Spreading Speeds and Traveling Waves for Periodic Evolution Systems

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Abstract The theory of spreading speeds and traveling waves for monotone autonomous semiflows is extended to periodic semiflows in the monostable case.

*Research supported in part by the NSF of China.
†Research supported in part by the NSF of USA grant DMS0204119.
‡Research supported in part by the NSERC of Canada and the MITACS of Canada.

Then these abstract results are applied to a periodic system modeling man-environment-man epidemics, a periodic time-delayed and diffusive equation, and a periodic reaction-diffusion equation on a cylinder.

Key words and phrases: Monotone systems, spreading speeds, periodic traveling waves.

2000 Math Subject Classification: 37C65, 37B55, 35K57, 35R10, 92D25

Short title for page headings: Spreading Speeds and Periodic Waves

1 Introduction

There have been extensive investigations on traveling waves and the asymptotic (long-time) behavior in terms of asymptotic speeds of spread for various evolution systems arising in applied sciences, see, e.g., [1]–[4], [6], [10]–[13], [18]–[20], [22]–[26], [29] and references therein. Asymptotic speed of spread (in short, spreading speed) was first introduced by Aronson and Weinberger [2] for reaction-diffusion equations. This concept has proved to be very important in the study of biological invasions and disease spread. There is an intuitive interpretation for the spreading speed c^* in a spatial epidemic model: if one runs at a speed $c > c^*$, then one will leave the epidemic behind; whereas if one runs at a speed $c < c^*$, then one will eventually be surrounded by the epidemic. Recently, the theory of asymptotic speeds of spread and traveling waves for monotone semiflows has been developed by Liang and Zhao [11] in such a way that it can be applied to various autonomous evolution equations admitting the comparison principle.

It is well known that interactive populations often live in a fluctuating environment. For example, physical environmental conditions such as temperature and humidity and the availability of food, water, and other resources usually vary in time with seasonal or daily variations. Therefore, more realistic models should be nonautonomous systems. In particular, if the data in a model are periodic functions of time with commensurate period, a periodic system arises;

if these periodic functions have different (minimal) periods, we get an almost periodic system. There are a few results on traveling waves for such systems: Alikakos, Bates and Chen [1], Bates and Chen [3], and Shen [20] established the existence and global stability of periodic traveling waves for periodic local, nonlocal and lattice equations with bistable nonlinearities, respectively; Shen [18] and Chen [6] also discussed almost periodic traveling waves for almost periodic local and nonlocal equations in the bistable case; and Shen [19] showed, among other things, the existence of a family of almost automorphic traveling waves for a class of almost periodic KPP-type reaction-diffusion equations. However, it seems that there are at present no exact results for asymptotic speeds of spread for periodic and almost periodic evolution systems with monostable nonlinearities. Our purpose in the current paper is to study spreading speeds and periodic traveling waves for monotone periodic semiflows in the monostable case and to apply the obtained results to three types of periodic evolution systems. Our results show that the spreading speed coincides with the minimal speed for monotone periodic traveling waves under reasonable assumptions.

Our approach is to apply the abstract results of [11] on monotone operators to the Poincaré (period) map associated with a given periodic semiflow. We should also point out that in the case of the continuous spatial habitat, the compactness of the operator with respect to the compact open topology is needed for the existence of traveling waves in [11] (see also [24, 10]). We will show that this compactness condition can be replaced with a much weaker one: the map is a contraction with respect to the Kuratowski measure of noncompactness (see Remarks 2.1 and 2.3). This new observation makes the developed theory applicable to some evolution systems consisting of reaction-diffusion equations coupled with ordinary differential equations (see, e.g., section 3).

The organization of this paper is as follows. In section 2, we summarize the abstract results for monotone maps (Theorems A, B, C and D) based on [11]. In order to weaken the compactness condition in [11], we present some properties of the Kuratowski measure of noncompactness on a Banach space (Lemma 2.1) and prove the asymptotic precompactness of a sequence of sets associated with the monotone map (Lemma 2.2). Then we show the existence of spreading speed (Theorem 2.1) for a monotone periodic semiflow, and its coincidence with the minimal wave speed for monotone periodic traveling waves (Theorems 2.2 and 2.3). In the rest of the paper we apply the general results of section 2 to three types of periodic differential systems: in section 3 to a periodic system modeling man-environment-man epidemics; in section 4 to a periodic time-delayed and diffusive equation; and in section 5 to a periodic reaction-diffusion equation on a cylinder.

2 Periodic semiflows

Let $(X, \|\cdot\|)$ be a Banach space over $\mathbb{R}$ or $\mathbb{C}$. For a bounded subset B of X, the Kuratowski measure of noncompactness of B is defined as

$$\alpha(B) = \inf \{r > 0 : \ B \text{ has a finite cover of diameter } \leq r\}.$$

Let B be covered by a finite number of subsets $\{M_1, \cdots, M_m\}$ of X each with diameter $\leq r$. Then $B = \cup_{i=1}^{m}(M_i \cap B)$ with the diameter of $M_i \cap B \leq r$. Thus, in the definition of $\alpha(B)$, we can always assume that each set in the finite cover is a subset of B. For various properties of the Kuratowski measure of noncompactness, we refer to [14]. The following lemma is a generalization of [14, Lemma I.5.3]. For the completeness, we provide a proof of it below.

Lemma 2.1. *Let d be the distance induced by the norm $\|\cdot\|$ on X. For two bounded subsets A, B of X, denote $\delta(B, A) := \sup_{x \in B} d(x, A)$. Let $\{A_n\}_{n=1}^{\infty}$ be a non-increasing family of non-empty, bounded and closed subsets (i.e., $m \geq n$ implies $A_m \subset A_n$). Assume that $\alpha(A_n) \to 0$ as $n \to +\infty$. Then $A_\infty = \bigcap_{n \geq 1} A_n$ is non-empty and compact, and $\delta(A_n, A_\infty) \to 0$ as $n \to +\infty$.*

Proof. Given a sequence of points $\{x_n\}$ with $x_n \in A_n$, $\forall n \geq 1$. Since

$$\alpha(\{x_n\}_{n\geq 1}) = \alpha(\{x_n\}_{n \geq m}) \leq \alpha(A_m) \to 0, \text{ as } m \to \infty,$$

we have $\alpha(\{x_n\}_{n\geq 1}) = 0$. It follows that $\overline{\{x_n : n \geq 1\}}$ is compact, and hence $\{x_n\}$ has a convergent subsequence. Since each A_n is nonempty, we can choose a sequence of points $y_n \in A_n$. It follows that there is a subsequence $n_k \to \infty$ such that $\lim\limits_{k\to\infty} y_{n_k} = y_0$. Thus, the closedness and monotonicity of A_n imply that $y_0 \in A_\infty$, and hence $A_\infty \neq \emptyset$. Clearly, A_∞ is closed. Given a sequence of points $\{z_n\} \subset A_\infty$, we have $z_n \in A_n, \forall n \geq 1$. By what we have proved, $\{z_n\}$ has a convergent subsequence. So A_∞ is compact. Assume, by contradiction, that $\lim\limits_{n\to\infty} \delta(A_n, A_\infty) \neq 0$. Then there exist a number $\epsilon_0 > 0$, a sequence of integers $m_k \to \infty$, and a sequence of points $w_{m_k} \in A_{m_k}$ such that $d(w_{m_k}, A_\infty) \geq \epsilon_0$ for all $k \geq 1$. Again by what we have proved, without loss of generality, we may assume that $\lim\limits_{k\to\infty} w_{m_k} = w_0$. Then we have $w_0 \in A_\infty$, and hence $\lim\limits_{k\to\infty} d(w_{m_k}, A_\infty) = d(w_0, A_\infty) = 0$, a contradiction. □

Let τ be a nonnegative real number and $\mathcal{C}$ be the set of all bounded and continuous functions from $[-\tau, 0] \times \mathcal{H}$ to $\mathbb{R}^k$, where $\mathcal{H} = \mathbb{R}$ or $\mathbb{Z}$. Clearly, any vector in $\mathbb{R}^k$ and any element in the Banach space $\bar{\mathcal{C}} := C([-\tau, 0], \mathbb{R}^k)$ can be regarded as the functions in $\mathcal{C}$.

For $u = (u^1, \cdots, u^k), v = (v^1, \cdots, v^k) \in \mathcal{C}$, we write $u \geq v (u \gg v)$ provided $u^i(\theta, x) \geq v^i(\theta, x) (u^i(\theta, x) > v^i(\theta, x)), \forall i = 1, \cdots k, \theta \in [-\tau, 0], x \in \mathcal{H}$; and $u > v$ provided $u \geq v$ but $u \neq v$. For any two vectors a, b in $\mathbb{R}^k$ or two functions $a, b \in \bar{\mathcal{C}}$, we can define $a \geq (>, \gg)\ b$ similarly. For any $r \in \bar{\mathcal{C}}$ with $r \gg 0$, we define $\mathcal{C}_r := \{u \in \mathcal{C} : r \geq u \geq 0\}$ and $\bar{\mathcal{C}}_r := \{u \in \bar{\mathcal{C}} : r \geq u \geq 0\}$.

We equip $\bar{\mathcal{C}}$ with the maximum norm topology and $\mathcal{C}$ with the compact open topology, that is, $v^n \to v$ in $\mathcal{C}$ means that the sequence of functions $v^n(\theta, x)$ converges to $v(\theta, x)$ uniformly for (θ, x) in every compact set. Moreover, we can define the metric function $d(\cdot, \cdot)$ in $\mathcal{C}$ with respect to this topology by

$$d(u, v) = \sum_{k=1}^{\infty} \frac{\max\limits_{|x|\leq k, \theta\in[-\tau,0]} |u(\theta, x) - v(\theta, x)|}{2^k}, \quad \forall u, v \in \mathcal{C}$$

such that $(\mathcal{C}, d)$ is a metric space.

Define the reflection operator $\mathcal{R}$ by $\mathcal{R}[u](\theta, x) = u(\theta, -x)$. Given $y \in \mathcal{H}$, define the translation operator T_y by $T_y[u](\theta, x) = u(\theta, x - y)$.

