DEGENERATE LOWER DIMENSIONAL TORI IN HAMILTONIAN SYSTEMS

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Abstract. We study the persistence of lower dimensional tori in Hamiltonian systems of the form $H(x,y,z) = \langle \omega, y\rangle + \frac{1}{2}\langle z, M(\omega)z\rangle + \varepsilon P(x,y,z,\omega)$, where $(x,y,z) \in T^n \times R^n \times R^{2m}$, ε is a small parameter, and $M(\omega)$ can be singular. We show under a weak Melnikov non-resonant condition and certain singularity-removing conditions on the perturbation that the majority of unperturbed n-tori can still survive from the small perturbation. As an application, we will consider persistence of invariant tori on certain resonant surfaces of a nearly integrable, properly degenerate Hamiltonian system for which neither the Kolmogorov nor the g-non-degenerate condition is satisfied.

1. Introduction and Main Results

We consider the Melnikov persistence problem of lower dimensional, possibly degenerate, invariant tori for Hamiltonian of the form

$$H = e(\omega) + \langle \omega, y\rangle + \frac{1}{2}\langle z, M(\omega)z\rangle + \varepsilon P(x,y,z,\omega,\varepsilon), \tag{1.1}$$

where $(x,y,z) \in T^n \times R^n \times R^{2m}$, ω is a parameter in a bounded closed region $\mathcal{O} \subset R^n$, $\varepsilon \in (0,1)$ is a small parameter, M is a real analytic, matrix-valued function on some complex neighborhood $\mathcal{O}(r) = \{\omega : \ |\mathrm{Im}\ \omega| < r\}$ of $\mathcal{O}$ taking values in the space of $2m \times 2m$ symmetric matrices, and P is real analytic in a complex neighborhood $D(r,s) \times \mathcal{O}(r) \times \Delta$ of $T^n \times \{0\} \times \{0\} \times \mathcal{O} \times (0,1)$ for some $D(r,s) = \{(x,y,z) : \ |\mathrm{Im}\ x| < r, |y| < s^2, |z| < s\}$. The Hamiltonian H is associated with the standard symplectic form

$$\sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i + \sum_{j=1}^{m} \mathrm{d}z_j \wedge \mathrm{d}z_{m+j}. \tag{1.2}$$

Clearly, the unperturbed system associated to (1.1) admits a family of invariant n-tori $T_\omega = T^n \times \{0\} \times \{0\}$ with linear flows which are parameterized by the toral frequency $\omega \in \mathcal{O}$.

The Melnikov persistence problem, initiated by Melnikov in [24, 25], concerns the persistence of the majority of the unperturbed n-tori T_ω under certain coupling non-resonance conditions, called Melnikov conditions, between the tangential frequencies ω and the normal ones associated to the eigenvalues of $M(\omega)$.

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Such persistence problem has been extensively studied in various non-degenerate cases (i.e. M is non-singular over $\mathcal{O}$), and also in infinite dimensional setting (see [3, 4, 5, 6, 7, 9, 11, 12, 14, 15, 16, 17, 19, 21, 28, 29, 30, 31, 33, 35, 36] and references therein).

A similar persistence problem was posted by Kuksin in [18] for the degenerate case when $M(\omega)$ becomes singular. The problem was studied in [21] under tangential non-degeneracy, i.e., the quadratic term in (1.1) has the form

$$\frac{1}{2}\langle \begin{pmatrix} y \\ z \end{pmatrix}, \mathcal{M}(\omega)\begin{pmatrix} y \\ z \end{pmatrix}\rangle$$

and $\mathcal{M}(\omega)$ is non-singular over $\mathcal{O}$. The aim of this paper is to consider the degenerate case without assuming tangential non-degeneracy. More precisely, we will study the Hamiltonian (1.1) and show that some non-degenerate conditions on the perturbation can remove the singularity and hence yield the persistence of the majority of invariant, quasi-periodic n-tori under a suitable non-resonance condition of Melnikov type.

For simplicity, we will use the same symbol $|\cdot|$ to denote an equivalent vector norm (and its induced matrix norm) in an Eucleadian space, absolute value of numbers, Lebesgue measure of sets, and l^1 norm of integer-valued vectors. Also, $|\cdot|_D$ will be used to denote the sup-norm of a function on a domain D.

Let $\lambda_1(\omega), \cdots, \lambda_{2m}(\omega)$ be eigenvalues of $JM(\omega)$, where J denotes the standard $2m \times 2m$ symplectic matrix. We assume the following conditions for (1.1):

H1) The set

$$\{\omega \in \mathcal{O} : \sqrt{-1}\langle k, \omega\rangle - \lambda_i(\omega) - \lambda_j(\omega) \neq 0, \ \forall k \in Z^n \backslash \{0\}, 1 \le i, j \le 2m\}$$

admits full Lebesgue measure relative to $\mathcal{O}$.

H2) There exists a real analytic family $z_\varepsilon : \mathcal{O}(r) \to D(s) = \{z : \ |z| < s\}$ such that

$$M(\omega)z_\varepsilon(\omega) + \varepsilon\partial_z[P](0, z_\varepsilon(\omega), \omega, 0) = 0,$$

for all $\omega \in \mathcal{O}(r)$, where $[P](y, z, \omega) = \int_{T^n} P(x, y, z, \omega, 0)dx$.

H3) There exists a constant $N_1 > 0$ such that the minimum $\lambda^\varepsilon_{\min}(\omega)$ among the absolute values of all eigenvalues of $M_\varepsilon(\omega) = M(\omega) + \varepsilon\partial_z^2[P](0, z_\varepsilon(\omega), \omega)$ satisfies $\lambda^\varepsilon_{\min}(\omega) > N_1\varepsilon$ for all $\omega \in \mathcal{O}(r)$.

Our main result states as the following.

Theorem 1. *Assume* H1) - H3). *Then there is an $\varepsilon_0 > 0$ and Cantor sets $\mathcal{O}_\varepsilon \subset \mathcal{O}$, $0 < \varepsilon < \varepsilon_0$, with $|\mathcal{O} \setminus \mathcal{O}_\varepsilon| \to 0$ as $\varepsilon \to 0$ such that for each $0 < \varepsilon < \varepsilon_0$ the Hamiltonian system* (1.1) *admits a Whitney smooth family of real analytic, quasi-periodic n-tori T^ε_ω, $\omega \in \mathcal{O}_\varepsilon$, which also varies smoothly in ε.*

We note that if $M(\omega)$ is non-singular over $\mathcal{O}$, then conditions H2) H3) are automatically satisfied. In the case that $M(\omega)$ becomes singular, invariant n-tori can be destroyed if the condition H2) fails. For example, it is easy to see that the Hamiltonian

$$H(x, y, u, v) = \langle \omega, y\rangle \pm \frac{1}{2}u^2 \pm \varepsilon v, \ \ (x, y, u, v) \in T^n \times R^n \times R^1 \times R^1$$

admits no invariant n-tori for any $\varepsilon > 0$ and H2) is not satisfied for this Hamiltonian. The condition H3) is of course not optimal for the persistence of invariant n-tori of Hamiltonian (1.1). In general, it should be possible to replace H3) by a weaker non-degenerate condition. This is certainly an interesting problem worthy for a further study.

The condition H1) is stronger than the first Melnikov non-resonance condition but is weaker than the second Melnikov non-resonance condition by allowing multiple normal eigenvalues of $JM(\omega)$. This condition was first introduced in [36] and has been employed in various studies on the persistence of lower dimensional tori in Hamiltonian systems (see [9, 21]).

Theorem 1 has no restriction on the invariant tori type, i.e., the perturbed tori can be normally hyperbolic, elliptic or of mixed type. However, unlike the non-degenerate cases considered in [21, 36], an unperturbed, persisted torus of (1.1) can change its type after perturbation in the case of normal degeneracy. Consider the following two Hamiltonians:

$$\begin{aligned} H_1 &= \langle \omega, y\rangle + u^2 + \varepsilon u - \varepsilon v^2 + \varepsilon \bar{P}_1(x,y,u,v), \\ &\equiv \langle \omega, y\rangle + \varepsilon P_1(x,y,z) \\ H_2 &= \langle \omega, y\rangle + \varepsilon u + \varepsilon v + \varepsilon u^2 + \varepsilon v^2 + \varepsilon \bar{P}_2(x,y,u,v), \\ &\equiv \langle \omega, y\rangle + \varepsilon P_2(x,y,z) \end{aligned}$$

where x, y, ω are as in (1.1), $z = (u, v) \in R^2$, and $[\bar{P}_i] = 0$, $i = 1, 2$. Clearly, $M = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ for H_1 and $M = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ for H_2. Hence the unperturbed n-tori in both cases are of degenerate elliptic types.

