1 map $\tilde{\Lambda}_1$ ([4]).

Let $Y^* = S^1 \times Y$ be given the flow $(\phi_0, y_0) \cdot t = (\phi_0 + \rho t \pmod 1), y_0 \cdot t)$, $t \in \mathbb{Z}$. Then $(Y^*, \mathbb{Z})$ is almost periodic (need not be minimal). Consider the skew-product flow $\Lambda_t^* : R^1 \times Y^* \to R^1 \times Y^*$:

$$\Lambda_t^*(x_0, \phi_0, y_0) = (\tilde{\phi}(\phi_0, y_0, t) - \phi_0 - \rho t + x_0, \phi_0 + \rho t \pmod 1), y_0 \cdot t), \quad t \in \mathbb{Z}.$$

It follows from Lemma 8.2 that each minimal set of Λ_t^* is an almost 1-1 extension of a minimal set in $(Y^*, \mathbb{Z})$. Using this fact and (8.9), the rest of the proof follows from that of Theorem 3.2 1) in [67] almost word by word.

2) Let ρ be the rotation number of (8.1) and M be a minimal set of (8.1) which is an almost N-1 extension of Y. We denote by $\mathcal{M}(M)$, $\mathcal{M}(Y)$ as the frequency modules of M, Y, respectively. Then by the general module containment property for almost automorphic minimal sets (e.g., [61]), N is the smallest positive integer such that $N\mathcal{M}(M) \subset \mathcal{M}(Y)$. But by 1), $\mathcal{M}(M)$ is generated by ρ and $\mathcal{M}(Y)$. The theorem follows. □

8.3. **APCF without mean motion.** We first prove a general result on positive transitivity and the uniqueness of minimal set in a SPCF $(S^1 \times Y, \mathbb{T})$.

Theorem 8.4. *If a SPCF* $(S^1 \times Y, \mathbb{T})$ *with minimal base flow* $(Y, \mathbb{T})$ *is positively transitive, then it has a unique minimal set.*

Proof. Suppose for contradiction that $(S^1 \times Y, \mathbb{T})$ has two distinct minimal sets M, M_0. We let $S(y)$, $S_0(y)$, ρ, ρ_0, Y^c and Y_0^c be defined as in the proof of Theorem 7.2 for the present M, M_0.

Since $(S^1 \times Y, \mathbb{T})$ is positively transitive, the set $Tran^+(S^1 \times Y)$ of positively transitive points in $S^1 \times Y$ is a residual subset of $S^1 \times Y$. Let

$$Y_T = \{y \in Y : Tran^+(S^1 \times Y) \cap (S^1 \times \{y\}) \text{ is a residual subset of } S^1 \times \{y\}\}.$$

Then by Lemma 5.1, Y_T is a residual subset of Y.

For a given $y_0 \in Y^c \cap Y_0^c \cap Y_T$, we have by the proof of Theorem 7.2 that $n =: |S_0(y_0)| = |S(y_0)| < +\infty$ and each of the n connected components of $S^1 \setminus S_0(y_0)$ contains precisely one point in $S(y_0)$. Thus there exists points $0 \le t_1 < r_1 < t_2 < r_2 < \cdots < t_n < r_n < 1 + t_1$ such that $S(y_0) = \{a_1, a_2, \cdots, a_n\}$ and $S_0(y_0) = \{b_1, b_2, \cdots, b_n\}$, where $a_j = e^{2\pi i t_j}$ and $b_j = e^{2\pi i r_j}$ for $j = 1, 2, \cdots, n$.

Since $Tran^+(S^1 \times Y) \cap (S^1 \times \{y_0\})$ is dense in $S^1 \times \{y_0\}$, there exists $w_1 \in (t_1, r_1)$ such that $(c_1, y_0) \in Tran^+(S^1 \times Y)$, where $c_1 = e^{2\pi i w_1}$. Take a number w_2 such that $w_2 \in (r_1, t_2)$ when $n \ge 2$ and $w_2 \in (r_1, 1 + t_1)$ when $n = 1$, and let $c_2 = e^{2\pi i w_2}$. Then $c_2 \in (b_1, a_2)$ when $n \ge 2$ and $c_2 \in (b_1, a_1)$ when $n = 1$. In any case, $c_2 \notin S(y_0) \cup S_0(y_0)$. Since $(c_1, y_0) \in Tran^+(S^1 \times Y)$, there exists a monotonically increasing, positive sequence $s_i \to +\infty$ such that $\lim_{i\to\infty}(c_1, y_0) \cdot s_i = (c_2, y_0)$. Without loss of generality, we assume that $\lim_{i\to\infty}(a_1, y_0) \cdot s_i = (a_j, y_0)$ for some $1 \le j \le n$. Consider the family of functions $f_t : S^1 \to S^1$: $u \mapsto \psi(u, y_0, t)$, $t \in \mathbb{T}$. Then each f_t is an orientation preserving homeomorphism. Since $\lim_{i\to\infty} f_{s_i}(c_1) = c_2$, $\lim_{i\to\infty} f_{s_i}(a_1) = a_j$, and $a_j \ne c_2$, we have by Lemma 5.3 (1) that $\lim_{i\to\infty} f_{s_i}([a_1, c_1]) = [a_j, c_2]$. Using the fact that $b_1 \in (a_1, c_2) \subseteq (a_j, c_2)$, we can find a sufficiently small open neighborhood V of b_1 in S^1 such that $\overline{V} \subset (a_j, c_2)$. Since $y_0 \in Y_0^c$ and $b_1 \in S_0(y_0)$, there exists an open neighborhood U of y_0 in Y such that $S_0(y) \cap V \ne \emptyset$ for each $y \in U$. Note that $f_{s_i}((a_1, c_1)) = (f_{s_i}(a_1), f_{s_i}(c_1))$, $\overline{V} \subset (a_j, c_2)$, and $y_0 \cdot s_j \to y_0$. It follows that there exists a positive integer N such that $f_{s_i}((a_1, c_1)) \supset V$ and $y_0 \cdot s_i \in U$ as $i \ge N$. Moreover, for a fixed $i \ge N$, there exists $b \in (a_1, c_1)$ such that $f_{s_i}(b) \in S_0(y_0 \cdot s_i) \cap V$. This implies that $(f_{s_j}(b), y_0 \cdot s_j) \in M_0$, i.e., $(b, y_0) \cdot s_i \in M_0$. Hence $(b, y_0) \in M_0$, i.e., $b \in S_0(y_0)$. This is a contradiction to the fact that $(a_1, c_1) \cap S_0(y_0) = \emptyset$. □

To prove Theorem 7, we need the following lemmas.

Lemma 8.3. *Consider the SPCF* (8.1) *with strictly ergodic base flow* $(Y, \mathbb{T})$. *Then for any* $\phi_1, \phi_2 \in R^1$, $y \in Y$, *and* $t \in \mathbb{T}$,

$$\tilde{\phi}(\phi_1, y, t) - \phi_1 \leq \tilde{\phi}(\phi_2, y, t) - \phi_2 + 2.$$

Proof. Let $\phi_1, \phi_2 \in R^1$, $y \in Y$, and $t \in \mathbb{T}$ be given. Since the function $\tilde{\phi}(\cdot, y, t) : R^1 \to R^1$ is strictly increasing, $\tilde{\phi}(\phi_1, y, t) \leq \tilde{\phi}(\phi_2 + [\phi_1 - \phi_2] + 1, y, t) = \tilde{\phi}(\phi_2, y, t) + [\phi_1 - \phi_2] + 1$, where for each $r \in R^1$, $[r]$ denotes the largest integer which is less than or equal to r. Thus $\tilde{\phi}(\phi_1, y, t) - \phi_1 \leq \tilde{\phi}(\phi_2, y, t) + [\phi_1 - \phi_2] - \phi_1 + 1 \leq \tilde{\phi}(\phi_2, y, t) + (\phi_1 - \phi_2 + 1) - \phi_1 + 1$, i.e., $\tilde{\phi}(\phi_1, y, t) - \phi_1 \leq \tilde{\phi}(\phi_2, y, t) - \phi_2 + 2$. □

Lemma 8.4. *Consider the SPCF* (8.1) *with strictly ergodic base flow* $(Y, \mathbb{T})$. *If there exists* $(\phi_*, y_*) \in R^1 \times Y$ *such that*

$$\limsup_{t \to +\infty} \Big(\tilde{\phi}(\phi_*, y_*, t) - \phi_* - \rho t\Big) = +\infty \ \text{(resp. } \liminf_{t \to +\infty} \Big(\tilde{\phi}(\phi_*, y_*, t) - \phi_* - \rho t\Big) = -\infty\text{)}, \tag{8.11}$$

then there exists a residual subset Y_* *of* Y *such that* $\limsup_{t \to +\infty} \Big(\tilde{\phi}(\phi_0, y_0, t) - \phi_0 - \rho t\Big) = +\infty$ (*resp.* $\liminf_{t \to +\infty} \Big(\tilde{\phi}(\phi_0, y_0, t) - \phi_0 - \rho t\Big) = -\infty$) *for all* $(\phi_0, y_0) \in R^1 \times Y_*$.

Proof. We only consider the case that

$$\limsup_{t \to +\infty} \Big(\tilde{\phi}(\phi_*, y_*, t) - \phi_* - \rho t)\Big) = +\infty.$$

Denote $\phi_s = \tilde{\phi}(\phi_*, y_*, s)$, $s \in R^1$. Let $M \in \mathbb{N}$ and $s \in \mathbb{T}$ be given. It follows from (8.11) that there exists $t(s) > M$ such that

$$\tilde{\phi}(\phi_*, y_*, s + t(s)) - \phi_* - \rho(s + t(s)) > M + 2 + \phi_s - \phi_* - \rho s,$$

i.e., $\tilde{\phi}(\phi_s, y_* \cdot s, t(s)) - \phi_s - \rho t(s) > M + 2$. By continuity, we let U_s^M be an open neighborhood of $y_* \cdot s$ such that $\tilde{\phi}(\phi_s, y, t(s)) - \phi_s - \rho t(s) > M + 2$ for all $y \in U_s^M$. Then by Lemma 8.3,

$$\tilde{\phi}(\phi, y, t(s)) - \phi - \rho t(s) > M$$

for all $(\phi, y) \in R^1 \times U_s^M$. Let $U_M = \bigcup_{s \in \mathbb{T}} U_s^M$. Since $\{y_* \cdot s\}_{s \in \mathbb{T}}$ is dense in Y, U_M is a dense open subset of Y. Moreover, for each $y \in U_M$ there exists $t > M$ such that $\tilde{\phi}(\phi, y, t) - \phi - \rho t > M$ for all $\phi \in R^1$.

Let $Y_* = \bigcap_{M \in \mathbb{N}} U_M$. Then Y_* is a residual subset of Y, and,

$$\limsup_{t \to +\infty} \Big(\tilde{\phi}(\phi_0, y_0, t) - \phi_0 - \rho t\Big) = +\infty$$

for any $(\phi_0, y_0) \in R^1 \times Y_*$. □

Lemma 8.5. *Consider the SPCF* (8.1) *with strictly ergodic base flow* $(Y, \mathbb{T})$. *Then there exist* $y_1, y_2 \in Y$ *such that*

$$\sup_{t \geq 1} (\tilde{\phi}(\phi, y_1, t) - \phi - \rho t) \leq 4, \tag{8.12}$$

$$\inf_{t \geq 1} (\tilde{\phi}(\phi, y_2, t) - \phi - \rho t) \geq -4 \tag{8.13}$$

for all $\phi \in R^1$.