Let $\beta \in \bar{\mathcal{C}}$ with $\beta \gg 0$ and $Q = (Q_1, \cdots, Q_k) : \mathcal{C}_\beta \to \mathcal{C}_\beta$. Assume that

(A1) $Q[\mathcal{R}[u]] = \mathcal{R}[Q[u]]$, $T_y[Q[u]] = Q[T_y[u]]$, $\forall y \in \mathcal{H}$.

(A2) $Q : \mathcal{C}_\beta \to \mathcal{C}_\beta$ is continuous with respect to the compact open topology.

(A3) One of the following two properties holds:

(a) $\{Q[u](\cdot, x) : u \in \mathcal{C}_\beta, x \in \mathcal{H}\}$ is a family of equicontinuous functions of $\theta \in [-\tau, 0]$.

(b) There is a nonnegative number $\varsigma < \tau$ such that $Q = S + L$, where

$$S[u](\theta, x) = \begin{cases} u(0, x), -\tau \leq \theta < -\varsigma \\ Q[u](\theta, x), -\varsigma \leq \theta \leq 0, \end{cases}$$

is a continuous operator on $\mathcal{C}_\beta$ and $\{S[u](\cdot, x) : u \in \mathcal{C}_\beta, x \in \mathcal{H}\}$ is a family of equicontinuous functions of $\theta \in [-\tau, 0]$, and

$$L[u](\theta, x) = \begin{cases} u(\theta + \varsigma, x) - u(0, x), -\tau \leq \theta < -\varsigma \\ 0, -\varsigma \leq \theta \leq 0. \end{cases}$$

(A4) $Q : \mathcal{C}_\beta \to \mathcal{C}_\beta$ is monotone (order-preserving) in the sense that $Q[u] \geq Q[v]$ whenever $u \geq v$ in $\mathcal{C}_\beta$.

(A5) $Q : \bar{\mathcal{C}}_\beta \to \bar{\mathcal{C}}_\beta$ admits exactly two fixed points 0 and β, and for any positive number ϵ, there is $\alpha \in \bar{\mathcal{C}}_\beta$ with $\|\alpha\| < \epsilon$ such that $Q^n[\alpha] \to \beta$ and $Q[\alpha] \gg \alpha$.

Theorem A. ([11, Theorems 2.11 and 2.15 and Corollary 2.16]) *Suppose that Q satisfies (A1)-(A5). Let $u_0 \in \mathcal{C}_\beta$ and $u_n = Q[u_{n-1}]$ for $n \geq 1$. Then there is a real number c^* such that the following statements are valid:*

(1) For any $c > c^*$, if $0 \leq u_0 \ll \beta$ and $u_0(\cdot, x) = 0$ for x outside a bounded interval, then $\lim\limits_{n\to\infty, |x|\geq nc} u_n(\theta, x) = 0$ uniformly for $\theta \in [-\tau, 0]$.

(2) For any $c < c^*$ and any $\sigma \in \bar{\mathcal{C}}_\beta$ with $\sigma \gg 0$, there exists $r_\sigma > 0$ such that if $u_0(\cdot, x) \geq \sigma(\cdot)$ for x on an interval of length $2r_\sigma$, then $\lim\limits_{n\to\infty, |x|\leq nc} u_n(\theta, x) = \beta(\theta)$ uniformly for $\theta \in [-\tau, 0]$. If, in addition, Q is subhomogeneous on $\mathcal{C}_\beta$, then r_σ can be chosen to be independent of $\sigma \gg 0$.

Remark 2.1. *Note that the assumption (A3)(a) is equivalent to that the set* $\{Q[u](\cdot, x) : u \in \mathcal{C}_\beta, x \in \mathcal{H}\}$ *is precompact in* $\bar{\mathcal{C}}$*. In the case where* Q *has the translation invariance property in (A1), we have* $T_y[\mathcal{C}_\beta] = \mathcal{C}_\beta$ *for any* $y \in \mathcal{H}$*. It then follows that* $\{Q[u](\cdot, x) : u \in \mathcal{C}_\beta\} = \{Q[u](\cdot, 0) : u \in \mathcal{C}_\beta\}$ *for any* $x \in \mathcal{H}$*, and hence* $\{Q[u](\cdot, x) : u \in \mathcal{C}_\beta, x \in \mathcal{H}\} = \{Q[u](\cdot, 0) : u \in \mathcal{C}_\beta\}$*. Theorem A is still valid if we replace (A3)(a) with the following weaker assumption (A3)(a′):*

(a′) There is a number $l \in [0, 1)$ *such that for any* $A \subset \mathcal{C}_\beta$ *and* $x \in \mathcal{H}$*,* $\alpha(\{Q[u](\cdot, x) : u \in A\}) \leq l\alpha(\{u(\cdot, x) : u \in A\})$*, where* α *is the Kuratowski measure of noncompactness on the Banach space* $\bar{\mathcal{C}}$*.*

To prove Theorem A in this case, it suffices to show that for any $s \in \mathbb{R}$ *the set* $\{a_n(c; \cdot, s) : n \geq 0\}$*, as defined in [11], is precompact in* $\bar{\mathcal{C}}$*. This can be done easily with the use of Lemma 2.1. For some details, see Lemma 2.2 and the arguments in Remark 2.3.*

Recall that a map $Q : \mathcal{C}_\beta \to \mathcal{C}_\beta$ is said to be subhomogeneous if $Q[\rho v] \geq \rho Q[v]$ for all $\rho \in [0, 1]$ and $v \in \mathcal{C}_\beta$. We call c^* in Theorem A the **asymptotic speed of spread** (in short, **spreading speed**) of a discrete-time semiflow $\{Q^n\}_{n=0}^{\infty}$ on $\mathcal{C}_\beta$. In order to estimate the spreading speed, we introduce the following notations and assumptions.

Let $M : \mathcal{C} \to \mathcal{C}$ be a linear operator with the following properties:

(C1) M is continuous with respect to the compact open topology.

(C2) M is a positive operator, that is, $M[v] \geq 0$ whenever $v > 0$.

(C3) M satisfies (A3) with $\mathcal{C}_\beta$ replaced by any subset of $\mathcal{C}$ consisting of uniformly bounded functions.

(C4) $M[\mathcal{R}[u]] = \mathcal{R}[M[u]], T_y[M[u]] = M[T_y[u]], \forall u \in \mathcal{C}, y \in \mathcal{H}$.

(C5) M can be extended to a linear operator on the linear space $\tilde{\mathcal{C}}$ of all function $v \in C([-\tau, 0] \times \mathcal{H}, \mathbb{R}^k)$ having the form

$$v(\theta, x) = v_1(\theta, x)e^{\mu_1 x} + v_2(\theta, x)e^{\mu_2 x}, v_1, v_2 \in \mathcal{C}, \mu_1, \mu_2 \in \mathbb{R},$$

such that if $v_n, v \in \tilde{\mathcal{C}}$ and $v_n(\theta, x) \to v(\theta, x)$ uniformly on any bounded set, then $M[v_n](\theta, x) \to M[v](\theta, x)$ uniformly on any bounded set.

We note that the hypothesis (C4) implies that $M[v] \in \bar{\mathcal{C}}$ whenever $v \in \bar{\mathcal{C}}$, and hence, M is also a linear operator on $\bar{\mathcal{C}}$.

Define the linear map $B_\mu : \bar{\mathcal{C}} \to \bar{\mathcal{C}}$ by

$$B_\mu[\alpha](\theta) = M[\alpha e^{-\mu x}](\theta, 0), \forall \theta \in [-\tau, 0].$$

In particular, $B_0 = M$ on $\bar{\mathcal{C}}$. If $\alpha_n, \alpha \in \bar{\mathcal{C}}$ and $\alpha_n \to \alpha$ as $n \to \infty$, then $\alpha_n(\theta)e^{-\mu x} \to \alpha(\theta)e^{-\mu x}$ uniformly on any bounded subset of $[-\tau, 0] \times \mathcal{H}$. Thus, $B_\mu[\alpha_n] = M[\alpha_n e^{-\mu x}](\cdot, 0) \to M[\alpha e^{-\mu x}](\cdot, 0) = B_\mu[\alpha]$, and hence B_μ is continuous. Moreover, B_μ is a positive operator on $\bar{\mathcal{C}}$. We assume that

(C6) For any $\mu \geq 0$, B_μ is a positive operator, and there is n_0 such that $B_\mu^{n_0} = \underbrace{B_\mu \circ \cdots \circ B_\mu}_{n_0}$ is a compact and strongly positive linear operator on $\bar{\mathcal{C}}$.

It then follows from [11, Lemma 3.1] that B_μ has a principal eigenvalue $\lambda(\mu)$ with a strongly positive eigenfunction. The following condition is needed for the estimate of the spreading speed c^*.

(C7) The principal eigenvalue $\lambda(0)$ of B_0 is larger than 1.

Theorem B. ([11, Theorem 3.10]) *Let Q be an operator on $\mathcal{C}_\beta$ satisfying (A1)–(A5) and c^* be its asymptotic speed of spread. Assume that the linear operator M satisfies (C1)–(C7) and that either M has compact support, or the infimum of $\Phi(\mu) := \frac{1}{\mu} \ln \lambda(\mu)$ is attained at some finite value μ^* and $\Phi(+\infty) > \Phi(\mu^*)$. Then the following statements are valid:*

(1) If $Q[u] \leq M[u]$ for all $u \in \mathcal{C}_\beta$, then $c^* \leq \inf_{\mu>0} \Phi(\mu)$.

(2) If there is some $\eta \in \bar{\mathcal{C}}$ with $\eta \gg 0$ such that $Q[u] \geq M[u]$ for any $u \in \mathcal{C}_\eta$, then $c^* \geq \inf_{\mu>0} \Phi(\mu)$.