Since M are constant matrices in both cases, H1) is satisfied for both H_1 and H_2. Moreover, it is easy to see that $z_\varepsilon = (-\frac{\varepsilon}{2}, 0)$ for H_1 and $z_\varepsilon = (-\frac{1}{2}, -\frac{1}{2})$ for H_2, i.e., H2) is satisfied in both cases. Since M_ε equals

$$\begin{pmatrix} 2 & 0 \\ 0 & -2\varepsilon \end{pmatrix}$$

and

$$\begin{pmatrix} 2\varepsilon & 0 \\ 0 & 2\varepsilon \end{pmatrix}$$

for H_1 and H_2 respectively, and $\lambda^\varepsilon_{\min} = 2\varepsilon$ in both cases, H3) is also satisfied for both H_1 and H_2. Hence Theorem 1 is applicable to both H_1 and H_2 to yield the persistence of two respective families of invariant, quasi-periodic n-tori.

However, for H_1,

$$JM_\varepsilon = \begin{pmatrix} 0 & -2\varepsilon \\ -2 & 0 \end{pmatrix}$$

has eigenvalues $\lambda_\pm = \pm 2\sqrt{\varepsilon}$, and, for H_2,

$$JM_\varepsilon = \begin{pmatrix} 0 & 2\varepsilon \\ -2\varepsilon & 0 \end{pmatrix}$$

has eigenvalues $\lambda_\pm = \pm 2\sqrt{-1}\varepsilon$. Thus the perturbed n-tori are all (non-degenerate) hyperbolic for H_1 and are all (non-degenerate) elliptic for H_2.

Normal degeneracy naturally occurs in a nearly integrable, properly degenerate Hamiltonian system. As an application of Theorem 1, we consider the following properly degenerate Hamiltonian

$$H(I,\theta,\varepsilon) = H_{00}(I_1,\cdots,I_r) + \varepsilon P(\theta,I,\varepsilon) \tag{1.3}$$

associated to the symplectic form

$$\sum_{i=1}^{d} \mathrm{d}\theta_i \wedge \mathrm{d}I_i,$$

where $(I,\theta) = (I_1,\cdots,I_d,\theta_1,\cdots,\theta_d) \in G\times T^d$, $G \subset R^d$ is a bounded closed region, $r < d$, H_{00}, P are real analytic, and H_{00} satisfies the Kolmogorov non-degenerate condition on $\tilde{G} = \{(I_1,\cdots,I_r):\ I \in G\}$, i.e.,

H3) the Hessian $(\frac{\partial^2 H_{00}}{\partial I_i \partial I_j})_{i,j=1}^r$ is non-singular on $\tilde{G}$.

The unperturbed system associated to (1.3) admits a family of invariant, resonant d-tori T_I parametrized by $I \in G$. Under the condition H3) and certain condition on the perturbation which removes the degeneracy of the unperturbed Hamiltonian, it was shown by Arnold ([2]) that there is a large subset of the phase space which is filled by invariant, quasi-periodic d-tori of the perturbed system exhibiting both fast and slow oscillations. However, if the perturbation fails to completely remove the degeneracy of the unperturbed Hamiltonian, then in general the unperturbed d-tori are expected to break up but some non-degenerate frictions or sub-tori of them can persist under certain Poincaré non-degenerate conditions on the perturbation. An extreme case is when r-dimensional sub-tori are considered. Let $y = (I_1,\cdots,I_r)$, $u = (I_{r+1},\cdots,I_d)$, $\phi = (\theta_1,\cdots,\theta_r)$, $\psi = (\theta_{r+1},\cdots,\theta_d)$, $z = (u,\psi)$, and $[P](y,z) = \int_{T^r} P(\phi,\psi,y,u,0)d\phi$ in (1.3). We assume the following Poincaré non-degenerate condition that

H4) $[P](y,\cdot)$ has a real analytic family of non-degenerate critical points, i.e., there exists a real analytic function $z_*: \tilde{G} \to R^{2m}$, where $m = d-r$, such that $\partial_z[P](y,z_*(y)) = 0$, $\det\partial_z^2[P](y,z_*(y)) \neq 0$, $y \in \tilde{G}$.

Now, for each $I = (y,u) \in \tilde{G}$, the unperturbed, resonant d-torus T_I is foliated into invariant r-tori $T_I^\psi = T^r \times \{\psi\}$ with frequencies $\omega_0(y) = \partial_y H_{00}(y)$, parameterized by $\psi \in T^m$.

The following result is a corollary of Theorem 1.

Theorem 2. *Assume* H3) *and* H4). *Then there is an* $\varepsilon_0 > 0$ *and Cantor sets* $\tilde{G}_\varepsilon \subset \tilde{G}$, $0 < \varepsilon < \varepsilon_0$, *with* $|\tilde{G} \setminus \tilde{G}_\varepsilon| \to 0$ *as* $\varepsilon \to 0$ *such that for each* $0 < \varepsilon < \varepsilon_0$ *the Hamiltonian system* (1.3) *admits a Whitney smooth family of real analytic, quasi-periodic* r*-tori* T_y^ε, $y \in \tilde{G}_\varepsilon$, *which also varies smoothly in* ε.

The persistence of sub-tori split from resonant tori of a nearly integrable Hamiltonian system has been studied in [10, 22, 20, 32] on any g-resonant surface under Poincaré non-degenerate conditions of the perturbation and Kolmogorov or g-non-degenerate condition of the unperturbed Hamiltonian. For the properly degenerate, nearly integrable Hamiltonian (1.3), Theorem 2 gives a result along the same line when neither the Kolmogorov nor the g-non-degenerate condition of the unperturbed Hamiltonian is satisfied. We note in the present case that the resonance group g is simply $\{0\} \times Z^m$, where 0 is the zero vector in Z^r, and $\tilde{G}$ is the r-dimensional g-resonant surface.

To prove Theorem 1, we will first reduce the Hamiltonian system (1.1) to the following normal form:

$$(1.4)\quad H(x,y,z) = e_\delta(\omega) + \langle \Omega_\delta(\omega), y\rangle + \frac{1}{2}\langle z, M_\delta(\omega) z\rangle + \delta P(x,y,z,\omega,\delta)$$

associated to the symplectic form (1.2), where $(x,y,z) \in T^n \times R^n \times R^{2m}$ and $\omega \in \mathcal{O}$ and $\delta \in [0,1)$ are parameters with $\mathcal{O} \subset R^n$ being a bounded closed region, $\Omega_\delta = id + O(\delta)$, and $M_\delta(\omega)$ is a $2m \times 2m$ symmetric matrix for each δ and ω. Moreover, for some complex neighborhoods Δ of $[0,1)$, $\mathcal{O}(r) = \{\omega : \ |\mathrm{Im}\ \omega| < r\}$ of $\mathcal{O}$, $D(r,s) = \{(x,y,z) : \ |\mathrm{Im}\ x| < r, |y| < s^2, |z| < s\}$ of $T^n \times \{0\} \times \{0\} \subset T^n \times R^n \times R^{2m}$, e, Ω, M are real analytic on $\Delta \times \mathcal{O}(r)$, and P is real analytic on $D(r,s) \times \mathcal{O}(r) \times \Delta$. We assume the following condition:

H5) There is a constant $\sigma > 0$ such that

$$\inf_{0<\delta<1} |\det \frac{1}{\delta} M_\delta| \geq \sigma > 0.$$

Clearly, when $P = 0$, the unperturbed system of (1.4) admits a family of invariant n-tori $T_\omega = T^n \times \{0\} \times \{0\}$ parametrized by the toral frequency $\omega \in \mathcal{O}$.

We will prove the following result from which Theorem 1 follows.

Theorem 3. *Assume* H5) *and that* H1) *holds for eigenvalues of* $JM_0(\omega)$. *Then there are* $\mu = \mu(r,s) > 0$, $\delta > 0$, $\gamma > 0$ *sufficiently small such that if*

$$(1.5)\qquad |P|_{D(r,s)\times\mathcal{O}(r)} < \gamma^{4m^2} s^2 \mu,$$

then there exists a Cantor set $\mathcal{O}_* \subset \mathcal{O}$, *with* $|\mathcal{O} \setminus \mathcal{O}_*| \to 0$ *as* $\gamma, \delta \to 0$, *for which the following holds. For each* δ *and* $\omega \in \mathcal{O}_*$, *the unperturbed torus* T_ω *persists and gives rise to a slightly deformed, analytic, quasi-periodic, invariant torus of the perturbed system* (1.4), *and moreover, these perturbed tori form a Whitney smooth family.*

The rest of paper is organized as follows. Section 2 is devoted to the proof of Theorem 3 via KAM method, in which we will give details for one KAM step, prove an iteration lemma, show convergence of KAM iterations, and conduct measure estimate. We will prove Theorems 1 in Section 3 by making a normal form reduction to (1.1) in order to remove the singularity of M and to improve the order of perturbation. Theorem 2 will also be proved in this section as a corollary of Theorem 1.