Proof. Suppose for contradiction that (8.12) is not true. Then for a given $y \in Y$ there exist $\phi_y \in R^1$ and $t_y \geq 1$ such that $\tilde{\phi}(\phi_y, y, t_y) - \phi_y - \rho t_y > 4$. By continuity, there exists an open neighborhood U_y of y such that $\tilde{\phi}(\phi_y, y', t_y) - \phi_y - \rho t_y > 4$ for all $y' \in U_y$. Since by Lemma 8.3 $\tilde{\phi}(\phi, z, t_y) - \phi \leq \tilde{\phi}(\phi_y, z, t_y) - \phi_y + 2$ for all $\phi \in R^1$ and $z \in Y$, we have

$$\tilde{\phi}(\phi, y', t_y) - \phi - \rho t_y > 2$$

for all $(\phi, y') \in R^1 \times U_y$.

Since $\{U_y\}_{y \in Y}$ is an open cover of Y and Y is compact, there exists a finite set $\{y_1, y_2, \cdots, y_k\} \subset Y$ such that $\bigcup_{i=1}^k U_{y_i} = Y$. We denote $U_i = U_{y_i}$, $t_i = t_{y_i}$ for short and let $T = \max\{t_1, t_2, \cdots, t_k\}$.

Given $(\phi_0, y_0) \in R^1 \times Y$, we inductively define sequences $\{T_i\} \subset \mathbb{T}$, $\{k_i\} \subset \{1, 2, \cdots k\}$, and $\{m_i\}$ by letting $T_0 = 0$, $y_0 \cdot T_i \in U_{k_i}$, $m_i = t_{k_i}$, and $T_{i+1} = T_i + m_i$. Then $\tilde{\phi}(\phi, y_0 \cdot T_i, m_i) - \phi - \rho m_i > 2$ for all $\phi \in R^1$. It follows that

$$\begin{aligned}\tilde{\phi}(\phi_0, y_0, T_{i+1}) - \phi_0 - \rho T_{i+1} &= (\tilde{\phi}(\phi(\phi_0, y_0, T_i), y_0 \cdot T_i, m_i) - \tilde{\phi}(\phi_0, y_0, T_i) - \rho m_i) \\ &\quad + (\tilde{\phi}(\phi_0, y_0, T_i) - \phi_0 - \rho T_i) \geq 2 + (\tilde{\phi}(\phi_0, y_0, T_i) - \phi_0 - \rho T_i).\end{aligned}$$

By induction,

$$\tilde{\phi}(\phi_0, y_0, T_i) - \phi_0 - \rho T_i \geq 2i$$

for all $i \in \mathbb{N}$. Since $i \leq T_i \leq Ti$ for all $i \in \mathbb{N}$, we have

$$\begin{aligned}\limsup_{i \to \infty} \frac{1}{T_i} \tilde{\phi}(\phi_0, y_0, T_i) &\geq \limsup_{i \to \infty} \frac{1}{T_i} (\phi_0 + \rho T_i + 2i) \\ &= \rho + \limsup_{i \to \infty} \frac{2i}{T_i} \geq \rho + \frac{2}{T} > \rho,\end{aligned}$$

which contradicts to the definition of the rotation number. This proves (8.12).

The proof of (8.13) is similar. □

Theorem 8.5. *Suppose that the SPCF* (8.1) *is an APCF which admits no mean motion. Then each of its minimal set is either the entire phase space $S^1 \times Y$ or is everywhere non-locally connected.*

Proof. We let d be an invariant compatible metric on Y, i.e., $d(y_1' \cdot t, y_2' \cdot t) = d(y_1', y_2')$ for all $y_1', y_2' \in Y$ and $t \in \mathbb{T}$. Since (8.1) admits no mean motion, the condition d) of Theorem 8.2 fails, i.e., there exists $(\phi_*, y_*) \in R^1 \times Y$ such that $\limsup_{t \to +\infty} (\tilde{\phi}(\phi_*, y_*, t) - \phi_* - \rho t) = +\infty$. It follows from Lemma 8.4 that there exists a residual subset Y_* of Y such that

$$\limsup_{t \to +\infty} (\tilde{\phi}(\phi_0, y_0, t) - \phi_0 - \rho t) = +\infty \tag{8.14}$$

for any $(\phi_0, y_0) \in R^1 \times Y_*$. By Lemma 8.5 (1), there exists $y_1 \in Y$ such that

$$\sup_{t \geq 1} (\tilde{\phi}(\phi, y_1, t) - \phi - \rho t) \leq 4 \tag{8.15}$$

for all $\phi \in R^1$.

Suppose that the entire phase space $S^1 \times Y$ is not minimal and let X be a minimal set of (8.1). Then there exist nonempty open subsets $U_2' \subset S^1$ and $V_2' \subset Y$ such that $X \cap (U_2' \times V_2') = \emptyset$. Since $(Y, \mathbb{T})$ is minimal, there exist $t_1', t_2', \cdots, t_\ell' \in \mathbb{T}$ such that $\mathcal{V}' = \{V_2' \cdot t_i\}_{i=1}^{\ell}$ is an open cover of Y. Let $\delta' > 0$ be the Lebesgue number of $\mathcal{V}'$ with respect to the metric d.

If X is not everywhere non-locally connected, then by Lemma 7.1 the set X_{lc} of locally connected points in X is an invariant residual subset of X. Since by Proposition 3.4 the projection $\pi : X \to Y$ is semi-open, it follows from Lemma 5.5 that

$$X_0 = \{x \in X : \text{for any open neighborhood } U \text{ of } x,\ \pi(U) \text{ is a neighborhood of } \pi(x)\}$$

is also a residual subset of X. Take $x_* \in X_{lc} \cap X_0$ and denote $y^* = \pi(x_*)$. Let $\phi_1 \in [0, 1]$ be such that $x_* = (a, y^*)$, where $a = e^{2\pi i \phi_1}$ and denote $A = [\phi_1 - \frac{1}{4}, \phi_1 + \frac{1}{4}]$. Also let V_1' be an open neighborhood of y^* such that $\operatorname{diam}(V_1') < \delta'$ and let $U_1' = \{e^{2\pi i \phi} : \phi \in A\}$. Since x_* is a locally connected point of X, there exists a connected closed neighborhood W of x_* in X such that $W \subseteq U_1' \times V_1'$. Since $x_* \in X_0$, $\pi(W)$ is also a closed neighborhood of y^* in Y. Using minimality of $(Y, \mathbb{T})$, we let $t^* \geq 1$ be such that $y_2 =: y_1 \cdot t^* \in \pi(W)$ and denote $C(t^*) = \max_{\phi \in R^1} |\tilde{\phi}(\phi, y_1, t^*) - \phi - \rho t^*|$. Then $C(t^*) < \infty$ and it follows from (8.15) that, for any $\phi \in R^1$,

$$\begin{aligned} 4 &\geq \sup_{t\geq 0}(\tilde{\phi}(\tilde{\phi}(\phi, y_1, t^*), y_2, t) - \phi - \rho(t + t^*)) \\ &= \sup_{t\geq 0}((\tilde{\phi}(\tilde{\phi}(\phi, y_1, t^*), y_2, t) - \tilde{\phi}(\phi, y_1, t^*) - \rho t^*) + (\tilde{\phi}(\phi, y_1, t^*) - \phi - \rho t^*)) \\ &\geq \sup_{t\geq 0}(\tilde{\phi}(\tilde{\phi}(\phi, y_1, t^*), y_2, t) - \tilde{\phi}(\phi, y_1, t^*) - \rho t^*) - C(t^*), \end{aligned}$$

i.e.,

$$\sup_{t\geq 0}(\tilde{\phi}(\phi, y_2, t) - \phi - \rho t) \leq 4 + C(t^*). \tag{8.16}$$

Since $y_2 \in \pi(W)$, there exists $\phi_2 \in A$ such that $(a_2, y_2) \in W$, where $a_2 = e^{2\pi i \phi_2}$.

For any $(b, y) \in W \subseteq U_1' \times V_1'$, there exists a unique $\phi(b) \in A$ such that $e^{2\pi i \phi(b)} = b$. Consider the map $h : W \to A \times V_1'$: $(b, y) \mapsto (\phi(b), y)$. Clearly, h is continuous. Let $F = h(W)$. Then $(\phi_2, y_2) \in F$ and it follows from the closeness and connectivity of W that F is a closed connected subset of $A \times V_1'$.

Since $\pi(W)$ is a closed neighborhood of y^*, $\pi(W) \cap Y_* \neq \emptyset$. Take $y_0 \in \pi(W) \cap Y_*$. Then there exists $\phi_0 \in A$ such that $(\phi_0, y_0) \in F$. Let $L' = \max_{i=1}^{\ell} \max\{\tilde{\phi}(\phi, y, t_i') - \phi - \rho t_i' : \phi \in R^1, y \in Y\}$. Then $L' < \infty$ and by (8.14), there exists $t' > \max\{t_1', t_2', \cdots, t_\ell'\}$ such that

$$\tilde{\phi}(\phi_0, y_0, t') - \phi_0 - \rho t' \geq 6 + L' + C(t^*).$$

Since $\mathrm{diam}(V_1' \cdot t') = \mathrm{diam}(V_1') < \delta'$, there exists $i \in \{1, 2, \cdots, \ell\}$ such that $V_1' \cdot t' \subseteq V_2' \cdot t_i'$, i.e., $V_1' \cdot (t' - t_i') \subseteq V_2'$. Denote $t_*' = t' - t_i'$. Then $t_*' > 0$, $V_1' \cdot t_*' \subseteq V_2'$, and

$$\begin{aligned} &\tilde{\phi}(\phi_0, y_0, t_*') - \phi_0 - \rho t_*' \\ &= (\tilde{\phi}(\phi_0, y_0, t') - \phi_0 - \rho t') - (\tilde{\phi}(\tilde{\phi}(\phi_0, y_0, t_*'), y_0 \cdot t_*', t_i') - \tilde{\phi}(\phi_0, y_0, t_*') - \rho t_i') \\ &\geq 6 + L' + C(t^*) - (\tilde{\phi}(\tilde{\phi}(\phi_0, y_0, t_*'), y_0 \cdot t_*', t_i') - \tilde{\phi}(\phi_0, y_0, t_*') - \rho t_i') \geq 6 + C(t^*), \end{aligned}$$

i.e.,

$$\tilde{\phi}(\phi_0, y_0, t_*') \geq (\phi_0 + \rho t_*') + 6 + C(t^*). \tag{8.17}$$

Consider the function $\Phi : F \to R^1$: $(\phi, y) \mapsto \tilde{\phi}(\phi, y, t_*')$. Obviously, Φ is continuous. Moreover, by noting that $\phi_2, \phi_0 \in A$, we have by (8.16) that $\Phi(\phi_2, y_2) \leq (\phi_2 + \rho t_*') + 4 + C(t^*) \leq (\phi_0 + \rho t_*') + 5 + C(t^*)$, and by (8.17) that $\Phi(\phi_0, y_0) \geq (\phi_0 + \rho t_*') + 6 + C(t^*)$. Since F is connected and $(\phi_0, y_0), (\phi_2, y_2) \in F$, we have that $\Phi(F) \supseteq [(\phi_0 + \rho t_*') + 5, (\phi_0 + \rho t_*') + 6]$. Hence there exists $(\phi_3, y_3) \in F$ such that $e^{2\pi i \Phi(\phi_3, y_3)} \in U_2'$. It follows from the definition of F that $(e^{2\pi i \phi_3}, y_3) \in W \subset X$.