Remark 2.2. *Theorem B is still valid if we replace (C6) with the following assumption:*

(C6 ′) *For any* $\mu \geq 0$, B_μ *is a positive operator, and there exist* n_0 *and* $l \in [0, 1)$ *such that* $B_\mu^{n_0} = \underbrace{B_\mu \circ \cdots \circ B_\mu}_{n_0}$ *is a strongly positive linear operator on* $\bar{\mathcal{C}}$ *and* $\alpha(B_\mu^{n_0}(A)) \leq l\alpha(A)$ *for any bounded subset* A *of* $\bar{\mathcal{C}}$.

To prove Theorem B in this case, it suffices to show that B_μ *has a principal eigenvalue. But this can be done by the use of a generalized Krein-Rutman theorem (see [16]).*

Recall that M is said to have compact support provided there is some ρ such that for any $\alpha \in \mathcal{C}$, $M[\alpha](\theta, x)$ only depends on the value of α in $[-\tau, 0] \times [x - \rho, x + \rho]$.

For any real number c, we define the set

$$\mathcal{D}_c := \{x - mc : \, x \in \mathcal{H}, m \in \mathbb{Z}\}.$$

We say that $W(\theta, x - nc)$ is a **traveling wave** of the map Q with the wave speed c if $W : [-\tau, 0] \times \mathcal{D}_c \to \mathbb{R}^k$ and $Q^n[W](\theta, x) = W(\theta, x - nc)$. We say that $W(\theta, x - nc)$ **connects** β **to** 0 if $W(\cdot, -\infty) = \beta$ and $W(\cdot, \infty) = 0$.

Theorem C. ([11, Theorem 4.1]) *Let* Q *satisfy (A1)–(A5), and* c^* *be its asymptotic speed of spread. Then for any* $c < c^*$, Q *has no traveling wave* $W(\theta, x - nc)$ *connecting* β *to* 0.

In order to obtain the existence of the traveling wave with the wave speed $c \geq c^*$, we need to strengthen the hypothesis (A3) into the following one.

(A6) One of the following two conditions holds:

- (a) $Q[\mathcal{C}_\beta]$ is precompact in $\mathcal{C}_\beta$.
- (b) There is a nonnegative number $\varsigma < \tau$ such that $Q[u](\theta, x) = u(\theta + \varsigma, x)$ for $-\tau \leq \theta < -\varsigma$, the operator

$$S[u](\theta, x) := \begin{cases} u(0, x), -\tau \leq \theta < -\varsigma \\ Q[u](\theta, x), -\varsigma \leq \theta \leq 0, \end{cases}$$

is continuous on $\mathcal{C}_\beta$, and $S[\mathcal{C}_\beta]$ is precompact in $\mathcal{C}_\beta$.

We note that (A6) is stronger than (A3) and if $\mathcal{H}$ is discrete, then the hypothesis (A3) on Q implies the hypothesis (A6). Moreover, if (A6)(b) holds and there is an integer n such that $n\varsigma \geq \tau$, then $\{Q^n[u] : u \in \mathcal{C}_\beta\}$ is precompact in $\mathcal{C}_\beta$.

Theorem D. ([11, Theorem 4.2]) *Let Q satisfy (A1)–(A6), and c^* be its asymptotic speed of spread. Then for any $c \geq c^*$, Q has a traveling wave $W(\theta, x - nc)$ connecting β to 0 such that $W(\theta, x)$ is nonincreasing in x. Moreover, if $\mathcal{H} = \mathbb{R}$, then $W(\theta, x)$ is continuous in (θ, x).*

Given a function $\phi \in \mathcal{C}_\beta$ and a bounded interval $I = [a, b] \subset \mathcal{H}$, we define a function $\phi_I \in C([-\tau, 0] \times I, \mathbb{R}^k)$ by $\phi_I(\theta, x) = \phi(\theta, x)$. Moreover, for any subset $\mathcal{D}$ of $\mathcal{C}_\beta$, we define

$$\mathcal{D}_I := \{\phi_I \in C([-\tau, 0] \times I, \mathbb{R}^k) : \phi \in \mathcal{D}\}.$$

Remark 2.3. *Note that the assumption (A6)(a) implies that for any interval $I = [a, b]$ of the length r, the set $(Q[\mathcal{C}_\beta])_I$ is precompact in the Banach space $C([-\tau, 0] \times I, \mathbb{R}^k)$, and hence $\alpha((Q[\mathcal{C}_\beta])_I) = 0$. Theorem D is still valid if we replace (A6)(a) with the following weaker assumption (A6)(a′):*

- *(a′) For any number $r > 0$, there exists $l = l(r) \in [0, 1)$ such that for any $\mathcal{D} \subset \mathcal{C}_\beta$ and any interval $I = [a, b]$ of the length r, we have $\alpha((Q[\mathcal{D}])_I) \leq l\alpha(\mathcal{D}_I)$, where α is the Kuratowski measure of noncompactness on the Banach space $C([-\tau, 0] \times I, \mathbb{R}^k)$.*

Let ϕ and $a_n(c, \kappa; \theta, s)$ be defined as in the proof of [11, Theorem 4.2]. Let $A_0 = \mathcal{C}_\beta$ and $A_i = \overline{\bigcup_{n=1}^{\infty} R_{c,1/n}[A_{i-1}]}$ for $i \geq 1$. To prove Theorem D in this case, it then suffices to show that the sequence of functions $a(c, 1/k; \cdot)$, $k \geq 1$, has a convergent subsequence in $\mathcal{C}_\beta$. For any interval $I = [a, b]$ of the length r, we define $A_I^* := \cap_{n=1}^{\infty} \overline{(A_n)_I}$, where the closure is taken in $C([-\tau, 0] \times I, \mathbb{R}^k)$. By Lemma 2.2 below, it follows that $A_{n+1} \subset A_n$ and $\lim_{n\to\infty} \alpha((A_n)_I) = 0$. Then Lemma 2.1 implies that A_I^* is a nonempty and compact set in $C([-\tau, 0] \times I, \mathbb{R}^k)$

and that $\lim_{n\to\infty} \delta((A_n)_I, A_I^*) = 0$. *Note that* $a_n(c,1/k;\cdot) \in A_n$ *for all* $k \geq 1$ *and hence* $(a_n(c,1/k;\cdot))_I \in (A_n)_I$. *Since for each* k *and* $x \in \mathbb{R}$, $a_n(c,1/k;x)$ *converges to* $a(c,1/k;x)$, *it follows from the compactness and attractivity of* A_I^* *that* $a(c,1/k;\cdot)_I \in A_I^*$ *for all* $k \geq 1$. *Thus, the family of functions* $a(c,1/k;\cdot)$ *with parameter* $k \geq 1$ *is equicontinuous for* (θ,s) *in any bounded subset of* $[-\tau,0]\times\mathbb{R}$. *In particular, the standard diagonal method implies that there exist* $k_m \to \infty$ *such that the subsequence* $a(c,1/k_m;\cdot)$ *converges with respect to the compact open topology.*

Lemma 2.2. *Let the assumption (A6)(a′) hold, and* $\phi \in \mathcal{C}_\beta$ *be fixed. For any* $c \in \mathbb{R}$ *and* $\kappa \in (0,1]$, *define an operator* $R_{c,\kappa}$ *on* $\mathcal{C}_\beta$ *by*

$$R_{c,\kappa}[a](\theta,x) := \max\{\kappa\phi(\theta,s), T_{-c}[Q[a]](\theta,x)\}.$$

Let $A_0 = \mathcal{C}_\beta$ *and* $A_i = \overline{\bigcup_{n=1}^{\infty} R_{c,1/n}[A_{i-1}]}$ *for* $i \geq 1$. *Then* $A_i \subset A_j$ *for any* $i > j$, *and* $\alpha((A_i)_I) \leq l(r)^i \alpha((A_0)_I)$ *for any interval* $I = [a,b]$ *of the length* r *and* $i \geq 1$.

Proof. The conclusion $A_i \subset A_j$ for $i > j$ follows easily from the induction argument. Let $l = l(r)$ and $m = \alpha((A_0)_I)$. Since $\lim_{n\to\infty} \phi_n = \phi$ in $\mathcal{C}_\beta$ implies $\lim_{n\to\infty} (\phi_n)_I = \phi_I$ with respect to the maximum norm, we have $(A_i)_I \subset \overline{\bigcup_{n=N}^{\infty} (R_{c,1/n}[A_i])_I}$.

Assume, by induction, that the conclusion holds for i, that is, $\alpha((A_i)_I) \leq l^i m$ for any interval I of length r. Now, we consider $i+1$. First, by our assumption, $\alpha((Q[A_i])_{I+c}) \leq l^{i+1}m$, where $I + c = \{x \in \mathbb{R}, x - c \in I\}$. This implies $\alpha((T_{-c}[Q[A_i]])_I) \leq l^{i+1}m$. Since

$$\begin{aligned}(R_{c,1/n}[A_i])_I &= \left\{\max\left(\frac{\phi_I}{n}, f_I\right) : f_I \in (T_{-c}[Q[A_i]])_I\right\}\\ &= \left\{\frac{\frac{\phi_I}{n} + f_I}{2} + \frac{|\frac{\phi_I}{n} - f_I|}{2} : f_I \in (T_{-c}[Q[A_i]])_I\right\},\end{aligned}$$

we have $\alpha(R_{c,1/n}[A_i])_I \leq l^{i+1}m$.