2. Proof of Theorem 3

We will prove Theorem 3 in this section by using KAM method, i.e., we will construct a symplectic transformation, consisting of infinitely many successive steps, called KAM steps, of iterations, so that the x-dependent terms are pushed into higher order perturbations after each step.

Initially, we set $e^0 = e_\delta$, $\Omega^0 = \Omega_\delta$, $M^0 = M_\delta$, $P_0 = P$, $\mathcal{O}_0 = \mathcal{O}$, $r_0 = r$, $s_0 = s$, $\mu_* = \mu$, $\gamma_0 = \gamma$, and

$$\begin{aligned} N_0 &= e^0 + \langle \Omega^0(\omega), y\rangle + \frac{1}{2}\langle z, M^0(\omega) z\rangle \\ H_0 &= N_0 + \delta P_0. \end{aligned}$$

For simplicity, we suspend the dependence of all quantities on δ in the rest of the section.

By (1.5) and Cauchy's estimate, we have that

$$|\partial_\omega^l P_0|_{D_0\times\mathcal{O}_0} < c_0\gamma_0^{4m^2} s_0^2\mu_*, \ |l| \le 4m^2$$

for some constant $c_0 > 0$ only depending on r_0. Let $\mu_0 = c_0\mu_*$. Then

$$|\partial_\omega^l P_0|_{D_0\times\mathcal{O}_0} < \gamma_0^{4m^2} s_0^2\mu_0, \ |l| \le 4m^2.$$

Suppose that after a νth KAM step, we arrive at a real analytic, parameter-dependent Hamiltonian

$$\begin{aligned} H &= H_\nu = N + \delta P, \\ N &= N_\nu = e(\omega) + \langle \Omega(\omega), y\rangle + \frac{1}{2}\langle z, M(\omega) z\rangle, \end{aligned} \tag{2.1}$$

where $(x, y, z) \in D = D_\nu = D(r, s)$, $r = r_\nu \le r_0, s = s_\nu \le s_0$, $\omega \in \mathcal{O} = \mathcal{O}_\nu \subset \mathcal{O}_0$, $e(\omega) = e^\nu(\omega), \Omega(\omega) = \Omega^\nu(\omega), M(\omega) = M^\nu(\omega), P = P_\nu(x, y, z, \omega)$ are real analytic in all their arguments and also depend on $\delta \in [0, 1)$ analytically, and moreover,

$$|\partial_\omega^l P|_{D\times\mathcal{O}} < \gamma^{4m^2} s^2\mu, \ |l| \le 4m^2,$$

for some $0 < \gamma = \gamma_\nu \le \gamma_0$, $0 < \mu = \mu_\nu \le \mu_0$.

We will construct a symplectic transformation $\Phi_+ = \Phi_{\nu+1}$, which, in smaller frequency and phase domains, carries the above Hamiltonian into the next KAM cycle. Thereafter, quantities (domains, normal form, perturbation, etc.) in the next KAM cycle will be simply indexed by $+$ ($=\ \nu+1$). Also, all constants $c_1 - c_9$ below are positive and independent of the iteration process.

2.1. **One step of KAM iteration.** Below, we will show detailed constructions of the KAM iteration for the Hamiltonian (2.1).

First, we expand the perturbation P into Taylor-Fourier series

$$P = \sum_{k\in Z^n, i\in Z_+^n, j\in Z_+^m} P_{kij} y^i z^j e^{\sqrt{-1}\langle k, x\rangle}$$

and let

$$R = \sum_{|k|\le K_+} (P_{k00} + \langle P_{k10}, y\rangle + \langle P_{k01}, z\rangle + \langle z, P_{k02} z\rangle) e^{\sqrt{-1}\langle k, x\rangle}, \tag{2.2}$$

$$I = \sum_{|k|>K_+} \sum_{i,j} P_{kij} y^i z^j e^{\sqrt{-1}\langle k, x\rangle}, \tag{2.3}$$

$$II = \sum_{|k|\le K_+} \sum_{2|i|+|j|\ge 3} P_{kij} y^i z^j e^{\sqrt{-1}\langle k, x\rangle}, \tag{2.4}$$

where

$$K_+ = \left(\left[\log\frac{1}{\mu}\right] + 1\right)^3.$$

Then

$$P - R = I + II.$$

Let

$$r_+ = \frac{r}{2} + \frac{r_0}{4}.$$

We now estimate $\partial_\omega^l(P-R)$, $\omega\in\mathcal{O}$, $|l|\le 4m^2$, on a smaller complex domain $D(r_*,\alpha s)$, where $\alpha=\mu^{1/3}$ and

$$r_*=r_++\frac{3}{4}(r-r_+).$$

For each $\omega\in\mathcal{O}$, $|l|\le 4m^2$, since

$$|\sum_{i\in Z_+^n,j\in Z_+^m}\partial_\omega^l P_{kij}y^iz^j|\le|\partial_\omega^l P|_D e^{-|k|r}$$

for all $|y|\le s^2, |z|\le s$, we have

$$\begin{aligned}|\partial_\omega^l I|_{D(r_*,s)} &\le \sum_{|k|>K_+}|\partial_\omega^l P|_{D(r,s)}e^{-\frac{|k|(r-r_+)}{4}}\\ &\le \gamma^{4m^2}s^2\mu\sum_{l>K_+}l^ne^{-\frac{l(r-r_+)}{4}}\\ &\le \gamma^{4m^2}s^2\mu\int_{K_+}^\infty\lambda^ne^{-\frac{\lambda(r-r_+)}{4}}\mathrm{d}\lambda\\ &\le \gamma^{4m^2}s^2\mu^2, \qquad (2.5)\end{aligned}$$

provided that

C1) $\displaystyle\int_{K_+}^\infty\lambda^ne^{-\frac{\lambda(r-r_+)}{4}}\mathrm{d}\lambda\le\mu.$

Hence

$$|\partial_\omega^l(P-I)|_{D(r_*,s)}\le|\partial_\omega^l P|_{D(r,s)}+|\partial_\omega^l I|_{D(r_*,s)}\le 2\gamma^{4m^2}s^2\mu$$

for all $\omega\in\mathcal{O}$.

By Cauchy's estimate, we also have

$$\begin{aligned}|\partial_\omega^l II|_{D(r_*,\alpha s)} &= |\int\frac{\partial^{|i|+|j|}}{\partial y^i\partial z^j}\sum_{|k|\le K_+|2i|+|j|\ge3}\partial_\omega^l P_{kij}e^{\sqrt{-1}\langle k,x\rangle}y^iz^j\mathrm{d}y\mathrm{d}z|_{D(r_*,\alpha s)}\\ &= |\int\frac{\partial^{|i|+|j|}}{\partial y^i\partial z^j}\partial_\omega^l(P-I)\mathrm{d}y\mathrm{d}z|_{D(r_*,\alpha s)}\\ &\le \int|\frac{\partial^{|i|+|j|}}{\partial y^i\partial z^j}\partial_\omega^l(P-I)|_{D(r_*,\alpha s)\mathrm{d}y\mathrm{d}z}\\ &\le c_1\gamma^{4m^2}\alpha^3s^2\mu\le c_1\gamma^{4m^2}s^2\mu^2, \qquad (2.6)\end{aligned}$$

for all $\omega\in\mathcal{O}$ and $|l|\le 4m^2$, where $|2i|+|j|=3$ and $\int=\int_0^y\cdots\int_0^y\int_0^z\cdots\int_0^z$ is the $2|i|+|j|$-fold integral. Thus by (2.5), (2.6),

$$\begin{aligned}|\partial_\omega^l(P-R)|_{D(r_*,\alpha s)\times\mathcal{O}} &\le |\partial_\omega^l I|_{D(r_*,s)\times\mathcal{O}}+|\partial_\omega^l II|_{D(r_*,\alpha s)\times\mathcal{O}}\\ &\le c_2\gamma^{4m^2}s^2\mu^2,\ |l|\le 4m^2. \qquad (2.7)\end{aligned}$$

It follows that

$$|\partial_\omega^l R|_{D(r_*,\alpha s)\times\mathcal{O}}\le c_3\gamma^{4m^2}s^2\mu,\ |l|\le 4m^2. \qquad (2.8)$$

Next, we construct a Hamiltonian F of the form

$$F=\sum_{0<|k|\le K_+,2|i|+|j|\le2}F_{kij}y^iz^je^{\sqrt{-1}\langle k,x\rangle}+\langle F_{001},z\rangle. \qquad (2.9)$$

such that the time 1-map $\Phi = \Phi_F^1$ generated by the Hamiltonian vector field $X_F = (F_y, -F_x, JF_z)^\top$ carries H into the Hamiltonian in the next KAM cycle.