Now, on one hand, $(e^{2\pi i \phi_3}, y_3) \cdot t_*' \in X$, and on the other hand, $(e^{2\pi i \phi_3}, y_3) \cdot t_*' = (e^{2\pi i \Phi(\phi_3, y_3)}, y_3 \cdot t_*') \in U_2' \times V_2'$ as $V_1' \cdot t_*' \subseteq V_2'$. This implies that $X \cap (U_2' \times V_2') \neq \emptyset$, a contradiction. □

Theorem 8.6. *Suppose that the SPCF* (8.1) *is an APCF with locally connected base space Y. If it admits no mean motion, then it is positively transitive and has only one minimal set.*

Proof. Let d, Y_* and y_1 be defined in the proof of Theorem 8.5.

Let $U_1, U_2 \subset S^1$ and $V_1, V_2 \subset Y$ be any nonempty open subsets. Since $(Y, \mathbb{T})$ is minimal, there exist $t_1, t_2, \cdots, t_k \in \mathbb{T}$ such that $\mathcal{V} = \{V_2 \cdot t_i\}_{i=1}^k$ is an open cover of Y. Let $\delta > 0$ be the Lebesgue number of $\mathcal{V}$ with respect to the metric d. Since $\{y_1 \cdot r\}_{r\geq 1}$ is dense in Y, there exists $r_1 \geq 1$ such that $y_1 \cdot r_1 \in V_1$, i.e., V_1 is an open neighborhood of $y_* = y_1 \cdot r_1$. Since Y is local connected, there exists a connected closed neighborhood V of y_* such that $V \subseteq V_1$ and $\mathrm{diam}(V) < \delta$.

By (8.15), we have

$$\sup_{t\geq 0}(\tilde{\phi}(\tilde{\phi}(\phi, y_1, r_1), y_*, t) - \tilde{\phi}(\phi, y_1, r_1) - \rho t) \leq 4 + |\tilde{\phi}(\phi, y_1, r_1) - \phi - \rho r_1|$$

for all $\phi \in R^1$. Let $C = \sup_{\phi\in R^1} |\tilde{\phi}(\phi, y_1, r_1) - \phi - \rho r_1| = \max_{0\leq\phi\leq 1} |\tilde{\phi}(\phi, y_1, r_1) - \phi - \rho r_1|$. Then $C < \infty$ and

$$\sup_{t\geq 0}(\tilde{\phi}(\phi, y_*, t) - \phi - \rho t) \leq 4 + C \tag{8.18}$$

for all $\phi \in R^1$. Take $z_1 \in Y_* \cap V$ and $\phi_1 \in [0, 1]$ such that $e^{2\pi i\phi_1} \in U_1$. Then

$$\limsup_{t\to+\infty}(\tilde{\phi}(\phi_1, z_1, t) - \phi_1 - \rho t) = +\infty.$$

Let $L = \max_{i=1}^k \max\{\tilde{\phi}(\phi, y, t_i) - \phi - \rho t_i : \phi \in R^1, y \in Y\}$. Then $L < \infty$. Take $t_0 > \max\{t_1, t_2, \cdots, t_k\}$ such that

$$\tilde{\phi}(\phi_1, z_1, t_0) - \phi_1 - \rho t_0 \geq 5 + C + L.$$

Since $\mathrm{diam}(V \cdot t_0) = \mathrm{diam}(V) < \delta$, there exists $i \in \{1, 2, \cdots, k\}$ such that $V \cdot t_0 \subseteq V_2 \cdot t_i$, i.e., $V \cdot (t_0 - t_i) \subseteq V_2$. Let $t_* = t_0 - t_i$. Then $t_* > 0$, $V \cdot t_* \subset V_2$, and

$$\begin{aligned}
&\tilde{\phi}(\phi_1, z_1, t_*) - \phi_1 - \rho t_* \\
=& (\tilde{\phi}(\phi_1, z_1, t_0) - \phi_1 - \rho t_0) - (\tilde{\phi}(\phi(\phi_1, z_1, t_*), z_1 \cdot t_*, t_i) - \tilde{\phi}(\phi_1, z_1, t_*) - \rho t_i) \\
\geq& 5 + C + L - (\tilde{\phi}(\phi(\phi_1, z_1, t_*), z_1 \cdot t_*, t_i) - \tilde{\phi}(\phi_1, z_1, t_*) - \rho t_i) \\
\geq& 5 + C.
\end{aligned}$$

Consider the function $\Psi : V \mapsto R^1$: $y \mapsto \tilde{\phi}(\phi_1, y, t_*)$. Clearly, Ψ is continuous, $\Psi(y_*) \leq 4 + C + \phi_1 + \rho t_*$, and $\Psi(z_1) \geq 5 + C + \phi_1 + \rho t_*$. Since V is connected, $\Psi(V) \supseteq [4 + C + \phi_1 + \rho t_*, 5 + C + \phi_1 + \rho t_*]$. Hence there exists $y_2 \in V$ such that $e^{2\pi i\Psi(y_2)} \in U_2$. This shows that $(e^{2\pi i\phi_1}, y_2) \in U_1 \times V_1$ and $(e^{2\pi i\phi_1}, y_2) \cdot t_* = (e^{2\pi i\Psi(y_2)}, y_2 \cdot t_*) \in U_2 \times V_2$ as $V \cdot t_* \subseteq V_2$. Therefore, $(e^{2\pi i\Psi(y_2)}, y_2 \cdot t_*) \in (U_1 \times V_1) \cdot t_* \cap (U_2 \times V_2) \neq \emptyset$. Since U_1, U_2, V_1, V_2 are arbitrary, the flow (8.1) is positively transitive.

It follows from Theorem 8.4 the flow (8.1) has a unique minimal set. □

Now, parts 1), 2) of Theorem 7 are just the respective Theorems 8.5, 8.6 above.

We note that using Theorem 6 1) and Theorem 7 2) we also obtain an alternative proof for Theorem 4 3) (Theorem 7.3).

8.4. **Quasi-periodically forced circle flows.** Our aim of this subsection is to prove Theorem 8 in which the base space Y is further assumed to be a torus. We will use a classical result of E. Cartan that every closed subgroup of a Lie group is also a Lie group. Hence any closed subgroup of a Lie group is a Lie group which cannot be a Cantor set.

The following theorem is just our main result Theorem 8.

Theorem 8.7. *Consider an APCF $(S^1 \times Y, \mathbb{T}) = (S^1 \times Y, \{\Lambda_t\}_{t\in\mathbb{T}})$ with Y being a torus (e.g., $(Y, \mathbb{T})$ is quasi-periodic) and suppose that the rotation number is rationally independent of the forcing frequencies. Then $(S^1 \times Y, \mathbb{T})$ has a unique minimal set M and M is either the entire phase space $S^1 \times Y$ or is everywhere non-locally connected. If, in addition, the APCF admits mean motion, then M is almost automorphic, and moreover, either $M = S^1 \times Y$ or M is an everywhere non-locally connected Cantorian.*

Proof. Since Y is local connected, it follows from Theorem 4 1), Theorem 6 2) in the case with mean motion and from Theorem 7 2) in the case without mean motion that $(S^1 \times Y, \mathbb{T})$ has a unique minimal set M. Suppose that $M \neq S^1 \times Y$. We want to show that M is everywhere non-locally connected.

In the case that the APCF $(S^1 \times Y, \mathbb{T})$ admits no mean motion, we have by Theorem 7 1) that M is everywhere non-locally connected.

We now consider the case that the APCF $(S^1 \times Y, \mathbb{T}) = (S^1 \times Y, \{\Lambda_t\}_{t\in\mathbb{T}})$ admits mean motion. By Theorem 3 and Theorem 6 1), M is both a Cantorian and an almost automorphic minimal set.

Suppose for contradiction that M has a locally connected point. Then the set M_{lc} of locally connected points in M is an invariant residual subset of M. Let Y^* be a maximal almost periodic factor of M and $p : (M, \mathbb{T}) \to (Y^*, \mathbb{T})$ be the almost 1-1 extension according to Theorem 3.2. Since the proximal relation

$$P(M) = \{(e_1, e_2) \in M \times M : \inf_{t\in\mathbb{T}} d(\Lambda_t(e_1), \Lambda_t(e_2)) = 0\},$$

where d denotes the standard metric on $S^1 \times Y$, is a closed (in particular, an equivalence), equivariance relation, Y^* can be identified to $M/P(M)$ with flow being induced by Λ_t.

Let $\pi : M \to Y$ be the natural projection. Then it is clear that

$$P(M) \subset R_\pi =: \{(e_1, e_2) \in M \times M : \pi(e_1) = \pi(e_2)\}.$$

Thus there exists an extension $\eta : (Y^*, \mathbb{T}) \to (Y, \mathbb{T})$ such that $\pi = \eta \circ p$. Since p is almost 1-1 and $M_{lc} \neq \emptyset$, we have by Theorem 7.1 2) that each fiber $p^{-1}(y^*)$, $y^* \in Y^*$, is connected. For each $y^* \in Y^*$, we note that

$$p^{-1}(y^*) \subseteq \pi^{-1}(\eta(y^*)) = \{(s, \eta(y^*)) : (s, \eta(y^*)) \in M\}.$$

It follows that for each $y^* \in Y^*$ there is a subinterval I_{y^*} (which can be degenerate) of S^1 such that $p^{-1}(y^*) = I_{y^*} \times \{\eta(y^*)\}$.

Since M is a Cantorian, there exists a residual subset Y_0 of Y such that for each $y \in Y_0$, $\pi^{-1}(y)$ is a Cantor set. For any $y \in Y_0$ and $y^* \in \eta^{-1}(y)$, since $p^{-1}(y^*) = I_{y^*} \times \{y\} \subseteq \pi^{-1}(y)$, $\pi^{-1}(y)$ is a Cantor set and I_{y^*} is a subinterval of S^1, it follows that I_{y^*} is a singleton. Thus for each $y \in Y_0$ the map $p : \pi^{-1}(y) \mapsto \eta^{-1}(y)$ is a homeomorphism, i.e., $\eta^{-1}(y)$ is also a Cantor set.

Fix $y_0 \in Y_0$ and $y_0^* \in \pi^{-1}(y_0)$. For any $y_1^*, y_2^* \in Y^*$, there exist sequences $\{t_i^j\}_{i=1}^\infty$, $j = 1, 2$, such that $\lim_{i\to\infty} y_0^* \cdot t_i^j = y_j^*$, $j = 1, 2$. We define

$$y_1^* \circ y_2^* = \lim_{i\to\infty} y_0^* \cdot (t_i^1 + t_i^2). \tag{8.19}$$

Since $(Y^*, \mathbb{T})$ is almost periodic, (8.19) is well-defined and is independent of the choose of sequences $\{t_i^j\}_{i=1}^\infty$, $j = 1, 2$. With the operation $y_1^* \circ y_2^*$, Y^* becomes a compact Abeliean topological group with unity y_0^* (see Theorem 3.2.1 in [62]). Using y_0, we can define an operation "$\circ$" on Y similarly so that it becomes a compact topological group with unity y_0.