By the discussion above, we can suppose that for any $\epsilon > 0$, $(T_{-c}Q[A_i])_I$ is covered by a finite number of sets with diameter less than $l^{i+1}m + \epsilon$. Denote these sets by $B_1, \cdots, B_p$. Moreover, there is some N such that $\|\phi_I\| \leq N l^{i+1} m$, that is,

$$\max\{\phi(\theta, x)/N : x \in I, \theta \in [-\tau, 0]\} \leq l^{i+1} m.$$

We claim that the diameter of the set $\bar{B}_i = \bigcup_{n=N}^{\infty} \{u = \max(v, \phi_I/n), v \in B_i\}$ is also less than $l^{i+1}m + \epsilon$, and hence $\bigcup_{n=N}^{\infty} (R_{c,1/n}[A_i])_I \subset \bigcup_{i=1}^{p} \bar{B}_i$. Moreover, $\bigcup_{n=1}^{\infty} (R_{c,1/n}[A_i])_I$ is covered by a finite number of sets with diameter less than $l^{i+1}m + \epsilon$. Since ϵ is arbitrary, we have

$$\alpha \left(\overline{\bigcup_{n=N}^{\infty} (R_{c,1/n}[A_i])_I} \right) = \alpha \left(\bigcup_{n=N}^{\infty} (R_{c,1/n}[A_i])_I \right) \leq l^{i+1} m$$

and hence, our lemma holds. It remains to prove our claim. For any $u_1, u_2 \in \bar{B}_i$, there is some $v_1, v_2 \in B_i$ and $n_1, n_2 \geq N$ such that $u_j = \max(v_j, \phi_I/n_j), j = 1, 2$. Then $|v_1(\theta, x) - v_2(\theta, x)| \leq l^{i+1}m, \theta \in [-\tau, 0], x \in I$. For any $\theta \in [-\tau, 0], x \in I$, one of the following three cases holds:

(1) $v_1(\theta, x) \geq \max(\phi_I(\theta, x)/n_1, v_2(\theta, x))$.

(2) $v_2(\theta, x) \geq \max(\phi_I(\theta, x)/n_1, v_1(\theta, x))$.

(3) $\phi_I(\theta, x)/n_1 \geq \max(v_1(\theta, x), v_2(\theta, x))$.

For case (1), $u_1(\theta, x) = v_1(\theta, x)$ and hence $|u_1(\theta, x) - v_2(\theta, x)| = |v_1(\theta, x) - v_2(\theta, x)| \leq l^{i+1}m$. For case (2), $|v_2(\theta, x) - u_1(\theta, x)| \leq |v_2(\theta, x) - v_2(\theta, x)| \leq l^{i+1}m$. For case (3), $|v_2(\theta, x) - u_1(\theta, x)| \leq |\phi_I(x)|/n_1$, and if $n \geq N$, then $|v_2(\theta, x) - u_1(\theta, x)| \leq l^{i+1}m$.

Furthermore, we also have one of the following three cases:

(a) $v_2(\theta, x) \geq \max(\phi_I(\theta, x)/n_2, u_1(\theta, x))$.

(b) $u_1(\theta, x) \geq \max(\phi_I(\theta, x)/n_2, v_2(\theta, x))$.

(c) $\phi_I(\theta, x)/n_2 \geq \max(u_1(\theta, x), v_2(\theta, x))$.

By similar arguments as above, we obtain $|u_2(\theta, x) - u_1(\theta, x)| \leq l^{i+1}m$. Thus, our claim holds. □

Let $\omega > 0$ and $r \in \bar{\mathcal{C}}$ with $r \gg 0$ be given. A family of mappings $\{Q_t\}_{t=0}^{\infty}$ is said to be an **ω-periodic semiflow** on $\mathcal{C}_r$ provided Q_t has the following properties:

(i) $Q_0[v] = v, \forall v \in \mathcal{C}_r$.

(ii) $Q_{t+\omega}[v] = Q_t[Q_\omega[v]], \forall t \geq 0, v \in \mathcal{C}_r$.

(iii) $Q(t, v) := Q_t(v)$ is continuous in (t, v) on $[0, \infty) \times \mathcal{C}_r$.

The mapping Q_ω is called the Poincaré map associated with this periodic semiflow.

It is easy to see that the property (iii) holds if $Q(\cdot, v)$ is continuous on $[0, +\infty)$ for each $v \in \mathcal{C}_r$, and $Q(t, \cdot)$ is continuous uniformly for t in bounded intervals in the sense that for any $v_0 \in \mathcal{C}_r$, bounded interval I and $\epsilon > 0$, there exists $\delta = \delta(v_0, I, \epsilon) > 0$ such that if $d(v, v_0) < \delta$, then $d(Q_t[v], Q_t[v_0]) < \epsilon$ for all $t \in I$.

Theorem 2.1. *Let $\{Q_t\}_{t=0}^{\infty}$ be an ω-periodic semiflow on $\mathcal{C}_r$ with two x-independent ω-periodic orbits $0 \ll \beta(t)$. Suppose that the Poincaré map $Q = Q_\omega$ satisfies all hypotheses (A1)–(A5) with $\beta = \beta(0)$, and Q_t satisfies (A1) for any $t > 0$. Let c^* be the asymptotic speed of spread for Q_ω. Then the following statements are valid:*

(1) For any $c > c^*/\omega$, if $v \in \mathcal{C}_\beta$ with $0 \leq v \ll \beta$, and $v(\cdot, x) = 0$ for x outside a bounded interval, then $\lim_{t\to\infty, |x|\geq tc} Q_t[v](\theta, x) = 0$ uniformly for $\theta \in [-\tau, 0]$.

(2) For any $c < c^*/\omega$ and $\sigma \in \bar{\mathcal{C}}_\beta$ with $\sigma \gg 0$, there is a positive number r_σ such that if $v \in \mathcal{C}_\beta$ and $v(\cdot, x) \gg \sigma(\cdot)$ for x on an interval of length $2r_\sigma$, then $\lim_{t\to\infty, |x|\leq tc} (Q_t[v](\theta, x) - \beta(t)(\theta)) = 0$ uniformly for $\theta \in [-\tau, 0]$. If, in addition, Q_ω is subhomogeneous, then r_σ can be chosen to be independent of $\sigma \gg 0$.

Proof. First, it is easy to see that for any $v_n \to 0$, $Q_t[v_n] \to 0$ uniformly for $t \in [0, \omega]$. In other words, for any $\epsilon > 0$ and any bounded interval I, there exist $\delta > 0$ and a sufficiently large positive number r such that if $v(\theta, x) < \delta$ for $x \in [-r, r], \theta \in [-\tau, 0]$, then $|Q_t[v](\theta, x)| < \epsilon$ for any $x \in I, \theta \in [-\tau, 0]$, and $t \in [0, \omega]$. In particular, since Q_t satisfies (A1), for any $\epsilon > 0$ we can find a sufficiently large positive number r such that for any $x_0 \in \mathbb{R}$, if $v(\theta, x) < \delta$ for $x \in [-r + x_0, r + x_0], \theta \in [-\tau, 0]$, then $|Q_t[v](\theta, x_0)| < \epsilon$ for any $\theta \in [-\tau, 0]$ and $t \in [0, \omega]$.

By Theorem A, it follows that for any $v \in \mathcal{C}_\beta$ with $0 \leq v \ll \beta$ and $v = 0$ outside a bounded subset of $[-\tau, 0] \times \mathbb{R}$ and any $c > c^*/\omega$, we have

$$\lim_{n \to \infty, |x| \geq n\omega c} Q_{n\omega}[v](\theta, x) = 0$$

uniformly for $\theta \in [-\tau, 0]$. Hence, for the positive number δ fixed above, we can find an integer N such that if $n \geq N$, then $|Q_{n\omega}[v](\theta, x)| < \delta$ for any $\theta \in [-\tau, 0]$ and $|x| \geq n\omega c$. Therefore, $|Q_t[v](\theta, x)| < \epsilon$ for any $n \geq N, t \in [n\omega, (n + 1)\omega]$ and $\theta \in [-\tau, 0], |x| \geq n\omega c + r$. For any $\rho > 0$, there is an integer N' such that if $n \geq N'$ and $t \in [n\omega, (n+1)\omega]$, then $t(c+\rho) > n\omega c+r$. Thus, $|Q_t[v](\theta, x)| < \epsilon$ for any $t \geq \max(N, N') \cdot \omega$ and $|x| \geq t(c + \rho)$. Since $c > c^*/\omega, \rho > 0$ are arbitrary, the conclusion (1) holds. The conclusion (2) can be proved in a similar way. □

We say that $W(\theta, t, x - ct)$ is a **periodic traveling wave** of the ω-periodic semiflow $\{Q_t\}_{t=0}^{\infty}$ if the vector-valued function $W(\theta, t, z)$ is ω-periodic in t and $Q_t[W(\cdot, 0, \cdot)](\theta, x) = W(\theta, t, x - ct)$, and that $W(\theta, t, x - ct)$ **connects** $\beta(t)$ **to** 0 if $W(\cdot, t, -\infty) = \beta(t)$ and $W(\cdot, t, +\infty) = 0$.

Theorem 2.2. *Suppose that $Q = Q_\omega$ satisfies the hypotheses (A1)–(A5) with $\beta = \beta(0)$, and let c^* be the asymptotic speed of spread of Q_ω. Then for any $0 < c < c^*/\omega$, $\{Q_t\}_{t=0}^{\infty}$ has no ω-periodic traveling wave $W(\theta, t, x - ct)$ connecting $\beta(t)$ to 0.*

Proof. If the periodic semiflow Q_t has a periodic traveling wave $W(\theta, t, x - ct)$, then $W(\theta, 0, x - c\omega n)$ is a traveling wave for Q_ω. Thus, Theorem C implies that Q_t admits no periodic traveling wave. □

Theorem 2.3. *Suppose that $\mathcal{H} = \mathbb{R}$ and Q_ω satisfies hypothesis (A1)–(A6) with $\beta = \beta(0)$, and let c^* be the asymptotic speed of spread of Q_ω. Moreover, assume that Q_t satisfies (A1) and (A4) for each $t > 0$. Then for any $c \geq c^*/\omega$, $\{Q_t\}_{t=0}^{\infty}$ has an ω-periodic traveling wave $U(\theta, t, x - ct)$ connecting $\beta(t)$ to 0 such that $U(\theta, t, s)$ is continuous, and nonincreasing in $s \in \mathbb{R}$.*

Proof. Given ω-periodic semiflow Q_t, $t \geq 0$, we define $P_t = T_{-ct}Q_t, t \geq 0$. Then we have

(1) $P_0[v] = T_0Q_0[v] = v$ for any $v \in \mathcal{C}_\beta$.

(2) $P_{t+\omega} = T_{-c(t+\omega)}Q_{t+\omega} = T_{-ct}T_{-c\omega}Q_tQ_\omega = T_{-ct}Q_tT_{-c\omega}Q_\omega = P_tP_\omega$ since (A1) holds for any $t \geq 0$.