Denote $[R] = \int_{T^n} R(x)dx$ and $\tilde{R} = R - [R]$. We let F be such that

$$\{N, F\} + \delta\tilde{R} + \delta\langle P_{001}, z\rangle = 0. \tag{2.10}$$

Then

$$\begin{aligned} H_+ &\equiv H \circ \Phi = H \circ \Phi_F^1 = (N + \delta R) \circ \Phi_F^1 + \delta(P - R) \circ \Phi_F^1 \\ &= N + \delta[R] - \delta\langle P_{001}, z\rangle + \delta \int_0^1 \{R_t, F\} \circ \Phi_F^t dt + \delta(P - R) \circ \Phi_F^1 \\ &= e^+ + \langle \Omega^+, y\rangle + \frac{1}{2}\langle z, M^+ z\rangle + \delta P_+ \\ &\equiv N_+ + \delta P_+, \end{aligned} \tag{2.11}$$

where

$$e^+ = e + \delta P_{000}, \tag{2.12}$$
$$\Omega^+ = \Omega + \delta P_{010}, \tag{2.13}$$
$$M^+ = M + \delta P_{002}, \tag{2.14}$$
$$R_t = (1 - t)([R] - R - \langle P_{001}, z\rangle) + R, \tag{2.15}$$
$$P_+ = \int_0^1 \{R_t, F\} \circ \Phi_F^t dt + (P - R) \circ \Phi_F^1. \tag{2.16}$$

Substituting (2.2) and (2.9) into (2.10) yields

$$\begin{aligned} &- \sum_{0<|k|\le K_+} \sqrt{-1}\langle k, \Omega\rangle (F_{k00} + \langle F_{k10}, y\rangle + \langle F_{k01}, z\rangle + \langle z, F_{k02} z\rangle) e^{\sqrt{-1}\langle k,x\rangle} \\ &+ \sum_{0<|k|\le K_+} (\langle Mz, JF_{k01}\rangle + 2\langle Mz, JF_{k02} z\rangle) e^{\sqrt{-1}\langle k,x\rangle} + \langle Mz, JF_{001}\rangle \\ = &- \delta \sum_{0<|k|\le K_+} (P_{k00} + \langle P_{k10}, y\rangle + \langle z, P_{k01}\rangle + \langle z, P_{k02} z\rangle) e^{\sqrt{-1}\langle k,x\rangle} - \delta\langle P_{001}, z\rangle. \end{aligned}$$

By comparing coefficients above, we obtain the following linear homological equations

$$\sqrt{-1}\langle k, \Omega\rangle F_{k00} = \delta P_{k00}, \tag{2.17}$$
$$\sqrt{-1}\langle k, \Omega\rangle F_{k10} = \delta P_{k10}, \tag{2.18}$$
$$A_{1k} F_{k01} = \delta P_{k01}, \tag{2.19}$$
$$A_{2k} F_{k02} = \delta P_{k02}, \tag{2.20}$$
$$M^\top J F_{001} = -\delta P_{001}, \tag{2.21}$$

where

$$\begin{aligned} A_{1k} &= \sqrt{-1}\langle k, \Omega\rangle I_{2m} - MJ, \\ A_{2k} &= \sqrt{-1}\langle k, \Omega\rangle I_{4m^2} - (MJ) \otimes I_{2m} - I_{2m} \otimes (MJ). \end{aligned}$$

Hereafter $\otimes$ denotes the tensor product of two matrices.

It is clear that the equations (2.17)-(2.20) are uniquely solvable on the new frequency domain

$$\begin{aligned}\mathcal{O}_+ &= \Big\{\omega\in\mathcal{O}:\ |\langle k,\Omega(\omega)\rangle|>\frac{\gamma}{|k|^{\tau}},\ |\mathrm{det}A_{1k}(\omega)|>\frac{\gamma^{2m}}{|k|^{2\tau m}},\\ &\qquad |\mathrm{det}A_{2k}(\omega)|>\frac{\gamma^{4m^2}}{|k|^{4\tau m^2}},\ 0<|k|\le K_+\Big\} \qquad (2.22)\end{aligned}$$

to yield the desired function F.

To estimate the transformation, we let

$$\begin{aligned}D_* &= D(r_*,s),\\ D_{\frac{i}{2}\alpha} &= D(r_++\frac{i(r-r_+)}{2},\frac{\alpha s}{2}),\ i=1,2.\end{aligned}$$

For each $\omega\in\mathcal{O}_+$ and $|l|\le 4m^2$, since by Cauchy's estimate,

$$|\partial_\omega^l P_{kij}|\le|\partial_\omega^l P|_{D\times\mathcal{O}}s^{-(2i+j)}e^{-|k|r}\le\gamma^{4m^2}s^{2-2i-j}\mu e^{-|k|r},\ 0\le 2i+j\le 2,$$

we have by (2.17)-(2.21) that

$$\begin{aligned}&|\frac{1}{\delta}\partial_\omega^l F_{k00}|\le c_4|k|^{\tau}s^2\mu e^{-|k|r},\\ &|\frac{1}{\delta}\partial_\omega^l F_{k10}|\le c_4|k|^{\tau}\mu e^{-|k|r},\\ &|\frac{1}{\delta}\partial_\omega^l F_{k01}|\le c_4|k|^{2\tau m}s\mu e^{-|k|r},\\ &|\frac{1}{\delta}\partial_\omega^l F_{k02}|\le c_4|k|^{4\tau m^2}\mu e^{-|k|r},\\ &|\partial_\omega^l F_{001}|\le s\mu e^{-|k|r}\le c_5 s\mu.\end{aligned}$$

By direct differentiation, we have

$$|\partial_x^i\partial_{(y,z)}^j\partial_\omega^l F|_{D_*}\le c_6\mu\Gamma(r-r_+)+c_6\mu,\ |i|+|j|\le 2,\ |l|\le 4m^2 \qquad (2.23)$$

for all $\omega\in\mathcal{O}_+$, where

$$\Gamma(r-r_+)=\sum_{k\in Z^n}|k|^{4\tau m^2+2}e^{-\frac{|k|(r-r_+)}{4}}.$$

Since

$$\Phi_F^t = id+\int_0^t JDF\circ\Phi_F^\lambda \mathrm{d}\lambda, \qquad (2.24)$$

$$D\Phi_F^t = I_{2(n+m)}+\int_0^t J(D^2F)D\Phi_F^\lambda \mathrm{d}\lambda, \qquad (2.25)$$

we have by (2.23) that

$$\Phi_F^t: D_{\frac{1}{2}\alpha}\to D_\alpha,$$

for each $\omega\in\mathcal{O}_+$, $0<t\le 1$, provided that

C2) $c_6\mu\Gamma(r-r_+)\le\frac{r-r_+}{2}$,
C3) $c_6\mu\Gamma(r-r_+)\le\frac{1}{4}\alpha$.

Moreover,

$$|\partial_\omega^l D^i(\Phi_F^t - id)|_{D_{\frac{1}{2}\alpha}\times\mathcal{O}_+} \le c_7\mu\Gamma(r-r_+) + c_7\mu,\ |l|\le 4m^2,\ i=0,1, \tag{2.26}$$

for all $0 < t \le 1$.

We now estimate the new Hamiltonian. It is clear from (2.12)-(2.14) that

$$|\partial_\omega^l(e-e^+)|_{\mathcal{O}_+} \le c_8\delta\gamma^{4m^2}s^2\mu, \tag{2.27}$$

$$|\partial_\omega^l(\Omega-\Omega^+)|_{\mathcal{O}_+} \le c_8\delta\gamma^{4m^2}s\mu, \tag{2.28}$$

$$|\partial_\omega^l(M-M^+)|_{\mathcal{O}_+} \le c_8\delta\gamma^{4m^2}s^2\mu \tag{2.29}$$

for all $|l|\le 4m^2$.