For any $y_1^*, y_2^* \in Y^*$, we take sequences $\{t_i^j\}_{i=1}^\infty$, $j = 1, 2$, such that $\lim_{i\to\infty} y_0^* \cdot t_i^j = y_j^*$, $j = 1, 2$. Then $\lim_{i\to\infty} y_0 \cdot t_i^j = \lim_{i\to\infty} \eta(y_0^* \cdot t_i^j) = \eta(y_j^*)$, $j = 1, 2$, and

$$\begin{aligned}\eta(y_1^* \circ y_2^*) &= \lim_{i\to\infty} \eta(y_0^* \cdot (t_i^1 + t_i^2)) = \lim_{i\to\infty} \eta(y_0^*) \cdot (t_i^1 + t_i^2)\\ &= \lim_{i\to\infty} y_0 \cdot (t_i^1 + t_i^2) = \eta(y_1^*) \circ \eta(y_2^*).\end{aligned}$$

Hence η is a group homomorphism from $(Y^*, \circ)$ to $(Y, \circ)$ and $\eta^{-1}(y_0) = ker(\eta)$ is a closed subgroup of $(Y^*, \circ)$.

Since p is semi-open, the set

$$M_0 = \{m \in M : \text{for any open neighborhood } U \text{ of } m,\ p(U) \text{ is a neighborhood of } p(m)\}$$

is a residual subset of M by Lemma 5.5. Hence $M_{lc} \cap M_0$ is also a residual subset of M. Since $\emptyset \neq p(M_{lc} \cap M_0) \subseteq Y_{lc}^*$, Y^* admits a locally connected point y_*. For each $y^* \in Y^*$, we consider the map $H_{y^*,y_*} : Y^* \to Y^*$: $y' \mapsto y' \circ ({y^*}^{-1} \circ y_*)$, where ${y^*}^{-1}$ is the inverse of y^*. Since H_{y^*,y_*}

is a homeomorphism and $H_{y^*,y_*}(y^*) = y_*$, y^* is also a locally connected point of Y^*. This shows that Y^* is locally connected. For any $y \in Y$ and $y^* \in \eta^{-1}(y)$, since $H_{y_0^*,y^*} : \eta^{-1}(y_0) \mapsto \eta^{-1}(y)$ is a homeomorphism, $\eta^{-1}(y)$ is a Cantor set and hence $\dim(\eta^{-1}(y)) = 0$, where $\dim(\cdot)$ denotes the covering dimension. It follows from Theorem VI. 7 in [26] that

$$\dim\,(Y^*) \leq \dim\,(Y) + \sup_{y\in Y} \dim(\eta^{-1}(y)) = \dim\,(Y) < \infty.$$

Summarizing up, we have shown that $(Y^*, \circ)$ is a locally connected, finite dimensional, compact Abelian topological group. It follows from a classical result of Pontrjagin (see Theorem 56 in [52]) that Y^* is a Lie group.

Now, since $\eta^{-1}(y_0)$ is a closed subgroup of the Lie group $(Y^*, \circ)$, $\eta^{-1}(y_0)$ is a Lie group, which is in particular not a Cantor set. This contradicts to the fact that $\eta^{-1}(y)$ is a Cantor set for all $y \in Y$. □

9. Projective bundle flows of $sl(2,\mathbb{R})$-valued cocycles

Let $T = \mathbb{R}$ or $\mathbb{Z}$ and $(Y, \mathbb{T})$ be an almost periodic minimal flow. We consider an almost periodic, $sl(2,\mathbb{R})$-valued cocycle $\{\Phi(y,t)\}_{y\in Y, t\in\mathbb{T}}$, i.e., the map $(y,t) \mapsto \Phi(y,t) \in sl(2,\mathbb{R})$ is continuous, $\Phi(y,0) \equiv I$ - the identity matrix, and $\{\Phi(y,t)\}_{y\in Y,t\in\mathbb{T}}$ satisfies the following cocycle property:

$$\Phi(y, t+s) = \Phi(y \cdot s, t)\Phi(y, s), \;\; y \in Y,\; t, s \in \mathbb{T}.$$

We refer the cocycle as *continuous cocycle* if $\mathbb{T} = \mathbb{R}$ and as *discrete cocycle* if $\mathbb{T} = \mathbb{Z}$. In the discrete case, we always assume that the cocycle is homotopic to identity. An important example of continuous, almost periodic, $sl(2,\mathbb{R})$-valued cocycles is the one generated from an almost periodic, 2-dimensional, linear system of ordinary differential equations:

$$x' = A(y \cdot t)x, \;\; x \in R^2,\; y \in Y,\; t \in \mathbb{R}, \tag{9.1}$$

where $tr\ A(y,t) \equiv 0$. In this case, the principal matrix solution of the linear system clearly forms a continuous cocycle.

The average exponential growth of the norm of $\{\Phi(y,t)\}$ is measured by the (maximal) Lyapunov exponent

$$\lambda = \lim_{t\to+\infty} \int_Y \frac{\log \|\Phi(y,t)\|}{t} d\mu(y) \geq 0,$$

where μ denotes the Haar measure on Y. We note that the limit exists by subadditivity, is independent of the matrix norm, and is non-negative because $\Phi(y,t) \in sl(2,\mathbb{R})$. By Kigman sub-additive ergodic theorem ([39]),

$$\lim_{t\to+\infty} \frac{\log \|\Phi(y,t)\|}{t} = \lambda, \;\; \mu - a.e\ y \in Y.$$

In fact, there exist $Y_* \subset Y$ with $\mu(Y_*) = 1$ and invariant, measurable line bundles $\{u^{\pm}(y)\}_{y\in Y_*} \subset R^2 \setminus \{(0,0)\}$ such that $\Phi(y,t)u^{\pm}(y) = u^{\pm}(y \cdot t)$, $y \in Y_*, t \in \mathbb{T}$, and

$$\lim_{t\to\infty} \frac{\log \|\Phi(y,t)u^{\pm}(y)\|}{t} = \pm\lambda, \;\; y \in Y_*. \tag{9.2}$$

We say that the cocycle $\{\Phi(y,t)\}_{y\in Y,t\in\mathbb{T}}$ is *elliptic* if $\sup_{t\in\mathbb{T}} \|\Phi(y,t)\| < +\infty$ for all $y \in Y$; *hyperbolic* if it admits an exponential dichotomy (or exponential splitting), *parabolic* if $\lambda = 0$ but the cocycle is not elliptic; and *partially hyperbolic* if $\lambda > 0$ but the cocycle is not hyperbolic. It is well-known that if the cocycle is hyperbolic, then the line bundles $\{u^{\pm}(y)\}$ can be extended continuously to the entire space Y and the limits above exist everywhere on Y. In term of Sacker-Sell spectrum theory ([57]) for the almost periodic linear system (9.1), hyperbolicity corresponds to the case with two-points spectrum, partially hyperbolicity corresponds to the case with non-degenerate interval spectrum, and ellipticity and parabolicity correspond to cases with zero spectrum.

Following works [6, 30] for continuous projective bundle flow generated from the linear differential system (9.1) and work [3] for discrete projective bundle flow with one forcing frequency, we will give a complete classification of minimal sets of the projective bundle flow generated from a general almost periodic, $sl(2,\mathbb{R})$-valued cocycle in both continuous and discrete cases. Such a classification will be particularly useful in characterizing dynamical and topological complexities of a SNA in a such projective bundle flow (see recent work [38] and references therein).

9.1. **A general classification of minimal sets.** Consider an almost periodic, $sl(2,\mathbb{R})$-valued cocycle $\{\Phi(y,t)\}_{y\in Y,t\in\mathbb{T}}$ and its generated linear skew-product flow $\pi_t : R^2\times Y \to R^2\times Y$:

$$\pi_t(v,y) = (\Phi(y,t)v, y\cdot t).$$

It is clear that the line bundle

$$V(l,y) = \{(v,y) : v \text{ is a vector in the line } l \text{ through the origin}\}$$

is invariant to π_t (or to the cocycle) in the sense that $\pi_t(V(l,y)) = V(\phi_t(l,y))$ for any line l through the origin and any $y\in Y$. Thus the cocycle generates a projective bundle flow $(P^1\times Y,\mathbb{T})$. For simplicity, we parameterize P^1 by angle $\theta\in[0,1]$ with $0,1$ being identified, i.e., we parameterize a line l through the origin with its angle $Arg(l) = \pi\theta$. Then the projective bundle flow can be defined as $\Lambda_t : P^1\times Y\to P^1\times Y$:

$$\Lambda_t(\theta,y) = (\frac{1}{\pi}Arg(\Phi(y,t)v), y\cdot t) =: (\tilde\theta(\theta,y,t), y\cdot t),\quad \theta\in R^1 \,(\text{mod } 1),\ y\in Y,\ t\in\mathbb{T},$$

where v is a vector in R^2 with angle $Arg(v) = \pi\theta$ and $\tilde\theta(\theta+1,y,t) = \tilde\theta(\theta,y,t)+1$.

Denote $r(\theta,y,t) = \|\Phi(y,t)\binom{\cos\pi\theta}{\sin\pi\theta}\|$.

Lemma 9.1. *Let $(\theta_1,y)\neq(\theta_2,y)\in P^1\times Y$. Then (θ_1,y), (θ_2,y) are proximal iff*

$$\sup_{t\in\mathbb{T}}\{r(\theta_1,y,t)r(\theta_2,y,t)\} = +\infty.$$

Proof. Without loss of generality, we let $0<\theta_1-\theta_2<1$. By taking determinant on both hand sides of the identity

$$\begin{pmatrix} r(\theta_1,y,t)\cos\pi\tilde\theta(\theta_1,y,t) & r(\theta_2,y,t)\cos\pi\tilde\theta(\theta_2,y,t)\\ r(\theta_1,y,t)\sin\pi\tilde\theta(\theta_1,y,t) & r(\theta_2,y,t)\sin\pi\tilde\theta(\theta_2,y,t)\end{pmatrix} = \Phi(y,t)\begin{pmatrix}\cos\pi\theta_1 & \cos\pi\theta_2\\ \sin\pi\theta_1 & \sin\pi\theta_2\end{pmatrix}$$

and using the fact that $\det\Phi(y,t)\equiv 1$, we have

$$r(\theta_1,y,t)r(\theta_2,y,t)\sin\pi(\tilde\theta(\theta_1,y,t)-\tilde\theta(\theta_2,y,t)) = \sin\pi(\theta_1-\theta_2)\neq 0,\quad y\in Y\ t\in\mathbb{T}. \tag{9.3}$$

Since $(\theta_1,y),(\theta_2,y)$ are proximal iff either $\inf_{t\in\mathbb{T}}(\tilde\theta(\theta_1,y,t)-\tilde\theta(\theta_2,y,t)) = 0$ or $\sup_{t\in\mathbb{T}}(\tilde\theta(\theta_1,y,t)-\tilde\theta(\theta_2,y,t)) = 1$, the lemma immediately follows from (9.3). □

Lemma 9.2. *Consider the cocycle $\{\Phi(y,t)\}_{y\in Y,t\in\mathbb{T}}$ and its generated projective bundle flow $(P^1\times Y,\mathbb{T})$. Then the followings are equivalent.*

a) *The cocycle is elliptic;*
b) *There exists $y_0\in Y$ such that $\sup_{t\geq 0}\|\Phi(y_0,t)\|<+\infty$;*
c) *There exist $(\theta_1,y_0)\neq(\theta_2,y_0)\in P^1\times Y$ such that $\sup_{t\geq0} r(\theta_i,y_0,t)<+\infty$, $i=1,2$;*
d) *$P^1\times Y$ is a distal extension of Y;*
e) *There exist $(\theta_i,y_0)\in P^1\times Y$, $i=1,2,3$, which are pairwise distal.*

Proof. It is clear that $a)\Longrightarrow b)$, $b)\Longrightarrow c)$, and $d)\Longrightarrow e)$.