(3) $P(t, v) = P_t(v) = T_{-ct}[Q_t[v]]$ is continuous in (t, v).

Thus, $P_t, t \geq 0$, is an ω-periodic semiflow on $\mathcal{C}_\beta$. Since $c\omega \geq c^*$, Theorem D implies that Q_ω admits a traveling wave $W(\theta, x - c\omega n)$, that is, $Q_\omega^n[W] = W(\theta, x-c\omega n), \forall n \geq 0$. Then $Q_\omega[W] = T_{c\omega}[W]$, that is, $P_\omega[W] = T_{-c\omega}Q_\omega[W] = W$. Thus, W is a fixed point of the Poincaré map, P_ω, of the periodic semiflow P_t.

It follows that $P_t[W]$ is an ω-periodic orbit of P_t, that is, $P_{t+\omega}[W] = P_t[W]$. Let $U(\theta, t, x) := P_t[W](\theta, x), \forall t \geq 0$. Clearly, $U(\theta, t, x)$ is continuous in (θ, x). We then have $Q_t[W](\theta, x) = T_{ct}P_t[W](\theta, x) = U(\theta, t, x - ct)$. Since $W(\theta, x)$ is nonincreasing in x and connects β to zero, and Q_t satisfies (A1) and (A4), it follows that $U(\theta, t, x)$ is nonincreasing in x and connects $\beta(t)$ to 0. □

Remark 2.4. *The above theorems are still valid provided that the interval $[-\tau, 0]$ is replaced with a compact metric space and that the hypotheses (A3) and (A6) are replaced with (A3)(a′) and (A6)(a′), respectively.*

3 A periodic epidemic model

We consider the following reaction-diffusion system modeling man-environment-man epidemics (see, e.g., [5])

$$\begin{cases} \frac{\partial u_1(x,t)}{\partial t} = d\frac{\partial^2 u_1(x,t)}{\partial x^2} - a_{11}u_1(x,t) + a_{12}u_2(x,t) \\ \frac{\partial u_2(x,t)}{\partial t} = -a_{22}u_2(x,t) + g(t, u_1(x,t)) \end{cases} \tag{3.1}$$

where d, a_{11}, a_{12} and a_{22} are positive constants, $u_1(x,t)$ denotes the spatial density of infectious agent at a point x in the habitat at time $t \geq 0$, and $u_2(x,t)$ denotes the spatial density of the infective human population at time t, $1/a_{11}$ is the mean lifetime of the agent in the environment, $1/a_{22}$ is the mean infectious period of the human infectives, a_{12} is the multiplicative factor of the infectious agent due to the human population, and $g(t,z)$ is the force of infection on the human population due to a concentration z of the infectious agent. In view of seasonal variations, we assume that $g(t+\omega, z) = g(t,z)$ for some $\omega > 0$. Note that system (3.1) models random dispersal of the pollutant while ignoring the small mobility of the infective human population. Mathematically it suffices to study the following dimensionless system

$$\begin{cases} \frac{\partial u_1(x,t)}{\partial t} = d\frac{\partial^2 u_1(x,t)}{\partial x^2} - u_1(x,t) + \alpha u_2(x,t) \\ \frac{\partial u_2(x,t)}{\partial t} = -\beta u_2(x,t) + g(t, u_1(x,t)) \end{cases} \tag{3.2}$$

with

$$\alpha = \frac{a_{12}}{a_{11}^2}, \qquad \beta = \frac{a_{22}}{a_{11}}.$$

Motivated by the biological interpretation of g, we assume that

(G1) $g \in C^1(\mathbb{R}_+^2, \mathbb{R}_+)$, $g(\cdot, 0) \equiv 0$, and $\frac{\partial g(t,z)}{\partial z} > 0$, $\forall (t,z) \in \mathbb{R}_+^2$.

Let ρ be the principal Floquet multiplier of the linear periodic cooperative and irreducible system

$$\begin{cases} u_1' = -u_1 + \alpha u_2 \\ u_2' = -\beta u_2 + \partial_z g(t,0) u_1, \end{cases} \tag{3.3}$$

that is, ρ is the principal eigenvalue of the strongly positive matrix $U(\omega)$, where $U(t)$ is the fundamental matrix solution of (3.3) with $U(0) = I$. In order to get a monostable case, we further make the following assumptions on $g(t, z)$:

(G2) $\rho > 1$, and $\frac{\bar{g}(\bar{z})}{\bar{z}} \leq \frac{\beta}{\alpha}$ for some $\bar{z} > 0$, where $\bar{g}(z) = \max_{t \in [0,\omega]} g(t, z)$.

(G3) For each $t \geq 0$, $g(t, \cdot)$ is strictly subhomogeneous on $\mathbb{R}_+$ in the sense that $g(t, sz) > sg(t, z)$, $\forall z > 0, s \in (0, 1)$.

It is easy to see that $u(t) = \bar{u} := (\bar{z}, \bar{z}/\alpha)$ is an upper solution of the periodic cooperative and irreducible system

$$\begin{cases} u_1' = -u_1 + \alpha u_2 \\ u_2' = -\beta u_2 + g(t, u_1). \end{cases} \tag{3.4}$$

By [28, Theorem 2.3.4], as applied to the Poincaré map associated with (3.4) on the order interval $[0, \bar{u}] \subset \mathbb{R}_+^2$ (see also [28, Theorem 3.1.2]), it follows that (3.4) has a positive ω-periodic solution $u^*(t)$, which is globally asymptotically stable in $[0, \bar{u}] \setminus 0$.

Let $\mathcal{C}$ be defined as in section 2 with $\tau = 0$, $\mathcal{H} = \mathbb{R}$ and $k = 2$, that is, $\mathcal{C}$ is the space of all bounded and continuous functions from $\mathbb{R}$ to $\mathbb{R}^2$ with the compact open topology. Let $T_1(t)$ be the semigroup generated by

$$\frac{\partial u_1(t, x)}{\partial t} = d\Delta u_1(t, x) - u_1(t, x),$$

and $T_2(t)\phi_2 = e^{-\beta t}\phi_2$. Then $T(t) = (T_1(t), T_2(t))$ is a linear semigroup on $\mathcal{C}$. Note that the reaction system (3.4) is cooperative. Using the standard linear semigroup theory (see, e.g., [17, 15]), we see that for any $\phi \in \mathcal{C}_{\bar{u}}$, (3.2) has a unique solution $u(t, \phi)$ with $u(0, \phi) = \phi$, which exists globally on $[0, +\infty)$. Define $Q_t(\phi) = u(t, \phi)$. It then follows that Q_t is a monotone periodic semiflow on $\mathcal{C}_{\bar{u}}$ (see, e.g., [15, Corollary 5] and the proof of [22, Theorem 2.2]). By the comparison method (or the integral form of (3.2)), we can further show that for each $t \geq 0$, Q_t is subhomogeneous on $\mathcal{C}_{\bar{u}}$. Let $\hat{Q}_t$ be the restriction of Q_t to $[0, \bar{u}]$. It is easy to see that $\hat{Q}_t$ is the periodic semiflow on $[0, \bar{u}]$ generated by the periodic cooperative and irreducible system (3.4). Thus, for each $t > 0$,

$\hat{Q}_t$ is strongly monotone on $[0, \bar{u}]$. As mentioned before, (3.4) has a positive ω-periodic solution $u^*(t)$, which is globally asymptotically stable in $[0, \bar{u}] \setminus 0$. By the Dancer-Hess connecting orbit lemma (see, e.g., [9, Proposition 2.1]), the map $\hat{Q}_\omega$ admits a strongly monotone full orbit connecting 0 to $u^* := u^*(0)$. Thus, assumption (A5) holds for the map Q_ω. The following result shows that Q_ω satisfies assumption (A6)(a$'$) on $\mathcal{C}_{u^*}$.

Lemma 3.1. *For any $\mathcal{D} \subset \mathcal{C}_{\bar{u}}$ and any bounded interval $I = [a, b]$, we have $\alpha((Q_t\mathcal{D})_I) \leq e^{-\beta t}\alpha(\mathcal{D}_I)$.*

Proof. Define a linear operator

$$S(t)\phi = (0, T_2(t)\phi_2), \quad \forall \phi = (\phi_1, \phi_2) \in \mathcal{C},$$

and a nonlinear map

$$U(t)\phi = \left(u_1(t, \cdot, \phi), \int_0^t T_2(t-s)g(s, u_1(s, \cdot, \phi))ds\right), \quad \forall \phi = (\phi_1, \phi_2) \in \mathcal{C}_{\bar{u}}.$$

It is easy to see that

$$Q_t\phi = S(t)\phi + U(t)\phi, \quad \forall \phi \in \mathcal{C}_{\bar{u}},\ t \geq 0.$$

Let $\|\cdot\|_I$ be the maximum norm associated with the Banach space $C(I, \mathbb{R}^2)$. Since $\|(S(t)\phi)_I\|_I \leq e^{-\beta t}\|\phi_I\|_I$, we have $\alpha((S(t)\mathcal{D})_I) \leq e^{-\beta t}\alpha(\mathcal{D}_I)$. Note that

$$u_1(t, \cdot, \phi) = T_1(t)\phi_1 + \alpha \int_0^t T_1(t-s)u_2(s, \cdot, \phi)ds.$$

Since for each $t > 0$, $T_1(t)$ is a compact map with respect to the compact open topology, so is $U(t) : \mathcal{C}_{\bar{u}} \to \mathcal{C}_{\bar{u}}$. This implies that $(U(t)\mathcal{D})_I$ is precompact in $C(I, \mathbb{R}^2)$, and hence $\alpha((U(t)\mathcal{D})_I) = 0$. Thus, we have

$$\alpha((Q_t\mathcal{D})_I) \leq \alpha((S(t)\mathcal{D})_I) + \alpha((U(t)\mathcal{D})_I) \leq e^{-\beta t}\alpha(\mathcal{D}_I), \quad \forall t > 0.$$

This completes the proof. □

Since Q_ω satisfies (A1)-(A6) with $\tau = 0$ and $\beta = u^*$, Theorem A implies that Q_ω admits a spreading speed c^*. Next we use Theorem B to obtain an explicit expression for c^*.