To estimate the new frequency domain, we let

$$\gamma_+ = \frac{\gamma_0}{4} + \frac{\gamma}{2}.$$

If we choose μ sufficiently small such that

C4) $3c_8\delta\mu K_+^{4m^2\tau+4m^2} < \min\{\frac{\gamma-\gamma_+}{\gamma_0}, \frac{\gamma^{2m}-\gamma_+^{2m}}{\gamma_0^{2m}}, \frac{\gamma^{4m^2}-\gamma_+^{4m^2}}{\gamma_0^{4m^2}}\}$,

then by (2.13), (2.14),

$$\begin{array}{rcl} |\langle k,\Omega^+\rangle| & \ge & |\langle k,\Omega\rangle| - \delta|\langle k,P_{010}\rangle| \\ & \ge & \dfrac{\gamma}{|k|^\tau} - \delta c_8\gamma^{4m^2}\mu|K_+| \ge \dfrac{\gamma_+}{|k|^\tau}, \end{array} \tag{2.30}$$

$$\begin{array}{rcl} |\det A_{1k}^+| & \ge & |\det A_{1k}| - |\sqrt{-1}\langle k,\delta P_{010}\rangle I_{2m} + \delta P_{002}J| \\ & > & \dfrac{\gamma^{2m}}{|k|^{2m\tau}} - (\delta c_7\gamma^{4m^2}\mu K_+)^{2m} - \delta c_8\gamma^{4m^2}\mu \\ & > & \dfrac{\gamma^{2m}}{|k|^{2m\tau}} - 2c_8\delta\gamma^{4m^2}\mu K_+^{2m} > \dfrac{\gamma_+^{2m}}{|k|^{2m\tau}}, \end{array} \tag{2.31}$$

$$\begin{array}{rcl} |\det A_{2k}^+| & \ge & |\det A_{2k}| - |\sqrt{-1}\langle k,\delta P_{010}\rangle I_{4m^2} \\ & - & (\delta P_{002}J)\otimes I_{2m} - I_{2m}\otimes(\delta P_{002}J)| \\ & > & \dfrac{\gamma^{4m^2}}{|k|^{4m^2\tau}} - (\delta c_7\gamma^{4m^2}\mu K_+)^{4m^2} - 2(\delta c_8\gamma^{4m^2}\mu)^{2m} \\ & > & \dfrac{\gamma^{4m^2}}{|k|^{4m^2\tau}} - 3c_8\delta\gamma^{4m^2}\mu K_+^{4m^2} > \dfrac{\gamma_+^{4m^2}}{|k|^{4m^2\tau}} \end{array} \tag{2.32}$$

for all $0<|k|\le K_+$, $\omega\in\mathcal{O}_+$.

To estimate the new perturbation, we let

$$\begin{array}{rcl} s_+ & = & \dfrac{1}{2}\alpha s, \\ D_+ & = & D(r_+,s_+). \end{array}$$

Then by (2.7),(2.8),(2.15),(2.16),(2.23) and Cauchy's estimate, we have

$$\begin{aligned} |\partial_\omega^l P_+|_{D_+\times\mathcal{O}_+} &\leq |\int_0^1 \partial_\omega^l \{R_t, F\}\circ\Phi_F^t dt| + |\partial_\omega^l (P-R)\circ\Phi_F^1|_{D_+\times\mathcal{O}_+} \\ &= |\int_0^1 \partial_\omega^l (\frac{\partial R_t}{\partial x}\frac{\partial F}{\partial y} - \frac{\partial R_t}{\partial y}\frac{\partial F}{\partial x} + \frac{\partial R_t}{\partial z} J \frac{\partial F}{\partial z})\circ\Phi_F^t dt|_{D_+\times\mathcal{O}_+} \\ &+ |\partial_\omega^l (P-R)\circ\Phi_F^1|_{D_+\times\mathcal{O}_+} \leq \frac{c\gamma^{4m^2}s^2\mu^2}{r-r_+}(\Gamma(r-r_+)+1)^{4m^2+1} \\ &\leq c_9 s_+^2 \gamma_+^{4m^2}(\Gamma(r-r_+)+1)^{4m^2+1}\mu^{\frac{4}{3}} \end{aligned} \tag{2.33}$$

for all $|l|\leq 4m^2$.

Finally, let

$$\mu_+ = (16c_0\alpha)^{\frac{1}{6}}\mu,$$

where

$$c_0 = \max\{c_1, \cdots, c_9\}.$$

If

C5) $c_9\mu^{\frac{4}{3}}(\Gamma(r-r_+)+1)^{4m^2+1} \leq \mu_+$,

then

$$|\partial_\omega^l P_+|_{D_+\times\mathcal{O}_+} \leq \gamma_+^{4m^2} s_+^2 \mu_+, \;\; |l|\leq 4m^2. \tag{2.34}$$

This completes one step of KAM iterations.

2.2. **Iteration lemma.** For each $\nu = 1, 2, \cdots$, we index all index-free quantities above by ν and index all '+'-indexed quantities above by $\nu+1$. This yields the following sequences

$$r_\nu, \gamma_\nu, , s_\nu, \alpha_\nu, \mu_\nu, H_\nu, N_\nu, P_\nu, e^\nu, \Omega^\nu, M^\nu, D_\nu, K_\nu, \mathcal{O}_\nu, \Phi_\nu.$$

In particular,

$$
\begin{aligned}
&r_\nu = r_0(1-\sum_{i=1}^{\nu}\frac{1}{2^{i+1}}),\\
&\gamma_\nu = \gamma_0(1-\sum_{i=1}^{\nu}\frac{1}{2^{i+1}}),\\
&s_\nu = \frac{1}{4}\alpha_{\nu-1}s_{\nu-1},\\
&\alpha_\nu = \mu_\nu^{\frac{1}{3}},\\
&\mu_\nu = (16c_0\alpha_{\nu-1})^{\frac{1}{6}}\mu_{\nu-1},\\
&H_\nu = N_\nu+\delta P_\nu,\\
&N_\nu = e^\nu+\langle\Omega^\nu,y\rangle+\frac{1}{2}\langle z,M^\nu z\rangle,\\
&D_\nu = D(r_\nu,s_\nu),\\
&K_\nu = \left(\left[\log\frac{1}{\mu_{\nu-1}}\right]+1\right)^3,\\
&\mathcal{O}_\nu = \Bigg\{\omega\in\mathcal{O}_{\nu-1}:\ |\langle k,\Omega\rangle|>\frac{\gamma_\nu}{|k|^\tau},\ |\det A_{1k}^\nu(\omega)|>\frac{\gamma_\nu^{2m}}{|k|^{2\tau m}},\\
&\qquad\qquad |\det A_{2k}^\nu(\omega)|>\frac{\gamma_\nu^{4m^2}}{|k|^{4\tau m^2}},\ 0<|k|<K_\nu\Bigg\},
\end{aligned}
$$

where

$$
\begin{aligned}
A_{1k}^\nu &= \sqrt{-1}\langle k,\Omega^\nu\rangle I_{2m}-M^\nu J,\\
A_{2k}^\nu &= \sqrt{-1}\langle k,\Omega^\nu\rangle I_{4m^2}-(M^\nu J)\otimes I_{2m}-I_{2m}\otimes(M^\nu J)
\end{aligned}
$$

for $0<|k|<K_\nu$.

The following iteration lemma ensures the validity of the KAM iteration for all steps.

Lemma 2.1. *If $\mu_0=\mu_0(r_0,s_0,\gamma_0)$ is sufficiently small, then the following holds for all $|l|\le 4m^2$ and $\nu=1,2,\cdots$.*

1) *$\Phi_\nu: D_\nu\to D_{\nu-1}$ are real analytic, symplectic, and C^{4m^2} depend on $\omega\in\mathcal{O}_\nu$. Moreover,*

$$H_\nu = H_{\nu-1}\circ\Phi_\nu = N_\nu+\delta P_\nu,$$

$$|\partial_\omega^l D^i(\Phi_\nu-id)|_{D_\nu\times\mathcal{O}_+}\le\frac{\mu_0^{\frac{1}{8}}}{2^\nu},\ |l|\le 4m^2,\ i=0,1. \tag{2.35}$$

2) *On $D_\nu\times\mathcal{O}_\nu$,*

$$|\partial_\omega^l(e^\nu-e^0)|,\ |\partial_\omega^l(\Omega^\nu-\Omega^0)|,\ |\partial_\omega^l(M^\nu-M^0)|\le c_0\delta\gamma_0^{4m^2}\mu_0, \tag{2.36}$$

$$|\partial_\omega^l(e^\nu-e^{\nu-1})|,\ |\partial_\omega^l(\Omega^\nu-\Omega^0)|,\ |\partial_\omega^l(M^\nu-M^{\nu-1})|\le\frac{c_0\delta\gamma_0^{4m^2}}{2^\nu}\mu_0, \tag{2.37}$$

$$|\partial_\omega^l P_\nu|\le\gamma_\nu^{4m^2}s_\nu^2\mu_\nu,\ |l|\le 4m^2. \tag{2.38}$$

3) $\mathcal{O}_\nu = \mathcal{O}_{\nu-1} \setminus \bigcup_{K_{\nu-1}<|k|\le K_\nu} R_k^\nu(\gamma_0)$, *where*

$$R_k^\nu(\gamma_0) = \Big\{\omega \in \mathcal{O}_{\nu-1}: \ |\langle k, \Omega^{\nu-1}(\omega)\rangle| \le \frac{\gamma_{\nu-1}}{|k|^\tau}, \quad \text{or } |\det A_{1k}^{\nu-1}(\omega)| \le \frac{\gamma_{\nu-1}^{2m}}{|k|^{2\tau m}}, \text{ or } |\det A_{2k}^{\nu-1}(\omega)| \le \frac{\gamma_{\nu-1}^{4m^2}}{|k|^{4\tau m^2}}\Big\}$$

for all $k \in Z^n \setminus \{0\}$.