$b)\Longrightarrow a)$: Let $K =: \sup_{t\geq0}\|\Phi(y_0,t)\|<+\infty$. Since $\det\Phi(y,s)\equiv1$, $K_1 =: \sup_{s\geq0}\|\Phi^{-1}(y_0,s)\|<+\infty$. Using the cocycle property

$$\Phi(y_0\cdot s,t) = \Phi(y_0,t+s)\Phi^{-1}(y_0,s),\quad s,t\in\mathbb{T},$$

we have that

$$\|\Phi(y_0\cdot s,t)\|\leq KK_1,\quad s\geq0,\ s+t\geq0.$$

For any $t\in\mathbb{T}$, $y\in Y$, we let $\{s_n\}$ be a positive sequence in $\mathbb{T}$ such that $y_0\cdot s_n\to y$. It follows from the above inequality that $\|\Phi(y,t)\|\leq KK_1$, i.e., a) holds.

c) $\Longrightarrow$ *b*): We note that there is a constant $c > 0$ such that

$$\|\Phi(y_0,t)\begin{pmatrix} \cos\pi\theta_1 & \cos\pi\theta_2 \\ \sin\pi\theta_1 & \sin\pi\theta_2 \end{pmatrix}\| \le c(r(\theta_1,y_0,t)+r(\theta_2,y_0,t)),$$

from which b) follows.

a) $\Longrightarrow$ *d*): Let $(\theta_1,y),(\theta_2,y)\in P^1\times Y$ be two distinct points. It follows from a) that

$$\sup_{t\in\mathbb{T}}\{r(\theta_1,y,t)r(\theta_2,y,t)\} < +\infty.$$

Hence by Lemma 9.1, $(\theta_1,y),(\theta_2,y)$ are distal.

e) $\Longrightarrow$ *c*): By Lemma 9.1, $\sup_{t\in\mathbb{T}}\{\{r(\theta_i,y_0,t)r(\theta_j,y_0,t)\} < +\infty$ for all $1\le i<j\le 3$. Consider the linear combination

$$\Phi(y_0,t)\begin{pmatrix}\cos\pi\theta_1\\ \sin\pi\theta_1\end{pmatrix} = c_1\Phi(y_0,t)\begin{pmatrix}\cos\pi\theta_2\\ \sin\pi\theta_2\end{pmatrix} + c_2\Phi(y_0,t)\begin{pmatrix}\cos\pi\theta_3\\ \sin\pi\theta_3\end{pmatrix},$$

where c_1,c_2 are constants. Then

$$\begin{aligned}\sup_{t\in\mathbb{T}}\{r^2(\theta_1,y,t)\} &\le \sup_{t\in\mathbb{T}}\{|c_1|r(\theta_1,y,t)r(\theta_2,y,t)+|c_2|r(\theta_1,y,t)r(\theta_3,y,t)\}\\ &\le |c_1|\sup_{t\in\mathbb{T}}\{r(\theta_1,y,t)r(\theta_2,y,t)\}+|c_2|\sup_{t\in\mathbb{T}}\{r(\theta_1,y,t)r(\theta_3,y,t)\}<\infty.\end{aligned}$$

It follows that $\{r(\theta_1,y_0,t)\}$ is bounded. Similarity, $\{r(\theta_i,y_0,t)\}, i=2,3$ are bounded, i.e., c) holds. ☐

Proposition 9.1. *If the cocycle* $\{\Phi(y,t)\}_{y\in Y,t\in\mathbb{T}}$ *is not elliptic, then its generated projective bundle flow has at most two minimal sets.*

Proof. Suppose that the projective bundle flow has three minimal sets M_i, $i=1,2,3$. Let $y_0\in Y$ and take $(\theta_i,y_0)\in M_i$, $i=1,2,3$. Then $\{(\theta_i,y_0): i=1,2,3\}$ are pairwise distal. It follows from Lemma 9.2 that the cocycle is elliptic, a contradiction. ☐

Lemma 9.3. *Consider the cocycle* $\{\Phi(y,t)\}_{y\in Y,t\in\mathbb{T}}$ *and its generated projective bundle flow* $(P^1\times Y,\mathbb{T})$. *Then the cocycle is hyperbolic iff* $\sup_{t\in\mathbb{T}} r(\theta,y,t)=+\infty$ *for all* $(\theta,y)\in P^1\times Y$.

Proof. It is a special case of the main result in [56]. ☐

Theorem 9.1. *Consider the cocycle* $\{\Phi(y,t)\}_{y\in Y,t\in\mathbb{T}}$ *and its generated projective bundle flow* $(P^1\times Y,\mathbb{T})$. *Then the following holds:*

1) *If the cocycle is elliptic, then either* $P^1\times Y$ *is minimal and distal or there is an integer* $N\ge 1$ *such that* $P^1\times Y$ *laminates into infinitely many minimal* N-1 *extensions of* Y (*hence they are almost periodic*).
2) *If the cocycle is hyperbolic, then* $(P^1\times Y,\mathbb{T})$ *has precisely two minimal sets and each of them is an* 1-1 *extension of* Y (*hence they are almost periodic*).

Proof. 1) Since, by Lemma 9.2, $P^1\times Y$ is a distal extension of Y, either i) it is minimal and distal; or ii) it laminates into infinitely many minimal sets ([11]). In the case ii), we have by Theorem 4 and distality that there is a positive integer N such that each minimal set is an N-1 extension of Y.

2) Let $\{u^\pm(y)\}_{y\in Y}\subset R^2$ be the continuous, invariant line bundles associated with hyperbolicity and let $\theta^\pm(y)=\frac{1}{\pi}Arg\ u^\pm(y)$, $y\in Y$. Then

$$M^\pm=\{(\theta^\pm(y),y): y\in Y\}$$

are two minimal sets of $(P^1\times Y,\mathbb{T})$ which are clearly 1-1 extensions of Y. By Proposition 9.1, the projective bundle flow $(P^1\times Y,\mathbb{T})$ cannot have more than two minimal sets in this case. ☐

We now exam minimal dynamics of the project bundle flow if the cocycle is either parabolic or partially hyperbolic.

Lemma 9.4. *The cocycle* $\{\Phi(y,t)\}_{y\in Y, t\in\mathbb{T}}$ *is either parabolic or partially hyperbolic iff there are* $(\theta_1^0, y_0), (\theta_2^0, y_0) \in P^1 \times Y$ *such that* $\sup_{t\geq 0} r(\theta_1^0, y_0, t) = +\infty$ *and* $\sup_{t\in\mathbb{T}} r(\theta_2^0, y_0, t) < +\infty$.

Proof. The lemma is a direct consequence of Lemmas 9.2, 9.3. □

We call an ordered pair $\{(\theta_+, y), (\theta_-, y)\}$ in $P^1 \times Y$ a *Morse pair* if

$$\limsup_{t\to\infty} \frac{r(\theta_+, y, t)}{r(\theta_-, y, t)} = +\infty.$$

This notion is a relaxed version of relative dichotomy or More decomposition in linear skew-product flows.

Let $\pi : P^1 \times Y \to Y$ be the natural projection.

Lemma 9.5. *If the cocycle* $\{\Phi(y,t)\}_{y\in Y, t\in\mathbb{T}}$ *is not elliptic and some fiber of its generated projective bundle flow* $(P^1 \times Y, \mathbb{T})$ *over* Y *admits a distal pair, then each fiber of* $(P^1 \times Y, \mathbb{T})$ *over* Y *admits a Morse pair.*

Proof. Suppose for contradiction that there is a fiber $\pi^{-1}(y_0)$ which admits no Morse pair. Since some fiber of $P^1\times Y$ over Y admits a distal pair, then all fibers of $P^1\times Y$ over Y admit distal pairs. In particular, $\pi^{-1}(y_0)$ admits a distal pair, say $\{(\theta_0^1, y_0), (\theta_0^2, y_0)\}$. It follows from Lemma 9.1 that

$$\sup_{t\in\mathbb{T}}\{r(\theta_0^1, y_0, t) r(\theta_0^2, y_0, t)\} < +\infty. \tag{9.4}$$

Since $(\theta_0^1, y_0), (\theta_0^2, y_0)$ do not form a Morse pair in both orders, there are positive constants K_1, K_2 such that

$$K_1 r(\theta_0^1, y_0, t) \leq r(\theta_0^2, y_0, t) \leq K_2 r(\theta_0^1, y_0, t), \ \ t\in\mathbb{T}.$$

It follows from (9.4) that both $r(\theta_0^1, y_0, t)$ and $r(\theta_0^2, y_0, t)$ are bounded. Hence by Lemma 9.2, the cocycle is elliptic, a contradiction. □

Lemma 9.6. *Let* $\{(\theta_+^0, y^0), (\theta_-^0, y^0)\}$ *be a Morse pair. Then the following holds:*

1) *For any* $\theta \neq \theta_-^0$, $\{(\theta, y^0), (\theta_-^0, y^0)\}$ *is a Morse pair.*
2) *If* $\theta_1, \theta_2 \neq \theta_-^0$, *then* $(\theta_1, y^0), (\theta_2, y^0)$ *are proximal.*
3) *Suppose* $\sup_{t\in\mathbb{T}}\{r(\theta_-^0, y^0, t)\} < \infty$. *Then for any* $\theta_1, \theta_2 \neq \theta_-^0$, $(\theta_1, y^0), (\theta_-^0, y^0)$ *are proximal iff* $(\theta_2, y^0), (\theta_-^0, y^0)$ *are proximal. In particular, for any* $\theta \neq \theta_0^-$, $(\theta, y^0), (\theta_-^0, y^0)$ *are proximal iff* $(\theta_+^0, y^0), (\theta_-^0, y^0)$ *are proximal.*

Proof. 1) Let $\theta \neq \theta_-^0 \in P^1$ and consider the linear combination

$$\Phi(y^0, t)\begin{pmatrix}\cos\pi\theta\\ \sin\pi\theta\end{pmatrix} = c_+\Phi(y^0, t)\begin{pmatrix}\cos\pi\theta_+^0\\ \sin\pi\theta_+^0\end{pmatrix} + c_-\Phi(y^0, t)\begin{pmatrix}\cos\pi\theta_-^0\\ \sin\pi\theta_-^0\end{pmatrix}, \tag{9.5}$$

where c_+, c_- are constants. Since $\theta \neq \theta_-^0$ and $c_+ \neq 0$, we have by (9.5) that

$$\limsup_{t\to\infty} \frac{r(\theta, y^0, t)}{r(\theta_-^0, y^0, t)} \geq \limsup_{t\to\infty}\{|c_+|\frac{r(\theta_+^0, y^0, t)}{r(\theta_-^0, y^0, t)} - |c_-|\} = +\infty,$$

i.e., $\{(\theta, y^0), (\theta_-^0, y^0)\}$ is a Morse pair.