Let $\rho(\mu)$ be the principal Floquet multiplier of the linear periodic cooperative and irreducible system

$$\begin{cases} u_1' = (d\mu^2 - 1)u_1 + \alpha u_2 \\ u_2' = -\beta u_2 + \partial_z g(t,0)u_1. \end{cases} \tag{3.5}$$

Let $v(t,w)$ be the solution of (3.5) satisfying $v(0,w) = w \in \mathbb{R}^2$. It is easy to see that $u(t,x) = e^{-\mu x}v(t,w)$ is the solution of the linear periodic system

$$\begin{cases} \frac{\partial u_1(x,t)}{\partial t} = d\frac{\partial^2 u_1(x,t)}{\partial x^2} - u_1(x,t) + \alpha u_2(x,t) \\ \frac{\partial u_2(x,t)}{\partial t} = -\beta u_2(x,t) + \partial_z g(t,0)u_1(x,t). \end{cases} \tag{3.6}$$

Let M_t be the solution map associated with (3.6), and B_μ^t be defined by M_t as in section 2. By above observation, it is easy to see that B_μ^t is just the solution map of the linear ordinary differential equation (3.5) on $\mathbb{R}^2$. It follows that $\rho(\mu)$ is the principal eigenvalue of B_μ^ω.

Define $\Phi(\mu) := \frac{\ln\rho(\mu)}{\mu}$. In order to use Theorem B, we show that $\Phi(\infty) = \infty$. Let $\lambda(\mu) = \frac{1}{\omega}\ln\rho(\mu)$. By the Floquet theory, there exists a positive ω-periodic function $w(t)$ such that $v(t) = e^{\lambda(\mu)t}w(t)$ is a solution of (3.5). Since $v_1'(t) \geq (d\mu^2 - 1)v_1(t)$, it follows that

$$\frac{w_1'(t)}{w_1(t)} \geq d\mu^2 - 1 - \lambda(\mu),\ \forall t \geq 0,$$

and hence

$$0 = \int_0^\omega \frac{w_1'(t)}{w_1(t)}dt \geq (d\mu^2 - 1 - \lambda(\mu))\omega.$$

Thus, we have

$$\Phi(\mu) = \frac{\ln\rho(\mu)}{\mu} = \frac{\omega\lambda(\mu)}{\mu} \geq \omega d\mu - \frac{\omega}{\mu},$$

which implies that $\Phi(\infty) = \infty$.

Since $g(t,\cdot)$ is subhomogeneous on $\mathbb{R}_+$, we have

$$g(t,z) \leq \partial_z g(t,0)z, \quad \forall (t,z) \in \mathbb{R}_+^2.$$

Then the comparison principle implies that $Q_\omega\phi \leq M_\omega\phi$ for all $\phi \in \mathcal{C}_{u^*}$. Thus, Theorem B (1) implies that $c^* \leq \inf_{\mu>0}\frac{\ln\rho(\mu)}{\mu}$.

For any $\epsilon \in (0,1)$, there exists $z_\epsilon > 0$ such that

$$g(t,z) \geq (1-\epsilon)\partial_z g(t,0)z, \quad \forall (t,z) \in \mathbb{R}_+ \times [0, z_\epsilon].$$

Let $\rho^\epsilon(\mu)$ be the principal Floquet multiplier of the linear periodic cooperative and irreducible system

$$\begin{cases} u_1' = (d\mu^2 - 1)u_1 + \alpha u_2 \\ u_2' = -\beta u_2 + (1-\epsilon)\partial_z g(t,0)u_1, \end{cases} \tag{3.7}$$

and let M_t^ϵ be the solution map associated with the linear periodic system

$$\begin{cases} \frac{\partial u_1(x,t)}{\partial t} = d\frac{\partial^2 u_1(x,t)}{\partial x^2} - u_1(x,t) + \alpha u_2(x,t) \\ \\ \frac{\partial u_2(x,t)}{\partial t} = -\beta u_2(x,t) + (1-\epsilon)\partial_z g(t,0)u_1(x,t). \end{cases} \tag{3.8}$$

By the comparison principle, there exists $\eta \gg 0$ in $\mathbb{R}^2$ such that for any $\phi \in \mathcal{C}_\eta$,

$$Q_t(\phi)(x) \leq u(t,\eta) \leq (z_\epsilon, z_\epsilon), \ \forall x \in \mathbb{R}, \ t \in [0,\omega],$$

where $u(t,\eta)$ is the solution of (3.4) with $u(0,\eta) = \eta$. Thus, we have $Q_t(\phi) \geq M_t^\epsilon(\phi)$, $\forall \phi \in \mathcal{C}_\eta$, $t \in [0,\omega]$. By an analysis similar to that of (3.6), it follows from Theorem B (2) that $c^* \geq \inf_{\mu>0} \frac{\ln \rho^\epsilon(\mu)}{\mu}$, and hence, letting $\epsilon \to 0$, we obtain $c^* \geq \inf_{\mu>0} \frac{\ln \rho(\mu)}{\mu}$. Consequently, $c^* = \inf_{\mu>0} \frac{\ln \rho(\mu)}{\mu}$.

Note that if $u(t,x)$ is a solution of (3.2) with $0 \leq u(0,x) < u^*, \forall x \in \mathbb{R}$, and $u(0,\cdot) \not\equiv 0$, then $u(t,x) > 0, \forall t > 0, x \in \mathbb{R}$ (see, e.g., the proof of [26, Lemma 3.1]).

As the consequences of Theorems 2.1, 2.2 and 2.3 with Remark 2.3, we have the following results.

Theorem 3.1. *Assume that (G1)–(G3) hold. Let* $c^* = \inf_{\mu>0} \frac{\ln \rho(\mu)}{\mu}$ *and* $u(t,x,\phi) = u(t,\phi)(x)$, $\phi \in \mathcal{C}_{u^*}$. *Then the following two statements are valid:*

(1) If $\phi(x) = 0$ *for* x *outside a bounded interval, then for any* $c > c^*/\omega$, $\lim_{t\to\infty, |x|\geq tc} u(t,x,\phi) = 0$.

(2) If $\phi(x) \not\equiv 0$*, then for any* $c < c^*/\omega$, $\lim_{t\to\infty, |x|\leq tc} (u(t,x,\phi) - u^*(t)) = 0$.

Theorem 3.2. *Assume that (G1)–(G3) hold, and let c^* be defined as in Theorem 3.1. Then for any $c \geq c^*/\omega$, (3.2) has a periodic traveling wave solution $U(t, x-tc)$ such that $U(t,s)$ is nonincreasing in $s \in \mathbb{R}$, and $\lim_{s\to-\infty} U(t,s) = u^*(t)$ and $\lim_{s\to\infty} U(t,s) = 0$. Moreover, for any $c < c^*/\omega$, (3.2) has no traveling wave $U(t, x-tc)$ connecting $u^*(t)$ to 0.*

We note that the autonomous version of system (3.2) was studied earlier in [29] for traveling waves and in [23] for spreading speeds and traveling waves.

4 A periodic delayed and diffusive equation

Let $\tau > 0$ be fixed and $\bar{\mathcal{C}} := C([-\tau, 0], \mathbb{R})$. For any $u \in C([-\tau, \sigma) \times \mathbb{R}, \mathbb{R})$ with $\sigma > 0$, and any $(t, x) \in [0, \sigma) \times \in \mathbb{R}$, we use $u_t(\cdot, x)$ to denote the member of $\bar{\mathcal{C}}$ defined by

$$u_t(\theta, x) = u(t + \theta, x), \ \ \forall \theta \in [-\tau, 0].$$

Consider a periodic delay differential equation with diffusion on $\mathbb{R}$:

$$\frac{\partial u(t,x)}{\partial t} = d\frac{\partial^2 u(t,x)}{\partial x^2} + f(t, u(t,x), u(t-\tau, x)), \quad t > 0,\, x \in \mathbb{R}, \tag{4.1}$$

where $d > 0$, $f \in C^1(\mathbb{R}^3_+, \mathbb{R})$, and $f(t,u,v)$ is ω-periodic in t for some $\omega > 0$. We need the following assumption on f to study the spreading speed and periodic traveling waves for (4.1).

(F) $f(\cdot, 0, 0) \equiv 0$, $\frac{\partial f(t,u,v)}{\partial v} > 0$, $\forall (t,u,v) \in \mathbb{R}^3_+$, and there is a real number $L > 0$ such that $f(t, L, L) \leq 0$ and for each $t \geq 0$, $f(t, \cdot, \cdot)$ is strictly subhomogeneous on $[0, L]^2$ in the sense that $f(t, \alpha u, \alpha v) > \alpha f(t, u, v)$ whenever $\alpha \in (0,1)$, $0 < u$, $v \leq L$.

Define $\tilde{f} : \mathbb{R}_+ \times \bar{\mathcal{C}} \to \mathbb{R}$ by

$$\tilde{f}(t, \phi) = f(t, \phi(0), \phi(-\tau)), \quad \forall (t, \phi) \in \mathbb{R}_+ \times \bar{\mathcal{C}}.$$

Then it is easy to see that for each $t \geq 0$, $\tilde{f}(t, \cdot)$ is quasimonotone on $\bar{\mathcal{C}}$ in the sense that $\tilde{f}(t, \phi) \leq \tilde{f}(t, \psi)$ whenever $\phi \leq \psi$ in $\bar{\mathcal{C}}$ and $\phi(0) = \psi(0)$.