Proof. We need to verify the conditions C1)-C5) for all $\nu = 0, 1, \cdots$.

First, we choose μ_0 sufficiently small such that

$$\frac{1}{2^{\nu+2}} \log \frac{1}{\mu_\nu} > 1.$$

Then

$$\begin{aligned}
&\log(n+1)! + n(\nu+2)\log 2 + n\log K_{\nu+1} - K_{\nu+1}\frac{1}{2^{\nu+2}} \\
&= \log(n+1)! + n(\nu+2)\log 2 + 3n\log(\log[\frac{1}{\mu_\nu}]+1) - \frac{1}{2^{\nu+2}}(\log[\frac{1}{\mu_\nu}]+1)^3 \\
&\le \log(n+1)! + n(\nu+2)\log 2 + 3n\log(\log\frac{1}{\mu_\nu}+2) - (\log\frac{1}{\mu_\nu})^2 \\
&\le -\log\frac{1}{\mu_\nu}.
\end{aligned}$$

Hence

$$\int_{K_{\nu+1}}^\infty \lambda^n e^{-\lambda(r_\nu - r_{\nu+1})}\mathrm{d}\lambda \le (n+1)!\frac{K_{\nu+1}^n}{(r_\nu - r_{\nu+1})^n} e^{-K_{\nu+1}(r_\nu - r_{\nu+1})} \le \mu_\nu,$$

i.e., C1) holds.

Next, we note that

$$\begin{aligned}
\Gamma(r_\nu - r_{\nu+1}) = \Gamma(\frac{1}{2^{\nu+2}}) &\le \int_1^\infty \lambda^{n+4\tau m^2+2} e^{-\lambda\frac{1}{2^{\nu+5}}}\mathrm{d}\lambda \\
&\le (n+4\tau m^2+2)!2^{(\nu+5)(n+4\tau m^2)}. \qquad (2.39)
\end{aligned}$$

Thus, to prove C2), it is sufficient to show that

$$c_0\mu_\nu(n+4\tau m^2+2)!2^{(\nu+5)(n+4\tau m^2)} \le \frac{1}{2^{\nu+2}}, \qquad (2.40)$$

which clearly holds for $\nu = 0$ if μ_0 is sufficiently small. We now consider $\nu \ge 1$. Since

$$\mu_\nu = (16c_0\alpha_{\nu-1})^{\frac{1}{6}}\mu_{\nu-1} = (16c_0)^{\frac{\nu-1}{6}}\mu_0^{\frac{19(\nu-1)}{18}},$$

(2.40) is equivalent to

$$(2^{\frac{5}{3}+n+4\tau m^2+2}c_0^{\frac{1}{6}}\mu_0^{\frac{19}{18}})^{\nu-1}c_0(n+4\tau m^2+2)!2^{5(n+4\tau m^2)} \le 1, \qquad (2.41)$$

which also holds if μ_0 sufficiently small. This proves C2).

C3) follows from (2.39) and a similar argument as above.

To prove C4), we note that for any constant $\beta > 0$, $\xi > 1$, $\mu^\beta(\log\frac{1}{\mu}+1)^\xi \to 0$ as $\mu \to 0$. Hence as μ_0 (hence μ_ν) sufficiently small, we have

$$3c_0\delta\mu K_{\nu+1}^{4m^2\tau+4m^2} = 3c_0\delta\mu_\nu([\log\frac{1}{\mu_\nu}]+1)^{3(4m^2\tau+4m^2)} < (1-\frac{1}{2^{4m^2}}),$$

i.e. C4) holds.

Note that C5) is equivalent to

$$\mu_\nu^{\frac{5}{18}}(\Gamma(r_\nu - r_{\nu+1})+1)^{4m^2+1} < \frac{1}{16}(16c_0)^{\frac{1}{6}}. \tag{2.42}$$

Since, by (2.39),

$$(\Gamma(r_\nu - r_{\nu+1})+1)^{4m^2+1} \leq ((n+4\tau m^2+2)!2^{(\nu+5)(n+4\tau m^2)+1})^{4m^2+1},$$

it is sufficient to show that

$$\mu_\nu^{\frac{5}{18}}(n+4\tau m^2+2)!2^{(\nu+5)(n+4\tau m^2+2)+1} < \frac{c_0^{\overline{\frac{1}{6(4m^2+1)}}}}{16^{\overline{\frac{7}{6(4m^2+1)}}}}. \tag{2.43}$$

Let $\lambda \gg 1$ be such that

$$\mu_0 < \frac{1}{(16c_0\lambda^{\frac{6\times 18}{5}})^3} \leq 1.$$

Then by induction

$$\mu_\nu = (16c\mu_{\nu-1}^{\frac{1}{3}})^{\frac{1}{6}}\mu_{\nu-1} < \cdots < \frac{1}{(\lambda^{\frac{18}{5}})^\nu}\mu_0. \tag{2.44}$$

Hence (2.43) holds if μ_0 is sufficiently small.

It follows that the KAM step is valid for all $\nu = 0, 1, \cdots$, from which 1) follows. In particular, (2.35) follows from (2.26), (2.42) and (2.44). Moreover, 2) follows from (2.27)-(2.29) and (2.44), and, 3) follows from (2.30)-(2.32). □

2.3. **Convergence and measure estimate.** Applying Lemma 2.1 inductively we obtain the following sequences:

$$\begin{aligned}
&\Psi_\nu = \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_\nu : D_\nu \times \mathcal{O}_\nu \to D_0,\\
&H_0 \circ \Psi_\nu = H_\nu = N_\nu + \delta P_\nu,\\
&N_\nu = e^\nu(\omega) + \langle \Omega^\nu(\omega), y\rangle + \frac{1}{2}\langle z, M^\nu(\omega)z\rangle,\ \nu = 1, 2, \cdots.
\end{aligned}$$

By (2.38) and Cauchy's estimate, we also have

$$|DP_\nu|_{D(r_\nu, \frac{s_\nu}{2})\times\mathcal{O}_\nu} \leq 2\gamma_\nu^{4m^2}\mu_\nu,\ \nu = 1, 2, \cdots. \tag{2.45}$$

Let

$$\mathcal{O}_* = \bigcap_{\nu=0}^{\infty}\mathcal{O}_\nu.$$

Then by Lemma 2.1 and (2.45), Ψ_ν, H_ν, N_ν, e^ν, Ω^ν, M^ν, P_ν converge uniformly on $D(\frac{r_0}{2}, 0)\times\mathcal{O}_*$, say, to Ψ_∞, H_∞, N_∞, e^∞, Ω^∞, M^∞, P_∞ respectively, and moreover,

$$H_\infty = N_\infty = e^\infty(\omega) + \langle\Omega^\infty(\omega), y\rangle + \frac{1}{2}\langle z, M^\infty(\omega)z\rangle.$$

Since $H_0 \circ \Psi_\nu = H_\nu$,

$$\Phi_{H_0}^t \circ \Psi_\nu = \Psi_\nu \circ \Phi_{H_\nu}^t.$$

It follows that

$$\Phi^t_{H_0} \circ \Psi_\infty = \Psi_\infty \circ \Phi^t_{H_\infty},$$

on $D(\frac{r_0}{2}, 0) \times \mathcal{O}_*$. This implies for each $\omega \in \mathcal{O}_*$ and $0 < \delta < 1$, (1.4) admits an invariant, quasi-periodic n-torus with the Diophantine frequency ω. The Whitney smoothness of these tori follows from a standard argument using the Whitney extension theorem (see [21, 27] and references therein).