2) Let $\theta_1, \theta_2 \neq \theta_-^0$ and consider the linear combination

$$\Phi(y^0, t)\begin{pmatrix}\cos\pi\theta_2\\ \sin\pi\theta_2\end{pmatrix} = c_1\Phi(y^0, t)\begin{pmatrix}\cos\pi\theta_1\\ \sin\pi\theta_1\end{pmatrix} + c_0\Phi(y^0, t)\begin{pmatrix}\cos\pi\theta_-^0\\ \sin\pi\theta_-^0\end{pmatrix}, \tag{9.6}$$

where c_1, c_0 are constants. Since $\theta_2 \neq \theta_-^0$ and $c_1 \neq 0$, we have by (9.3) that $r(\theta_1, y^0, t)r(\theta_-^0, y^0, t)$ admits a positive lower bound, say $c(\theta_1)$. Then by (9.6),

$$\begin{aligned} r(\theta_1, y^0, t)r(\theta_2, y^0, t) &\geq r(\theta_1, y^0, t)r(\theta_-^0, y^0, t)\max\{0, |c_1|\frac{r(\theta_1, y^0, t)}{r(\theta_-^0, y^0, t)} - |c_0|\} \\ &\geq c(\theta_1)\max\{0, |c_1|\frac{r(\theta_1, y^0, t)}{r(\theta_-^0, y^0, t)} - |c_0|\}. \end{aligned}$$

It follows that $\sup_{t\in T}\{r(\theta_1, y^0, t)r(\theta_2, y^0, t)\} = +\infty$ since $\{(\theta_1, y^0), (\theta_-^0, y^0)\}$ is a Morse pair. Hence by Lemma 9.1, $(\theta_1, y^0), (\theta_2, y^0)$ are proximal.

3) By symmetry, it is sufficient to show that if $(\theta_1, y^0), (\theta_-^0, y^0)$ are proximal, then so are $(\theta_2, y^0), (\theta_-^0, y^0)$.

Assume that $(\theta_1, y^0), (\theta_-^0, y^0)$ are proximal, i.e., $\sup_{t\in\mathbb{T}}\{r(\theta_1, y^0, t)r(\theta_-^0, y^0, t)\} = +\infty$. Then by (9.6),

$$r(\theta_2, y^0, t)r(\theta_-^0, y^0, t) \geq |c_1|r(\theta_1, y^0, t)r(\theta_-^0, y^0, t) - |c_0|r(\theta_-^0, y^0, t)^2.$$

It follows that $\sup_{t\in\mathbb{T}}\{r(\theta_1, y^0, t)r(\theta_-^0, y^0, t)\} = +\infty$ since $\sup_{t\in\mathbb{T}}\{r(\theta_1, y^0, t)r(\theta_-^0, y^0, t)\} = +\infty$ and $\sup_{t\in\mathbb{T}}\{r(\theta_-^0, y^0, t)\} < \infty$. By Lemma 9.1, $(\theta_2, y^0), (\theta_-^0, y^0)$ are proximal.

□

Theorem 9.2. *Let the cocycle $\{\Phi(y, t)\}_{y\in Y, t\in\mathbb{T}}$ be either parabolic or partially hyperbolic. Then the following holds for its generated projective bundle flow $(P^1 \times Y, \mathbb{T})$:*

1) *$(P^1 \times Y, \mathbb{T})$ admits at most two minimal sets.*
2) *If $(P^1 \times Y, \mathbb{T})$ admits two minimal sets, then each minimal set is an almost 1-1 extension of Y (hence they are almost automorphic).*
3) *If $(P^1 \times Y, \mathbb{T})$ admits only one minimal set M, then precisely one of the following holds:*
   i) *M is almost automorphic and is either an almost 1-1 or almost 2-1 extension of Y;*
   ii) *M is an everywhere non-locally connected Cantorian and is residually Li-Yorke chaotic;*
   iii) *M is the entire space $P^1 \times Y$ and is residually Li-Yorke chaotic.*

   Moreover, in cases ii) *and* iii)*, M is a proximal extension of Y which is not almost 1-1 (hence M is not almost automorphic).*

Proof. 1) is clear by Proposition 9.1.

2) It follows from Theorem 4 that each minimal set is an almost N-1 extension of Y for some positive integer N. We note that an N-1 extension of Y must be a distal extension. But by Lemma 9.2 there are no three points on a same fiber which can be pair-wise distal. Hence $N = 1$.

3) By Lemma 9.4, we let $(\theta_+(y_0), y_0), (\theta_-(y_0), y_0) \in P^1 \times Y$ be such that $\sup_{t\geq 0} r(\theta_+(y_0), y_0, t) = +\infty$ and $\sup_{t\in\mathbb{T}} r(\theta_-(y_0), y_0, t) < +\infty$. It is clear that $\{(\theta_+(y_0), y_0), (\theta_-(y_0), y_0)\}$ is a Morse pair.

Case 1. $(\theta_+(y_0), y_0), (\theta_-(y_0), y_0)$ are proximal.

In this case, since $\sup_{t\in\mathbb{T}}\{r(\theta_-(y_0), y_0, t)\} < \infty$, we have by Lemma 9.6 (3) that any two points on the fiber $p^{-1}(y_0)$ are proximal. This particularly implies that M is a proximal extension of Y. Hence if M is point-distal, then it must be an almost 1-1 extension of Y.

Now suppose that M is not point-distal. Then by Theorem 2, it must be residually Li-Yorke chaotic, and by Theorem 6, the corresponding projective bundle flow admits no mean motion (because M is not almost automorphic). Moreover, since M is not an almost N-1 extension of Y for any positive integer N, it follows from Theorem 3 that M is either a Cantorian or the entire phase space, and, in the case that M is a Cantorian, we have by Theorem 7 that it is everywhere non-locally connected.

Case 2. $(\theta_+(y_0), y_0), (\theta_-(y_0), y_0)$ are distal.

We note that $\sup_{t\in\mathbb{T}}\{r(\theta_-(y_0), y_0, t)\} < \infty$. It follows from Lemma 9.6 that $(\theta, y_0), (\theta_-(y_0), y_0)$ are distal if $\theta \neq \theta_-(y_0)$ and $(\theta_1, y_0), (\theta_2, y_0)$ are proximal if $\theta_1, \theta_2 \neq \theta_-(y_0)$. According to a general result due to Auslander ([2]), there exists $(\theta_*, y_0) \in M$ such that $(\theta_-(y_0), y_0), (\theta_*, y_*)$ are proximal. Since by Lemma 9.6, $(\theta, y_0), (\theta_-(y_0), y_0)$ are distal for any $\theta \neq \theta_-(y_0)$, we have that $\theta_* = \theta_-(y_0)$, i.e., $(\theta_-(y_0), y_0) \in M$. Applying the result of Auslander to $(\theta_+(y_0), y_0)$, we also find a point $(\theta(y_0), y_0) \in M$ such that $(\theta(y_0), y_0), (\theta_+(y_0), y_0)$ are proximal. Clearly, $\theta(y_0) \neq \theta_-(y_0)$ and $(\theta(y_0), y_0), (\theta_-(y_0), y_0)$ are distal. Hence $\pi^{-1}(y_0) \cap M$ admits a distal pair. It follows that all fibers $\pi^{-1}(y) \cap M$, $y \in Y$, admit distal pairs. It now follows from Lemma 9.5 that each fiber $\pi^{-1}(y)$ admits a Morse pair $\{(\theta_+(y), y), (\theta_-(y), y)\}$. Hence by Lemma 9.6 (2), $(\theta_1, y), (\theta_2, y)$ are proximal for any $\theta_1, \theta_2 \neq \theta_-(y)$. Since $\pi^{-1}(y) \cap M$ admits a distal pairs, $(\theta_-(y), y) \in M$ and there exists $(\theta(y), y) \in M$ such that $(\theta(y), y), (\theta_-^0(y), y)$ are distal. Let $\delta = \inf_{t \in \mathbb{T}} |\tilde{\theta}(\theta(y_0), y_0, t) - \tilde{\theta}(\theta_-(y_0), y_0, t)|$. It is clear that $\delta > 0$.

Claim 1. For any $(\theta, y) \in M$ with $\theta \neq \theta_-(y)$, $(\theta, y), (\theta_-(y), y)$ are distal and $|\theta - \theta_-(y)| \geq \delta$.

Since $(\theta(y_0), y_0) \in M$, there exists a sequence $\{t_n\} \subset \mathbb{T}$ such that $\lim_{n\to\infty} \Lambda_{t_n}(\theta(y_0), y_0) = (\theta, y)$ and $\lim_{n\to\infty} \Lambda_{t_n}(\theta_-(y_0), y_0) = (\theta_*(y), y)$ for some $(\theta_*(y), y) \in M$. Clearly, $(\theta, y), (\theta_*(y), y)$ are distal and $|\theta - \theta_*(y)| \geq \delta$. Since $(\theta_1, y), (\theta_2, y)$ are proximal for any $\theta_1, \theta_2 \neq \theta_-(y)$, we have $\theta_*(y) = \theta_-(y)$. This proves the claim.

Claim 2. There is a residual set $Y_0 \subset Y$ such that $|\pi^{-1}(y) \cap M| = 2$ for all $y \in Y_0$.

We use the argument in the proof of Theorem 7.4 in [30]. By Claim 1, there exist $0 \leq \theta_1 \leq \theta_2 < 1 + \theta_1$ such that $(\theta_1, y_0), (\theta_2, y_0) \in M$ and $([\theta_1, \theta_2] \times \{y_0\}) \cap M = \pi^{-1}(y_0) \cap M \setminus \{(\theta_-(y_0), y_0)\}$. Let Y_0 be the set of all continuity points of the upper semi-continuous map $y \mapsto \pi^{-1}(y) \cap M$. Then Y_0 is a residual subset of Y. For any $y \in Y_0$, since $(\theta_1, y_0), (\theta_2, y_0)$ are proximal, there exists a sequence $\{t_n\} \subset \mathbb{T}$ such that $\lim_{n\to\infty} \Lambda_{t_n}(\theta_1, y_0) = \lim_{n\to\infty} \Lambda_{t_n}(\theta_2, y_0) = (\theta(y), y)$ and $\lim_{n\to\infty} \Lambda_{t_n}(\theta_-(y_0), y_0) = (\theta^*(y), y)$ for some $(\theta^*(y), y) \in M$. Since $(\theta_1, y_0), (\theta_-(y_0), y_0)$ are distal, so are $(\theta(y), y), (\theta^*(y), y)$. Hence $\theta^*(y) = \theta_-(y)$. It follows from Lemma 5.3 (2) that either $\lim_{n\to\infty} \Lambda_{t_n}([\theta_1, \theta_2] \times \{y_0\}) = \{(\theta(y), y)\}$ or $\lim_{n\to\infty} \Lambda_{t_n}([\theta_2, \theta_1] \times \{y_0\}) = \{(\theta(y), y)\}$ by taking subsequences if necessary.