Let $\mathcal{C}$ be defined as in section 2 with $\mathcal{H} = \mathbb{R}$ and $k = 1$, that is, $\mathcal{C}$ is the space of all bounded and continuous functions from $[-\tau, 0] \times \mathbb{R}$ to $\mathbb{R}$ with the compact open topology. Using the semigroup generated by the heat equation and [15, Corollary 5] (see, e.g., the proof of [22, Theorem 2.2]), we can show that (4.1) generates a monotone periodic semiflow $Q_t : \mathcal{C}_L \to \mathcal{C}_L$ defined by

$$Q_t(\phi)(\theta, x) = u_t(\theta, x, \phi), \ \forall (\theta, x) \in [-\tau, 0] \times \mathbb{R}, \ \phi \in \mathcal{C}_L,$$

where $u(t, x, \phi)$ is the unique solution of (4.1) satisfying $u_0(\cdot, \cdot, \phi) = \phi \in \mathcal{C}_L$.

Let $\hat{Q}_t$ be the restriction of Q_t to $\bar{\mathcal{C}}_L$. It is easy to see that $\hat{Q}_t : \bar{\mathcal{C}}_L \to \bar{\mathcal{C}}_L$ is the periodic semiflow generated by the following periodic delay differential equation

$$\frac{du(t)}{dt} = f(t, u(t), u(t-\tau)), \ t \geq 0, \tag{4.2}$$

with initial date $u_0 = \phi \in \bar{\mathcal{C}}_L$. By the nonautonomous version of [21, Theorem 5.3.4], it follows that the map $\hat{Q}_t$ is strongly monotone for $t \geq 2\tau$. Let r_0 be the spectral radius of the Poincaré map associated with the linear periodic delay differential equation

$$\frac{du(t)}{dt} = f'_u(t, 0, 0)u(t) + f'_v(t, 0, 0)u(t-\tau), \ t \geq 0. \tag{4.3}$$

Assume that $r_0 > 1$. Then [27, Theorem 2.1] implies that system (4.2) has a positive ω-periodic solution $\beta(t)$, which is globally asymptotically stable in $\bar{\mathcal{C}}_L \setminus \{0\}$. By the Dancer-Hess connecting orbit lemma (see, e.g., [9, Proposition 2.1]), the map $\hat{Q}_\omega$ admits a strongly monotone full orbit connecting 0 to $\beta := \beta(0)$. Thus, assumption (A5) holds for the map Q_ω.

Define the linear operator $L(t) : \mathcal{C} \to \mathcal{C}$, $t \geq 0$, by the relation

$$L(t)\phi(\theta, x) = \begin{cases} \phi(t+\theta, x) - \phi(0, x), & t + \theta < 0, x \in \mathbb{R} \\ 0, & t + \theta \geq 0, \ -\tau \leq \theta \leq 0, x \in \mathbb{R}. \end{cases}$$

Clearly, $L(t) = 0$ for $t \geq \tau$. Define $S(t) := Q_t - L(t)$, $t \geq 0$. By the smoothing property of the semigroup associated with the heat equation, it then follows that Q_t satisfies (A6)(a) for $t \geq \tau$, and (A6)(b) with $\varsigma = t$ for $t \in (0, \tau)$ (see also the proof of [8, Theorem 6.1]). Now it is easy to see that the map Q_ω satisfies all assumptions (A1)–(A6).

Let $r(\mu)$ be the spectral radius of the Poincaré map associated with the following linear periodic delay differential equation

$$\frac{dv(t)}{dt} = d\mu^2 v(t) + f'_u(t,0,0)v(t) + f'_v(t,0,0)v(t-\tau). \tag{4.4}$$

Then [27, Theorem 2.1] implies that $r(\mu) > 0$. Furthermore, we have the following result on the spreading speed c^* of the Poincaré map Q_ω associated with (4.1).

Lemma 4.1. *Let c^* be the asymptotic speed of spread of the map Q_ω. Then $c^* = \inf_{\mu>0} \frac{\ln r(\mu)}{\mu}$.*

Proof. Since $f(t,\cdot)$ is subhomogeneous on $[0,L]^2$, it follows from [28, Lemma 2.3.2] that

$$f(t,u,v) \le f'(t,0,0)u + f'_v(t,0,0)v, \quad \forall (u,v) \in [0,L]^2.$$

We fix a positive number α such that $\alpha + f'_u(t,0,0) > 0$, $\forall t \in [0,\omega]$. Let $\bar{f}(t,u,v) := \alpha u + f(t,u,v)$. Then $\bar{f}'_u(t,0,0) > 0$, $\bar{f}'_v(t,0,0) > 0$, $\forall t \in [0,\omega]$. It is easy to see that for any $\epsilon \in (0,1)$, there exists $\delta = \delta(\epsilon) \in (0,L)$ such that

$$\bar{f}(t,u,v) \ge (1-\epsilon)\bar{f}'_u(t,0,0)u + (1-\epsilon)\bar{f}'_v(t,0,0)v, \quad \forall (u,v) \in [0,\delta]^2.$$

Since $f(t,u,v) = -\alpha u + \bar{f}(t,u,v)$, we further have

$$f(t,u,v) \ge [(1-\epsilon)f'(t,0,0) - \epsilon\alpha]u + (1-\epsilon)f'_v(t,0,0)v, \quad \forall (u,v) \in [0,\delta]^2.$$

Let $v(t,\phi)$ be the solution of the linear periodic equation (4.4) satisfying $v_0 = \phi \in \bar{\mathcal{C}}$. It is easy to see that $u(t,x) = e^{-\mu x}v(t,\phi)$ is the solution of the linear periodic delay differential equation with diffusion

$$\frac{\partial u}{\partial t} = d\frac{\partial^2 u(t,x)}{\partial x^2} + f'_u(t,0,0)u(t,x) + f'_v(t,0,0)u(t-\tau,x). \tag{4.5}$$

Let M_t be the solution map associated with (4.5), and B^t_μ be defined by M_t as in section 2. By above observation, it is easy to see that B^t_μ is just the solution map of the linear functional differential equation (4.4) on $\bar{\mathcal{C}}$.

By the proof of [27, Theorem 2.1], it follows that there exists a positive ω-periodic function $w(t)$ such that $v(t) = e^{\lambda(\mu)t}w(t)$ is a solution of (4.4), where $\lambda(\mu) = \frac{1}{\omega}\ln r(\mu)$. Define $\psi \in \bar{\mathcal{C}}$ by $\psi(\theta) = e^{\lambda(\mu)\theta}w(\theta)$, $\forall \theta \in [-\tau, 0]$. Clearly, $v(t,\psi) = e^{\lambda(\mu)t}w(t)$, $\forall t \geq 0$. Then we have

$$B_\mu^t(\psi)(\theta) = v(t+\theta,\psi) = e^{\lambda(\mu)t}e^{\lambda(\mu)\theta)}w(t+\theta), \quad \forall \theta \in [-\tau,0],\, t \geq 0.$$

By the ω-periodicity of $w(t)$, it follows that

$$B_\mu^\omega(\psi)(\theta) = e^{\lambda(\mu)\omega}e^{\lambda(\mu)\theta}w(\theta) = e^{\lambda(\mu)\omega}\psi(\theta), \quad \forall \theta \in [-\tau,0],$$

that is, $B_\mu^\omega(\psi) = e^{\lambda(\mu)\omega}\psi$. This implies that $e^{\lambda(\mu)\omega}$ is the principal eigenvalue of B_μ^ω with positive eigenfunction ψ. Then we have

$$\Phi(\mu) := \frac{1}{\mu}\ln\left(e^{\lambda(\mu)\omega}\right) = \frac{\lambda(\mu)\omega}{\mu} = \frac{\ln r(\mu)}{\mu}.$$

By a similar argument as in section 3, we can show that $\Phi(\infty) = \infty$. Thus, Theorem B (1) implies that $c^* \leq \inf_{\mu>0}\frac{\ln r(\mu)}{\mu}$.

For any $\epsilon \in (0,1)$, let $r^\epsilon(\mu)$ be the spectral radius of the Poincaré map associated with the linear periodic delay differential equation

$$\frac{dv(t)}{dt} = d\mu^2 v(t) + [(1-\epsilon)f'(t,0,0) - \epsilon\alpha]v(t) + (1-\epsilon)f'_v(t,0,0)v(t-\tau).$$

By an analysis similar to that of (4.4), it follows from Theorem B (2) that $c^* \geq \inf_{\mu>0}\frac{\ln r^\epsilon(\mu)}{\mu}$, and hence, letting $\epsilon \to 0$, we have $c^* \geq \inf_{\mu>0}\frac{\ln r(\mu)}{\mu}$. Consequently, $c^* = \inf_{\mu>0}\frac{\ln r(\mu)}{\mu}$. □

By Theorems 2.1, 2.2 and 2.3, we then have the following result.

Theorem 4.1. *Assume that (F) holds and $r_0 > 1$. Let c^* be defined as in Lemma 4.1. Then the following statements are valid:*

(1) For any $c > c^*/\omega$, if $\phi \in \mathcal{C}_\beta$ with $0 \leq \phi \ll \beta$, and $\phi(\cdot,x) = 0$ for x outside a bounded interval, then $\lim_{t\to\infty,|x|\geq tc} u(t,x,\phi) = 0$.

(2) For any $c < c^*$, if $\phi \in \mathcal{C}_\beta$ with $\phi \not\equiv 0$, then $\lim_{t\to\infty,|x|\leq tc}(u(t,x,\phi) - \beta(t)) = 0$.

Theorem 4.2. *Assume that (F) holds and $r_0 > 1$. Let c^* be defined as in Lemma 4.1. Then for any $c \geq c^*/\omega$, (4.1) has a periodic traveling wave solution $U(t, x-ct)$ connecting $\beta(t)$ to 0 such that $U(t,s)$ is continuous and nonincreasing in $s \in \mathbb{R}$. Moreover, for any $c < c^*/\omega$, (4.1) has no traveling wave $U(t, x-ct)$ connecting $\beta(t)$ to 0.*

5 A reaction-diffusion equation in a cylinder

We consider the ω-periodic reaction-diffusion equation in a cylinder

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \Delta_y u + ug(t,y,u), \ x \in \mathbb{R}, y = (y_1 \cdots, y_m) \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = 0 \ on\, \mathbb{R} \times \partial\Omega \times (0, +\infty), \end{cases} \tag{5.1}$$

where Ω is a bounded domain in $\mathbb{R}^m$ with smooth boundary $\partial\Omega$, $\Delta_y = \sum_{i=1}^{m} \frac{\partial^2}{\partial y_i^2}$, and ν is the outer unit normal vector to $\partial\Omega \times \mathbb{R}$. Assume that

(G) $g \in C^1(\mathbb{R}_+ \times \overline{\Omega} \times \mathbb{R}_+, \mathbb{R})$, ω-periodic in t, $\frac{\partial g}{\partial u} < 0, \forall (t,y,u) \in \mathbb{R}_+ \times \overline{\Omega} \times \mathbb{R}_+$, and there is $K > 0$ such that $g(t,y,K) \leq 0, \forall (t,y) \in \mathbb{R}_+ \times \overline{\Omega}$.