For measure estimate, we need the following lemma from ([21]).

Lemma 2.2. *Let $M(\omega)$, $\omega \in \mathcal{O}$, be a family of symmetric, $2m \times 2m$ matrices and $\lambda_1(\omega), \cdots, \lambda_{2m}(\omega)$ be the eigenvalues of $JM(\omega)$ satisfying the Melnikov non-resonant condition* H1). *Denote*

$$\begin{aligned} A_{1k}(\omega) &= \sqrt{-1}\langle k, \omega\rangle I_{2m} - M(\omega)J, \\ A_{2k}(\omega) &= \sqrt{-1}\langle k, \omega\rangle I_{4m^2} - (M(\omega)J) \otimes I_{2m} - I_{2m} \otimes (M(\omega)J),\ \omega \in \mathcal{O},\ k \in Z^n \setminus \{0\}. \end{aligned}$$

Then the following hold.

1) *For every $k \in Z^n \setminus \{0\}$,*

$$\det A_{1k} = \prod_{i=1}^{2m} (\sqrt{-1}\langle k, \omega\rangle - \lambda_i),$$

$$\det A_{2k} = \prod_{i,j=1}^{2m} (\sqrt{-1}\langle k, \omega\rangle - \lambda_i - \lambda_j).$$

2) *The set*

$$\{\omega \in \mathcal{O} :\ \langle k, \omega\rangle \neq 0, \det A_{1k}(\omega) \neq 0, \det A_{2k}(\omega) \neq 0,\ \forall k \in Z^n \setminus \{0\}\}$$

admits full Lebesgue measure relative to $\mathcal{O}$.

We are now ready to estimate the measure $|\mathcal{O} \setminus \mathcal{O}_*|$. Since

$$\mathcal{O}_{\nu+1} = \mathcal{O}_\nu \setminus \bigcup_{K_\nu < |k| \le K_{\nu+1}} R^{\nu+1}_k(\gamma),\ \nu = 0, 1, \cdots,$$

we have

$$\mathcal{O} \setminus \mathcal{O}_* = \bigcup_{\nu=0}^{\infty} \bigcup_{K_\nu < |k| \le K_{\nu+1}} R^{\nu+1}_k(\gamma).$$

By (2.36), we have that

$$\begin{aligned} |\partial^{2m}_\omega A^\nu_{1k}(\omega)| &= |k|^{2m}((2m)! + O(\frac{1}{|k|+1}) + O(\delta + \mu)), \\ |\partial^{4m^2}_\omega A^\nu_{2k}(\omega)| &= |k|^{4m^2}((4m^2)! + O(\frac{1}{|k|+1}) + O(\delta + \mu)), \end{aligned}$$

where $O(\frac{1}{|k|+1})$ and $O(\delta + \mu)$ are independent of ν, ω. It follows from ([34]), Lemma 2.2 that there is a positive integer n_0 and a positive constant c such that

$$|R^{\nu+1}_k(\gamma)| \le c\frac{\gamma}{|k|^\tau},$$

for all ν and $|k| \geq n_0$. Let ν_0 be such that $K_\nu > n_0$ as $\nu \geq \nu_0$. Then

$$|\bigcup_{\nu=\nu_0}^{\infty} \bigcup_{K_\nu < |k| \leq K_{\nu+1}} R_k^{\nu+1}(\gamma)| \leq c\gamma \sum_{\nu=\nu_0}^{\infty} \sum_{K_\nu < |k| \leq K_{\nu+1}} \frac{1}{|k|^\tau} = O(\gamma). \tag{2.46}$$

To estimate $R_k^{\nu+1}$ for $0 < |k| \leq K_\nu$, $\nu \leq n_0$, we let γ, δ be sufficiently small such that

$$\begin{aligned} R_k^{\nu+1}(\gamma) \quad \subset \quad R_k^*(\gamma) &= \{\omega \in \mathcal{O}_\nu : \ |\langle k, \omega\rangle| \leq \frac{2\gamma_\nu}{|k|^\tau}, \text{ or } |\det A_{1k}^0(\omega)| \leq \frac{2\gamma_\nu^{2m}}{|k|^{2\tau m}}, \\ &\quad \text{or } |\det A_{2k}^0(\omega)| \leq \frac{2\gamma_\nu^{4m^2}}{|k|^{4\tau m^2}}\} \end{aligned}$$

for all $0 < |k| \leq K_\nu$, $\nu \leq n_0$. Then by H1) and Lemma 2.2, $|R_k^{\nu+1}(\gamma)| \leq |R_k^*(\gamma)| \to 0$ as $\gamma, \delta \to 0$, uniformly for all $0 < |k| \leq K_\nu$, $\nu \leq n_0$. Consequently,

$$|\bigcup_{\nu=0}^{\nu_0} \bigcup_{0<|k|\leq K_\nu} R_k^{\nu+1}(\gamma)| \to 0,$$

as $\gamma, \delta \to 0$. Combining this with (2.46), we have that

$$|\mathcal{O}_0 \setminus \mathcal{O}_*| \leq |\bigcup_{\nu=0}^{\nu_0} \bigcup_{0<|k|\leq K_\nu} R_k^{\nu+1}(\gamma)| + |\bigcup_{\nu=\nu_0}^{\infty} \bigcup_{K_\nu<|k|\leq K_{\nu+1}} R_k^{\nu+1}(\gamma)| \to 0,$$

as $\gamma, \ \delta \to 0$.

The proof of Theorem 3 is now completed.

3. Proof of Theorems 1 and 2

3.1. **Reduction to normal form.** Consider Hamiltonian (1.1). In order to apply Theorem 3, we need to first remove the singularity of $M(\omega)$ by considering $M_\varepsilon(\omega)$ as in H3).

Let $z_\varepsilon(\omega)$ be as in H2) and consider the translation $\phi : x = x, y = y, z \to z + z_\varepsilon$. Then

$$\begin{aligned} \tilde{H} &= H \circ \phi(x, y, z) \\ &= \tilde{e}_\varepsilon(\omega) + \langle \Omega_\varepsilon, y\rangle + \frac{1}{2}\langle z, M_\varepsilon(\omega)\rangle + \varepsilon \tilde{P}(x, y, z, \omega), \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} \tilde{e}_\varepsilon(\omega) &= \varepsilon[P](0, z_\varepsilon(\omega)), \\ \Omega_\varepsilon(\omega) &= \omega + \varepsilon\partial_y[P](0, z_\varepsilon(\omega)), \\ M_\varepsilon(\omega) &= M(\omega) + \varepsilon\partial_z^2[P](0, z_\varepsilon(\omega)), \\ \tilde{P}(x, y, z, \omega) &= O((|y| + |z|)^2) + \sum_{k\neq 0}\sum_{i,j} P_{kij} y^i z^j e^{\sqrt{-1}\langle k, x\rangle}, \end{aligned}$$

where $O((|y| + |z|)^2)$ is independent of x.

The Hamiltonian (3.1) is in the form (1.4) when $\delta = \varepsilon$ but the order of $\tilde{P}$ needs to be improved in order for the condition (1.5) to satisfy. To improve the order of $\tilde{P}$, a crucial idea is to perform one step of KAM iteration similar to that in Section 2. Write

$$R = \sum_{0<|k|<K_1, 2|i|+|j|\leq 2} P_{kij} y^i z^j e^{\sqrt{-1}\langle k,x\rangle},$$
$$I = \sum_{0<|k|<K_1, 2|i|+|j|\geq 3} P_{kij} y^i z^j e^{\sqrt{-1}\langle k,x\rangle},$$
$$II = \sum_{|k|\geq K_1, i,j} P_{kij} y^i z^j e^{\sqrt{-1}\langle k,x\rangle}$$

for some $K_1 > 0$ to be determined later. Then

$$\tilde{P} = O((|y|+|z|)^2) + R + I + II.$$

Consider re-scaling $y \to \varepsilon^{\frac{1}{3}} y, z \to \varepsilon^{\frac{1}{6}} z, \tilde{H} \to \varepsilon^{\frac{1}{3}} \tilde{H}$. Then the re-scaled Hamiltonian reads

$$\begin{aligned}
\bar{H} &= \frac{\tilde{H}(x, \varepsilon^{\frac{1}{3}} y, \varepsilon^{\frac{1}{6}} z)}{\varepsilon^{\frac{1}{3}}} = \bar{N} + \bar{P} \\
\bar{N} &= \bar{e}_\varepsilon + \langle \Omega_\varepsilon, y\rangle + \frac{1}{2}\langle z, M_\varepsilon(\omega) z\rangle \\
\bar{P} &= \varepsilon^{\frac{7}{6}} O((|y|+|z|)^2) + \varepsilon^{\frac{2}{3}} \bar{R} + \varepsilon^{\frac{7}{6}} \bar{I} + \varepsilon^{\frac{2}{3}} \bar{I}I,
\end{aligned}$$

where $\bar{e}_\varepsilon, \bar{R}, \bar{I}, \bar{I}I$ are obtained from their respective terms above via re-scaling. We choose K_1 such that

$$|\bar{I}I|_{D(r,s)\times \mathcal{O}(r)} \leq \varepsilon. \tag{3.2}$$

Then there is a constant $c > 0$ such that

$$|\bar{P} - \varepsilon^{\frac{2}{3}} \bar{R}|_{D(r,s)\times\mathcal{O}(r)} \leq c\varepsilon^{\frac{7}{6}} \tag{3.3}$$

for some constant $c > 0$. We note that $K_1 \to \infty$ as $\varepsilon \to 0$.