Since $\theta_-(y_0) \in [\theta_2, \theta_1]$ and $\lim_{n\to\infty} \Lambda_{t_n}(\theta_-(y_0), y_0) = (\theta_-(y), y) \neq (\theta(y), y)$, we have $\lim_{n\to\infty} \Lambda_{t_n}([\theta_1, \theta_2] \times \{y_0\}) = \{(\theta(y), y)\}$. Thus

$$\begin{aligned} \pi^{-1}(y) \cap M &= \lim_{n\to\infty} \pi^{-1}(y_0 \cdot t_n) \cap M \subseteq \lim_{n\to\infty} \Lambda_{t_n}([\theta_1, \theta_2] \times \{y_0\}) \cup \Lambda_{t_n}(\theta_-(y_0), y_0) \\ &= \{(\theta(y), y), (\theta_-(y), y)\}. \end{aligned}$$

It follows that $|\pi^{-1}(y) \cap M| = 2$ for all $y \in Y_0$.

Now, we have by Claim 2 that M is an almost 2-1 extension of Y. To show that M is almost automorphic, we note that it easily follows from Claim 1 and Lemma 9.6 (2) that the proximal relation $P(M)$ is closed. Hence $M/P(M)$ is a compact Hausdroff space and there is a natural flow $(M/P(M), \mathbb{T})$ induced from the flow $(P^1 \times Y, \mathbb{T})$. By Lemma 9.1, $M/P(M)$ is a 2-1 extension of Y, hence it is almost periodic minimal. Let $p : M \to M/P(M)$ be the natural projection. Then it follows from Claim 2 that $p : (M, \mathbb{T}) \to (M/P(M), \mathbb{T})$ is an almost 1-1 extension. Hence M is almost automorphic. □

In the partially hyperbolic case, more precise information can be obtained as follows.

Theorem 9.3. *Let the cocycle $\{\Phi(y,t)\}_{y \in Y, t \in \mathbb{T}}$ be partially hyperbolic. Then its generated projective bundle flow $(P^1 \times Y, \mathbb{T})$ admits a unique minimal set M. Moreover, the following holds:*

a) *M is characterized precisely by one of the case* i)-iii) *in* Theorem 9.2 *but it is nether almost periodic nor an almost* 2-1 *extension of Y;*
b) *M is non-uniquely ergodic and admits precisely two ergodic sheets $\{(u^{\pm}(y), y)\}_{y \in Y^*}$ as in* (9.2).
c) *There is a residual set $Y_0 \subset Y$ such that for each $(\theta, y) \in M \cap p^{-1}(y)$, $r(\theta, y, t)$ oscillates between $-\infty$ and $+\infty$ as $t \to \pm\infty$.*

Proof. The fact that M cannot be an almost 2-1 extension of Y was proved in [30] for the linear system (9.1). b) and c) were given in Theorem 4.10 of [38] also for the linear system (9.1). The proof for the general situation follows from similar arguments. Since M is not uniquely ergodic, it cannot be almost periodic. □

Examples of continuous, almost periodic, $sl(2,\mathbb{R})$-valued cocycles whose projective bundle flows have the property i) stated in Theorem 9.2 are well-known (see [6, 30]). Also there are many continuous, almost periodic, $sl(2,\mathbb{R})$-valued cocycles whose projective bundle flows have the property iii) stated in Theorem 9.2 (see [33, 45]). An interesting question is whether case ii) in Theorem 9.2 can really occur in a projective bundle flow. The following result shows that the answer to this question is negative when the forcing space in a projective bundle flow is locally connected.

Theorem 9.4. *Let Y be locally connected and the cocycle $\{\Phi(y,t)\}_{y\in Y,t\in\mathbb{T}}$ be either parabolic or partially hyperbolic. Then its generated projective bundle flow $(P^1\times Y,\mathbb{T})$ admits no Cantorian minimal set.*

Proof. If $(P^1\times Y,\mathbb{T})$ admits mean motion, then we have by Theorem 6 that all its minimal sets are almost automorphic hence they are not residually Li-Yorke chaotic. This show that case ii) in Theorem 9.2 does not occur. In particular, $(P^1\times Y,\mathbb{T})$ admits no Cantorian minimal set.

If $(P^1\times Y,\mathbb{T})$ admits no mean motion, then by Theorem 7 2) (Theorem 8.6) it is topologically transitive. It then follows from almost exact proof of Proposition 4.6 in [3] that if the entire phase space $P^1\times Y$ is not minimal, then a minimal set M of $(P^1\times Y,\mathbb{T})$ is either an almost 1-1 or an almost 2-1 extension of Y. In particular, M is not a Cantorian. □

Remark. 1) Using the same argument as the above, one sees that if a projective bundle flow admits mean motion, then case ii) in Theorem 9.2 cannot occur regardless whether the base Y is locally connected or not (this has already been shown in [6] for the continuous case).

It then remains an open question whether case ii) in Theorem 9.2 can occur in a projective bundle flow without mean motion when the base is not locally connected. We think that the answer to this question should be affirmative.

2) Suppose that the projective bundle flow of a partially hyperbolic, quasi-periodically forced cocycle $\{\Phi(y,t)\}_{y\in Y,t\in\mathbb{T}}$ admits a globally attracting SNA, say $\mathcal{A}$. Then $\mathcal{A}$ cannot be the entire phase space, and by Theorem 9.3, $\mathcal{A}$ is made up by a unique minimal set M along with its "homoclinic orbits" (in the sense of proximality). Now, by Theorems 9.2, 9.4, M is non-almost-periodic, almost automorphic extension of Y. If we further assume that the rotation number of the projective bundle flow is rationally independent of the forcing frequencies, then by Theorem 8, M is everywhere non-locally connected.

All these simply suggests an important role played by almost aotumorphic dynamics to such a SNA: topologically the minimal set in the SNA is everywhere non-local connected and an almost 1-cover of the forcing space, and dynamically the minimal set in the SNA is almost automorphic.

9.2. **Cases with mean motion properties.** With the classification given in the above, it is important to know when or how often almost automorphic dynamics can occur in the projective bundle flow of an almost periodic, $sl(2,\mathbb{R})$-valued cocycle in the non-parabolic case. Some affirmative answers to this problem was given in [38] with respect to extreme points of spectral gaps of the following almost periodic Schrödinger and Schrödinger-like operators:

$$L_q=-\frac{d^2}{dt^2}+q(y\cdot t):\ L^2(\mathbb{R})\to L^2(\mathbb{R});$$
$$L_Q=J(\frac{d}{dt}-Q(y\cdot t)):\ L^2(\mathbb{R},R^2)\to L^2(\mathbb{R},R^2);$$
$$L_v=-A+v(y\cdot n):\ L^2(\mathbb{Z})\to L^2(\mathbb{Z}),$$

where $(Y, \mathbb{T})$ is almost periodic minimal for $T = \mathbb{R}$ or $\mathbb{Z}$, q, v are continuous functions on Y, Q is a 2×2 matrix-valued continuous function on Y, J is the standard 2×2 symplectic matrix, and A is the operator defined by $Az(n) = z(n+1) + z(n-1)$.

Of course, when automorphic dynamics exist in the projective bundle flow of an almost periodic, $sl(2, \mathbb{R})$-valued cocycle in the non-hyperbolic case, it is also interesting to know whether the corresponding projective bundle flow admits mean motion.

Consider the spectral problem

$$L_q x(t) = \lambda x(t), \tag{9.7}$$

$$L_Q X(t) = \lambda X(t), \tag{9.8}$$

$$L_v z(n) = \lambda z(n). \tag{9.9}$$

Each linear equation (9.7)-(9.9) generates an almost periodic, $sl(2, \mathbb{T})$-valued cocycle for $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$ in the natural way, which gives rise to a projective bundle flow $\Pi_\lambda = (P^1 \times Y, \mathbb{T})$, where P^1 is parametrized by $\phi = -\frac{2}{\pi} Arg\binom{x'}{x}$, $\phi = \frac{2}{\pi} Arg X$, $\phi = -\frac{2}{\pi} Arg\binom{z(n+1)}{z(n)}$ for (9.7), (9.8), (9.9) respectively. For each $L = L_q, L_Q, L_v$, according to the Gap Labeling Theorem ([37]), the rotation number $\rho(\lambda)$ of Π_λ is monotonically increasing and increases precisely on the spectrum Σ_L of L which is contained in a half line $[\lambda^*, +\infty)$. For each λ in the resolvent of L, it is well-known that the corresponding cocycle is hyperbolic, hence Π_λ admits exactly two minimal sets which are all 1-1 extensions of Y (hence they are almost periodic).

Proposition 9.2. *Consider $L = L_q, L_Q, L_v$ and let λ_0 be a finite extreme point of a spectral gap (i.e., a maximal open interval in the resolvent of L). Then Π_{λ_0} admits mean motion. Consequently, each minimal set of Π_{λ_0} is almost automorphic and in fact an almost 1-1 extension of Y.*

Proof. We only give the proof for the case of L_q. The other two cases can be treated similarly.

Observe from (9.7) that $\phi = -\frac{2}{\pi} Arg\binom{x'}{x}$ satisfies the equation

$$\phi' = \frac{\lambda - q(y \cdot t) - 1}{\pi} + \frac{\lambda - q(y \cdot t) + 1}{\pi} \cos \pi\phi. \tag{9.10}$$

We denote $\tilde{\phi}_\lambda(\phi, y, t)$ as the solution of (9.10) corresponding to λ, y and with initial value ϕ.

Let λ_0 be the left end point of a spectral gap I in the resolvant of L_q. Then the rotation number of (9.10) is a constant over $\bar{I}$, which we denote by ρ.

By elementary theory of ordinary differential equations, we see from (9.10) that

$$\frac{\partial \tilde{\phi}_\lambda(\phi, y, t)}{\partial \lambda} \geq 0 \tag{9.11}$$

for all $\lambda, \phi \in R^1$, $y \in Y$, and $t \geq 0$. For any $(\phi_0, y_0) \in R^1 \times Y$ and a given $\lambda_* \in I$, we denote $\phi_0(t) = \tilde{\phi}_{\lambda_0}(\phi_0, y_0, t)$, $\phi_*(t) = \tilde{\phi}_{\lambda_*}(\phi_0, y_0, t)$. Using (9.11) and the comparison principle of scalar ordinary differential equations, it is easy to see that $\phi_0(t) \leq \phi_*(t)$ for all $t \geq 0$. It follows that

$$\phi_0(t) - \phi_0 - \rho t \leq \phi_*(t) - \phi_0 - \rho t$$

for all $t \geq 0$. Since Π_{λ_*} admits almost periodic motion, it admits mean motion. Hence

$$\sup_{t \geq 0}(\phi_0(t) - \phi_0 - \rho t) \leq \sup_{t \geq 0}(\phi_*(t) - \phi_0 - \rho t) < \infty.$$

Since (ϕ_0, y_0) is arbitrary, we have by Theorem 8.2 d) that Π_{λ_0} admits mean motion. It follows from Theorem 6 that each minimal set of Π_{λ_0} is almost automorphic. Since almost periodic minimal sets of Π_{λ_*} are all 1-1 extensions of Y, we have by Theorem 6 2) that ρ is contained in the frequency module of the forcing. Applying Theorem 6 2) again, we conclude that any minimal set of Π_{λ_0} cannot be an almost 2-1 extension of Y.