Let μ_0 be the principal eigenvalue of the periodic-parabolic eigenvalue problem

$$\begin{cases} \frac{\partial v}{\partial t} = \Delta_y v + vg(t,y,0) + \mu v, \ y \in \Omega, \\ \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega, \\ v \quad \omega - \text{periodic in } t \end{cases} \tag{5.2}$$

with a positive eigenfunction $\varphi(t,y)$, ω-periodic in t (see [9, Section II.14]). Assume that $\mu_0 < 0$. By [28, Theorem 3.1.5], it then follows that the reaction-diffusion equation

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta_y u + ug(t,y,u), \ y \in \Omega, t > 0, \\ \frac{\partial v}{\partial \nu} = 0 \ on\ \partial\Omega \times (0, +\infty), \end{cases} \tag{5.3}$$

admits a unique positive periodic solution $\beta(t,y)$, which is globally asymptotically stable in $C(\overline{\Omega}, \mathbb{R}_+) \setminus \{0\}$. Moreover, the Dancer-Hess connecting orbit lemma (see, e.g., [9, Proposition 2.1]) implies that the Poincaré map associated with (5.3) admits a strongly monotone full orbit connecting 0 to $\beta := \beta(0, \cdot)$.

Let $\mathcal{C}$ be the set of all bounded and continuous functions from $\mathbb{R} \times \overline{\Omega}$ to $\mathbb{R}$. We consider the linear equation

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \Delta_y u, \ x \in \mathbb{R}, y \in \Omega, t > 0, \\ \frac{\partial v}{\partial \nu} = 0 \ on \ \mathbb{R} \times \partial\Omega \times (0, +\infty). \end{cases} \tag{5.4}$$

Let $G(t, y, w)$ be the Green function of the equation (see, e.g., [7])

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta_y u, \ y \in \Omega, t > 0, \\ \frac{\partial v}{\partial \nu} = 0 \ on \ \partial\Omega \times (0, +\infty). \end{cases} \tag{5.5}$$

Then it is easy to verify that $e^{-\frac{(x-z)^2}{4\pi t}} G(t, y, w)$ is the Green function of equation (5.4), that is, the solution of (5.4) with initial value $u(0, \cdot) = \phi(\cdot) \in \mathcal{C}$ can be expressed as

$$u(t, x, y, \phi) \quad = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} \int_{\Omega} e^{-\frac{(x-z)^2}{4\pi t}} G(t, y, w) \phi(z, w) dwdz.$$

Define $T(t)\phi = u(t, \cdot, \phi)$, $\forall \phi \in \mathcal{C}$. It then follows that $\{T(t)\}_{t=0}^{\infty}$ is a linear semigroup on the space $\mathcal{C}$ with respect to the compact open topology. For any $a, b \in \mathcal{C}$, define $[a, b]_{\mathcal{C}} := \{\phi \in \mathcal{C}, a \leq \phi \leq b\}$. For any $t > 0$ and $a, b \in \mathcal{C}$, it is easy to verify that $T(t)[a, b]_{\mathcal{C}}$ is a family of equicontinuous functions.

Now we write (5.1) subject to $u(0, \cdot) = \phi \in \mathcal{C}$ as an integral equation

$$u(t, x, y) = T(t)[\phi](x, y) + \int_0^t T(s) f(t-s, y, u(t-s, x, y)) ds, \tag{5.6}$$

where $f(t, y, u) = ug(t, y, u)$. Using the standard linear semigroup theory (see, e.g., [17, 15]), we see that for any $\phi \in \mathcal{C}_K$, (5.1) has a unique solution $u(t, \phi)$ with $u(0, \phi) = \phi$, which exists globally on $[0, +\infty)$. Define $Q_t(\phi) = u(t, \phi)$. With the expression of the semigroup $T(t)$ and (5.6), we can show that $\{Q_t\}_{t=0}^{\infty}$ is a subhomogeneous ω-periodic semiflow on $\mathcal{C}_K$. Moreover, for each $t > 0$, Q_t satisfies hypotheses (A1),(A2), (A3)(a), (A4), (A5) and (A6)(a) with $[-\tau, 0]$ replaced by $\overline{\Omega}$.

Let $\{M_t\}_{t=0}^{\infty}$ be the periodic semiflow associated with the linear equation

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \Delta_y u + ug(t, y, 0), \ x \in \mathbb{R}, y \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = 0 \ on \ \mathbb{R} \times \partial\Omega \times (0, +\infty). \end{cases} \tag{5.7}$$

Since $g(t, y, 0) \geq g(t, y, u)$, we have $M_t[\phi] \geq Q_t[\phi]$ for any $\phi \in \mathcal{C}_\beta$. Let M_t^ϵ be the periodic semiflow associated with the linear equation

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \Delta_y u + (1-\epsilon) u g(t, y, 0),\ x \in \mathbb{R}, y \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = 0 \ on\ \mathbb{R} \times \partial\Omega \times (0, +\infty). \end{cases} \tag{5.8}$$

Then for any ϵ, there is a $\delta \gg 0$ such that $M_t^\epsilon[\phi] \leq Q_t[\phi]$ for any $\phi \in \mathcal{C}_\delta$ and $t \in [0, \omega]$.

Let $\rho \in \mathbb{R}$ be a parameter. It is easy to see that if $\eta(t, y)$ is a solution of the linear equation

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta_y u + u g(y, 0) + \rho^2 u,\ y \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = 0 \ on\ \partial\Omega \times (0, +\infty), \end{cases} \tag{5.9}$$

then $u(t, x, y) = \eta(t, y) e^{-\rho x}$ is a solution of (5.7). Define

$$B_\rho^t[\alpha](y) = M_t[\alpha(y) e^{-\rho x}](0, y), \quad \forall \alpha \in \bar{\mathcal{C}} := C(\overline{\Omega}, \mathbb{R}),\ y \in \overline{\Omega}.$$

It follows that B_μ^t is the solution map associated with (5.9). Let $\mu_\rho := \mu_0 - \rho^2$. It is easy to verify that $e^{-\mu_\rho t} \varphi(t, y)$ is a solution of (5.9). Thus, we have

$$B_\rho^t[\varphi(0, \cdot)] = e^{-\mu_\rho t} \varphi(t, \cdot), \quad \forall t \geq 0,$$

and hence

$$B_\rho^\omega[\varphi(0, \cdot)] = e^{-\mu_\rho \omega} \varphi(\omega, \cdot) = e^{-\mu_\rho \omega} \varphi(0, \cdot).$$

This implies that $e^{-\mu_\rho \omega}$ is the principal eigenvalue of B_ρ^ω with positive eigenfunction $\varphi(0, \cdot)$. Define

$$\Phi(\rho) := \frac{\ln e^{-\mu_\rho \omega}}{\rho} = \left(\rho - \frac{\mu_0}{\rho}\right) \omega.$$

Clearly, $\Phi(\infty) = \infty$. Let c^* be the spreading speed of Q_ω. Note that that M_ω satisfies (C1)-(C7). By similar arguments as in sections 3 and 4, it follows from Theorem B that $c^* = \inf_{\rho > 0} \Phi(\rho) == 2\omega\sqrt{-\mu_0}$.

Note that if $u(t, x, y)$ is a solution of (5.1) with $0 \leq u(0, x, y) < \beta(y), \forall y \in \Omega, x \in \mathbb{R}$, and $u(0, x, y) \not\equiv 0$, then $u(t, x, y) > 0, \forall t > 0, y \in \Omega, x \in \mathbb{R}$ (see, e.g., the proof of [26, Lemma 3.1]).

As the consequences of Theorems 2.1, 2.2 and 2.3 with Remark 2.4, we have the following results.

Theorem 5.1. *Assume that (G) holds and $\mu_0 < 0$. Let $u(t,x,y)$ be a solution of (5.1) with $u(0,\cdot) \in \mathcal{C}_\beta$. Then the following two statements are valid:*

(1) If $u(0,x,y) = 0$ for $y \in \Omega$ and x outside a bounded interval, then for any $c > 2\sqrt{-\mu_0}$, $\lim_{t\to\infty, |x|\geq tc} u(t,x,y) = 0$ uniformly for $y \in \Omega$.

(2) If $u(0,x,y) \not\equiv 0$, then for any $c < 2\sqrt{-\mu_0}$, $\lim_{t\to\infty, |x|\leq tc} (u(t,x,y) - \beta(t,y)) = 0$ uniformly for $y \in \Omega$.

Theorem 5.2. *Assume that (G) holds and $\mu_0 < 0$. For any $c \geq 2\sqrt{-\mu_0}$, (5.1) has a periodic traveling wave solution $U(t, x - tc, y)$ such that $U(t,s,y)$ is nonincreasing in $s \in \mathbb{R}$, and $\lim_{s\to-\infty} U(t,s,y) = \beta(t,y)$ and $\lim_{s\to\infty} U(t,s,y) = 0$ uniformly for $y \in \Omega$. Moreover, for any $c < 2\sqrt{-\mu_0}$, (5.1) has no traveling wave $U(t, x - tc, y)$ connecting $\beta(t,\cdot)$ to 0.*

We should mention that some autonomous parabolic equations in cylinders were studied earlier in [4, 13] for traveling waves and in [11] for spreading speeds and traveling waves.

Acknowledgment: X.-Q. Zhao would like to thank Georgia Institute of Technology for its kind hospitality during his leave there.

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