Next, similar to Section 2, we eliminate $\varepsilon^{\frac{2}{3}} \bar{R}$ by the symplectic transformation Φ_F^1, where

$$F(x, y, z) = \sum_{0<|k|<K_1, 2|i|+|j|\leq 2} F_{kij} y^i z^j e^{\sqrt{-1}\langle k,x\rangle} \tag{3.4}$$

satisfies

$$\{\bar{N}, F\} + \varepsilon^{\frac{2}{3}} \bar{R} = 0. \tag{3.5}$$

Similar to (2.17)-(2.20), the equation (3.5) is equivalent to the following system of homological equations

$$\begin{aligned}
\sqrt{-1}\langle k, \Omega_\varepsilon\rangle F_{k00} &= \varepsilon^{\frac{2}{3}} \bar{P}_{k00}, \\
\sqrt{-1}\langle k, \Omega_\varepsilon\rangle F_{k10} &= \varepsilon^{\frac{2}{3}} \bar{P}_{k10}, \\
A_{1k}^{\varepsilon} F_{k01} &= \varepsilon^{\frac{2}{3}} \bar{P}_{k01}, \\
A_{2k}^{\varepsilon} F_{k02} &= \varepsilon^{\frac{2}{3}} \bar{P}_{k02},
\end{aligned}$$

which can be uniquely solved on the open domain

$$\mathcal{O}_0 = \Big\{\omega\in\mathcal{O}:\ |\langle k,\Omega_\varepsilon(\omega)\rangle| > \frac{\gamma}{|k|^\tau},\ |\det A^\varepsilon_{1k}(\omega)| > \frac{\gamma^{2m}}{|k|^{2\tau m}},\ |\det A^\varepsilon_{2k}(\omega)| > \frac{\gamma^{4m^2}}{|k|^{4\tau m^2}},\ 0<|k|\le K_1\Big\},$$

where

$$A^\varepsilon_{1k} = \sqrt{-1}\langle k,\Omega_\varepsilon\rangle I_{2m} - M_\varepsilon J,$$
$$A^\varepsilon_{2k} = \sqrt{-1}\langle k,\Omega_\varepsilon\rangle I_{4m^2} - (M_\varepsilon J)\otimes I_{2m} - I_{2m}\otimes(M_\varepsilon J).$$

This yields a real analytic function F of the form (3.4) which also depends on ω real analytically. We note by H1) and Lemma 2.2 that $|\mathcal{O}\setminus\mathcal{O}_0|\to 0$ as $\varepsilon\to 0$.

Similar to (2.23), we also have

$$|\partial^i_x\partial^j_{(y,z)}F|_{D(\frac{3r}{4},s)\times\mathcal{O}_0(\frac{r}{4})} \le c(r)\frac{\varepsilon^{\frac{2}{3}}}{\gamma^{4m^2}},\ |i|+|j|\le 2, \tag{3.6}$$

for some continuous function $c(r)>0$. It follows from (2.24), (2.25) that if ε is sufficiently small, then

$$\phi^t_F : D(\frac{r}{4},\frac{s}{2})\times\mathcal{O}_0(\frac{r}{4})\to D(\frac{3r}{4},s),\ 0<t\le 1,$$

and moreover,

$$|\{R,F\}|_{D(\frac{3r}{4},s)\times\mathcal{O}_0(\frac{r}{4})}\le c\frac{\varepsilon^{\frac{2}{3}}}{s^2\gamma^{4m^2}} \tag{3.7}$$

for some constant $c>0$.

Now,

$$H_0\equiv \bar{H}\circ\Phi^1_F = N_0+\varepsilon P_0, \tag{3.8}$$

where

$$N_0 = \bar{N}$$
$$P_0 = \frac{1}{\varepsilon^{\frac{1}{3}}}\int_0^1\{tR,F\}\circ\phi^t_F dt + (\bar{P}-\varepsilon^{\frac{2}{3}}R)\circ\phi^1_F.$$

If we let $\delta=\varepsilon$, $r_0=\frac{r}{4}$, $s_0=\frac{s}{2}$, then (3.8) is in the normal form (1.4), and by (3.3), (3.7),

$$|P_0|_{D(r_0,s_0)\times\mathcal{O}_0(r_0)}\le c(\varepsilon^{\frac{1}{6}}+\frac{\varepsilon^{\frac{1}{3}}}{s_0^2\gamma^{4m^2}})$$

for some constant $c>0$.

3.2. **Proof of Theorem 1.** Let $0<a<\frac{1}{12}$, $0<b<\frac{1}{6}-2a$, $0<\beta<\frac{1}{6}-2a-b$ be fixed constants and let ε be small such that $s_0\ge\varepsilon^a$. Define $\gamma=\varepsilon^{\frac{b}{4m^2}}$ and $\mu=2c\varepsilon^\beta$. Then

$$|P_0|_{D(r_0,s_0)\times\mathcal{O}_0(r_0)}\le\gamma^{4m^2}s_0^2\mu.$$

Since H3) implies H5), all conditions of Theorem 3 are satisfied. Applying Theorem 3, we obtain a subset $\mathcal{O}_*$ of $\mathcal{O}_0$, with $|\mathcal{O}_0\setminus\mathcal{O}_*|\to 0$ as $\varepsilon\to 0$, which parametrizes a Whitney smooth family of quasi-periodic n-tori of (1.1). Since $|\mathcal{O}\setminus\mathcal{O}_0|\to 0$ as $\varepsilon\to 0$, we have $|\mathcal{O}\setminus\mathcal{O}_*|\to 0$ as $\varepsilon\to 0$. This proves Theorem 1.

3.3. **Proof of Theorem 2.** Let

$$\begin{array}{rcl} y & = & (I_1, \cdots, I_r), \\ x & = & (\theta_1, \cdots, \theta_r), \\ z & = & (I_{r+1}, \cdots, I_d, \theta_{r+1}, \cdots, \theta_d). \end{array}$$

We associate $\omega \in \mathcal{O}$ with $y_0 \in G$ by $\omega = \partial_y H_{00}(y_0)$ through the diffeomorphism between G and $\mathcal{O} \equiv \partial_y H_{00}(G)$. Then up to a constant the Hamiltonian (1.3) under the translation $y \to y_+ y_0$ reads

$$H = \langle \omega, y \rangle + \varepsilon P(x, y + y_0, z, \varepsilon) + O(|y|^2).$$

After re-scaling $y \to \varepsilon^{\frac{2}{3}} y, H \to \varepsilon^{\frac{2}{3}} H$, we have

$$H = \langle \omega, y \rangle + \varepsilon^{\frac{1}{3}} P(x, y, \omega, \varepsilon),$$

where

$$P(x, y, \omega, \varepsilon) = P(x, \varepsilon^{\frac{2}{3}} y + y_0, z, \varepsilon) + \varepsilon^{\frac{2}{3}} O(|y|^2).$$

Replacing $\varepsilon^{\frac{1}{3}}$ by a parameter, again called ε, we obtain the Hamiltonian

$$\begin{array}{rcl} H & = & \langle \omega, y \rangle + \varepsilon P(x, y, \omega, \varepsilon), \\ P(x, y, \omega, \varepsilon) & = & P(x, \varepsilon^2 y + y_0, z, \varepsilon) + \varepsilon^2 O(|y|^2) \end{array}$$

which is in the form (1.1) with $M \equiv 0$. Hence the Melnikov condition H1) holds automatically, and H2), H3) are implied by H4), H5) respectively. Applying Theorem 1, Theorem 2 follows.

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