The case when λ_0 is the right end point (including λ^*) of a spectral gap in the resolvant of L_q is similar. □

Dynamics of Π_λ when λ entering the spectrum through λ_0 are expected to be more complicated due to the possible loss of mean motion property. This can be viewed as another intermittency phenomenon characterized by almost automorphic intermediate dynamics.

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References

[1] V. I. Arnold, Small divisors, I. On mapings of a circle onto itself, *Izv. Akad. Nauk SSSR, Ser Math.* **25** (1961), 21-86.
[2] J. Auslander, Minimal Flows and Their Extensions, North-Holland Mathematics Studies, **153**, Amsterdam, 1988.
[3] F. Béguin, S. Croviser, T. H. Jäger, and F. Le Roux, Denjoy construction for fibered homeomorphism of the torus, *Trans. Amer. Math. Soc.* to appear.
[4] A. Berger, S. Siegmund and Y. Yi, On almost automorphic dynamics in symbolic lattices, *Ergod. Th. & Dynam. Sys.* **24** No. 3 (2004), 677-696.
[5] K. Bjerklov, Positive Lyapunov exponent and minimality for a class of one-dimensional quasi-periodic Schrödinger equations, *Ergod. Th. & Dynam. Sys.* **25** No. 4 (2005), 1015-1045.
[6] K. Bjerklov and R. A. Johnson, Minimal subsets of projective flows, *Discrete Cont. Dyanm. Sys., Series B* **9** No. 3 & 4 (2008), 495-516.
[7] F. Blanchard, E. Glasner, S. Kolyada and A. Maass, On Li-Yorke pairs, *J. Reine Angew. Math.* **547** (2002), 51-68.
[8] R. Bowen, Entropy for group endormorphisms and homogeneous spaces, *Trans. Amer. Math. Soc.* **153** (1971), 401-414.
[9] M. Denker, C. Grillenberger and C. Sigmund, Ergodic Theory on Compact Spaces, Lecture Notes in Math. **527**, Springer-Verlag, New York, 1976.
[10] R. Ellis, Distal transformation groups, *Pacific J. Math.* **8** (1958), 401-405.
[11] R. Ellis, Lectures on Topological Dynamics, W. A. Benjamin, Inc., New York, 1969.
[12] A. Fathi, Weak KAM Theorem in Lagrangian Dynamics, Cambridge Studies in Advanced Mathematics, **88**, Cambridge University Press, Cambridge, 2007.
[13] U. Feudel, S. Kuznetsov, and A. Pikovsky, Strange Nonchaotic Attactors, World Scientific, 2006.
[14] H. Furstenberg, Strictly ergodicity and transformations of the torus, *Amer. J. Math.* **83** (1961), 573-601.
[15] H. Furstenberg, Recurrence in Ergodic Theory and Combinatorial Number Theory, Princeton Univ. Press, 1981.
[16] H. Furstenberg and B. Weiss, On almost 1-1 extensions, *Israel J. Math.* **65** (1989), 311–322.
[17] E. Glasner, Ergodic Theory via Joinings, Mathematical Surveys and Monographs, **101**, American Mathematical Society, Providence, RI, 2003.
[18] E. Glanser, Topological weak mixing and quasi-Bohr systems, *Israel J. Math.* **148** (2005), 277-304.
[19] E. Glasner, W. Huang, S. Shao, and X. Ye, What transitive system can have chaotic properties? preprint, 2007.
[20] E. Glasner, J. P. Thouvenot, and B. Weiss, Entropy theory without a past, *Ergod. Th. & Dynam. Sys.* **20** (2000), 1355-1370.
[21] P. Glendinning, T. H. Jäger, and G. Keller, How chaotic are strange non-chaotic attractors? *Nonlinearity* **19** (2006), 2005-2022.
[22] W. Gottschalk and G. Hedlund, Topological Dynamics, Amer. Math. Soc. Colloq., **36**, 1955.
[23] C. Grebogi, E. Ott, S. Pelikan, and J. A. Yorke, Strange attractors that are not chaotic, *Physica D* **13** (1984), 261-268.
[24] M. R. Herman, Une méthode pour minorer les exposants de Lyapunov et quelques, *Comment. Math. Helvetici* **58** (1983), 453-502.
[25] P. Hulse, Sequence entropy relative to an invariant σ-algebra, *J. London Math. Soc.* **33** (1986), 59-72.
[26] W. Hurewicz and H. Wallman, Dimension Theory, Princeton Mathematical Series, **4**, Princeton University Press, Princeton, New Jersery, 1941.
[27] R. Iturriaga, Minimizing measures for time-dependent Lagrangians, *Proc. London Math. Soc.* **73** (1996), 216-240.
[28] T. H. Jäger and G. Keller, The Denjoy type of argument for quasiperiodically forced circle diffeomorphism, *Ergod. Th. & Dynam. Sys.* **26** No. 2 (2006), 447-465.
[29] T. H. Jäger and J. Stark, Towards a classification for quasi-periodically forced circle homeomorphism, *J. London Math. Soc.* **77** (2006), 727-744.

[30] R. A. Johnson, On a Floquet theory for almost-periodic, two-dimensional linear systems, *J. Differential Equations* **37** (1980), 184-205.

[31] R. A. Johnson, Minimal functions with unbounded integral, *Israel J. Math.* **31** (1978), 133-141.

[32] R. A. Johnson, A linear, almost periodic equation with an almost automorphic solution, *Proc. Amer. Math. Soc.* **82** (1981), 199-205.

[33] R. A. Johnson, Two-dimensional, almost periodic linear systems with proximal and recurrent behavior, *Proc. Amer. Math. Soc.* **82** (1981), 417-422.

[34] R. A. Johnson, Bounded solutions of scalar, almost periodic linear equations, *Illinois J. Math.* **25** (1981), 632-643.

[35] R. A. Johnson, On almost-periodic linear differential systems of Millionščikov and Vinograd, *J. Math. Anal. App.* **85** (1982), 452-460.

[36] R. A. Johnson, An example concerning the geometric significance of the rotation numbers-integrated density of states, *Proc. Breman Conf., Lect. Notes Math.,* **1186**, Springer-Verlag, 1984.

[37] R. A. Johnson and J. Moser, The rotation number for almost periodic potentials,*Comm. Math. Phys.* **84** (1982), 403-438.

[38] A. Jorba, C. Nunez, R. Obaya, and J. C. Tatjer, Old and new results on SNAs on the real line, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **17** No. 11 (2007), 3895-3928.

[39] J. F. C. Kingman, Subadditive ergodic theory, *Ann. Prob.* **1** (1973), 883-909.

[40] T. Y. Li and J. A. Yorke, Period three implies chaos, *Amer. Math. Monthly* **82** (1975), 985-992.

[41] J. C. Martin, Substitution minimal flows, *Amer. J. Math.***93** (1971), 503-526.

[42] J. Mather, Minimal measures, *Comment. Math. Helv.* **64** (1989), 375-394.

[43] J. Mather, Minimal action measures for positive-definite Lagrangian systems, IXth International Congress on Mathematical Physics (Swansea, 1988), 466-468, Hilger, Bristol, 1989.

[44] J. Mather, Action minimizing invariant measures for positive definite Lagrangian systems, *Math. Z.* **207** (1991), 169-207.

[45] M. Nerurka, On the construction of smooth ergodic skew-products, *Ergod. Th. & Dynam. Sys.* **8** (1988), 311-326.

[46] J. Milnor, Lectures on Characteristic Classes, Princeton University, 1957.

[47] J. Moser, On the theory of quasi-periodic motions, *SIAM Review* **8** (1966), 145-172.

[48] J. Mycielski, Independent sets in topological algebra, *Fund. Math.* **55** (1964), 139-147.

[49] K. Namakura, On bicompact semigroup, *Math. J. Okayama Univ.* **1** (1952), 99-108.

[50] W. Parry, Topics in ergodic theory, Cambridge Tracts in Mathematics, **75**, Cambridge University Press, Cambridge-New York, 1981.

[51] V. A. Pliss and G. R. Sell, Planetary motions and climate of the earth, preprint 2006.

[52] L. Pontrjagin, Topological Groups, Princeton Math. Series, **2**, Princeton University Press, Princeton, 1939.

[53] F. P. Ramsey, On a problem of formal logic, *Proc. London Math. Soc.* (2) **30** (1930), 264-286.

[54] V. A. Rohlin, On the fundament ideas of measure theory, *Math. Sb.* (N.S.) **25** (67) 1 (1949), 107-150. Engl. transl., Amer. Math. Soc. Translations, Ser.1, **10** (1962), 1-54.

[55] F. J. Romeiras and E. Ott, Strange non-chaotic attractors of the damped pendulum with quasiperiodic forcing, *Phy. Rev. A* **35** (1987), 4404-4413.

[56] R. J. Sacker and G. R. Sell, Existence of Dochotomies and invariant splitting for linear systems I, *J. Differential Equations* **15** (1974), 429-458.

[57] R. J. Sacker and G. R. Sell, A spectral theory for linear differential systems, *J. Differential equations* **27** (1978), 320-358.

[58] W. Shen, Global attractor in quasi-periodically forced Josephson junctions, *Far East J. Dyn. Syst.* **3** (2001), 51-80.

[59] W. Shen and Y. Yi, Dynamics of almost periodic scalar parabolic equations, *J. Differential Equations* **122** (1995), 114-136.

[60] W. Shen and Y. Yi, On minimal sets of scalar parabolic equations with skew-product structures, *Trans. Amer. Math. Soc.* **347** (1995), 4413-4431.

[61] W. Shen and Y. Yi, Almost Automorphy and Skew-product Semi-flow, *Mem. Amer. math. Soc.* **136** No. 647, 1998.

[62] W. A. Veech, Almost automorphic functions on groups, *Amer. J. Math.* **87** (1965), 719-751.

[63] W. A. Veech, Point-distal flows, *Amer. J. Math.* **92** (1970), 205-242.

[64] W. A. Veech, Topological dynamics, *Bull. Amer. Math. Soc.* **83** (1977), 775-830.

[65] P. Walters, An Introduction to Ergodic Theory, Graduate Texts in Math., **79**, Springer-Verlag, New York-Berlin, 1982.

[66] Y. Yi, A generalized integral manifold theorem, *J. Differential Equations* **102** (1993), 153-187.

[67] Y. Yi, On almost automorphic oscillations, *Fields Inst. Commun.* **42** (2004), 75-99.

[68] R. J. Zimmer, Extensions of ergodic group actions, *Illinois J. Math.* **20** (1976), 373-409.

[69] R. J. Zimmer, Ergodic actions with generalized discrete spectrum, *Illinois J. Math.* **20** (1976), 555-588.

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