The Cyclicity of Period Annulus of Degenerate Quadratic Hamiltonian System with Elliptic Segment

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Abstract

We study the cyclicity of period annuli (or annulus) for general degenerate quadratic Hamiltonian systems with an elliptic segment or a saddle loop, under quadratic perturbations. By using geometrical arguments and studying the respective Abelian integral based on the Picard-Fuchs equation, it is shown that the cyclicity of period annuli or annulus for such systems equals two. This result, together with those of [8],[10],[11],[18],[19], gives a complete solution to the infinitesimal Hilbert 16th problem in the case of degenerate quadratic Hamiltonian systems under quadratic perturbations.

Keywords. Abelian integral, Cyclicity of period annulus, Degenerate quadratic Hamiltonian system, Infinitesimal Hilbert 16th problem.

AMS(MOS). Mathematics Subject Classification. 58F21, 34C05, 58F27, 58F30.

1 Introduction and main results

The second part of the well-known Hilbert 16th problem is to find the maximal number and the positions of the limit cycles for planar autonomous differential equations of the following form

$$\begin{cases} \dot{x} = X_n(x, y), \\ \dot{y} = Y_n(x, y), \end{cases} \tag{1.1}$$

where X_n and Y_n are polynomials of degree n. To date, this problem remains unsolved even for the quadratic case, i.e., $n = 2$. In [1], V. I. Arnold proposed a weaker version of the problem by restricting (1.1) to the form

$$\begin{cases} \dot{x} = \dfrac{\partial H(x, y)}{\partial y} + \epsilon f(x, y), \\[2ex] \dot{y} = -\dfrac{\partial H(x, y)}{\partial x} + \epsilon g(x, y), \end{cases} \tag{1.2}$$

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where ϵ is a small parameter, H is a polynomial of degree $n+1$, and, f and g are polynomials of degree less than or equal to n. Thus, (1.2) is a perturbation of the Hamiltonian system

$$\begin{cases} \dot{x} = \dfrac{\partial H(x,y)}{\partial y}, \\ \dot{y} = -\dfrac{\partial H(x,y)}{\partial x}. \end{cases} \tag{1.3}$$

To tackle the Hilbert 16th problem for (1.2), besides the Hopf bifurcation and homoclinic (or heteroclinic) bifurcation analysis, a crucial step is to study the *cyclicity of period annulus* of X_H which is roughly the total number of limit cycles (counting multiplicity) that can be bifurcated from a period annulus (or annuli) of (1.3) (see Section 2 for precise definition of cyclicity of period annulus).

In this work, we will give a complete answer to the cyclicity of period annulus (or annuli) for the degenerate cases of (1.3) when $n = 2$. Recall that quadratic planar systems (1.1) with at least one center are always integrable. Based on their algebraic invariants, systems of such nature can be classified into the following four classes: Hamiltonian (Q_3^H), reversible (Q_3^R), Lotka-Volterra type (Q_3^{LV}) and co-dimension 4 type (Q_4) (see [20]). In particular, a vector field X_H belonging to Q_3^H must have the form (1.3), where $H(x,y)$ is a real cubic polynomial. Such a vector field $X_H \in Q_3^H$ is said to be *generic* if it does not belong to other integrable classes. Otherwise, it is called *non-generic* or *degenerate*.

It is shown by E. Horozov and I. D. Iliev in [9] that any cubic Hamiltonian can be transformed into the following normal form

$$H(x,y) = \frac{1}{2}(x^2+y^2) - \frac{1}{3}x^3 + axy^2 + \frac{1}{3}by^3,$$

where a, b are parameters lying in the region

$$G = \left\{(a,b) : -\frac{1}{2} \le a \le 1,\ 0 \le b \le (1-a)\sqrt{1+2a}\right\},$$

and moreover, their respective vector fields X_H are generic if $(a,b) \in G^* = G \setminus \partial G$ and degenerate if $X_H \in \partial G$. Figure 1 is adopted from [9] which shows all possible phase portraits of X_H for different ranges of a, b, where G^* is divided into three parts G_1, G_2 and G_3 by two curves l_2 and l_∞. Along l_2 two singularities of X_H coincide, and along l_∞ one singularity of X_H tends to infinity. Hence, besides the two critical situations along l_2 and l_∞, X_H has one, two or three saddle points for $(a,b) \in G_1, G_2$ and G_3, respectively. It is shown in [9] that if $(a,b) \in G_3$, then the cyclicity of period annulus of X_H under quadratic perturbations equals two. A recent work [7] concludes the same for $(a,b) \in G_1 \cup G_2$. If $(a,b) \in \partial G$, then in suitable coordinates all X_H's have an axis of symmetry, and as shown in Figure 1, their respective basic dynamics can be classified into the following eight types consisting of:

(1) a saddle loop with a double singularity at infinity if $(a,b) = O = (0,0)$;
(2) a saddle loop with two more saddles if (a,b) lies on the segment OT;
(3) a triangular heteroclinic loop if $(a,b) = T = (1,0)$;

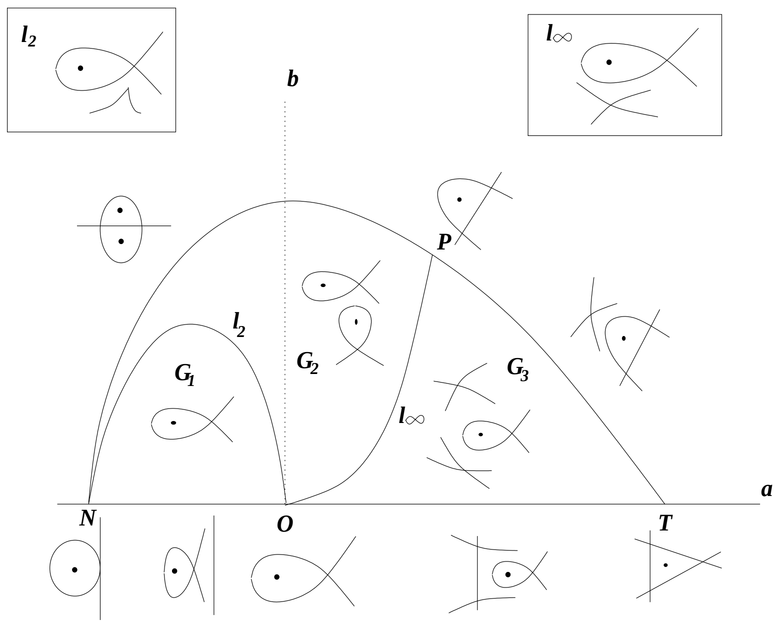


Figure 1 The phase portraits of $X_H \in Q_3^H$

(4) a hyperbolic segment loop if (a, b) lies on the arc TP;
(5) a parabolic segment loop if $(a, b) = P$;
(6) an elliptic segment loop if (a, b) lies on the arc PN;
(7) a non-Morsean point if $(a, b) = N = (-\frac{1}{2}, 0)$;
(8) a saddle loop and a pair of complex singularities if (a, b) lies on the segment NO.

The case (1), referred to as the standard elliptic Hamiltonian having the normal form $H(x, y) = y^2 - x^3 + x$, has been extensively studied in [2],[13]-[16],[10]. For the cases (2)-(8), X_H admit invariant straight lines, hence their respective Hamiltonian functions can be transformed into the following normal form

$$H = x\left[y^2 + Ax^2 - 3(A-1)x + 3(A-2)\right], \tag{1.4}$$

where $A \in \mathbf{R}$ is a parameter (see [10]). The cases (2)-(8) in the above correspond to the parameter ranges $A \in (-\infty, -1)$, $A = -1$, $A \in (-1, 0)$, $A = 0$, $A \in (0, 2)$, $A = 2$ and $A \in (2, \infty)$, respectively.

It has been proved in [11] that the cyclicity of period annulus of the Hamiltonian triangle (case (3)) is 3. The cases (1), (4), (5), (7) and (8) are studied in [10] (Theorem 3), [19],[10] (Theorem 4), [18] and [8], respectively, and the cyclicities of all these cases are known to be 2.

In this paper, we will study the remaining cases (6) and (2), i.e., $A \in (0, 2)$ and $A \in (-\infty, -1)$, which respective phase portraits are shown in Figures 2 and 3 respectively.

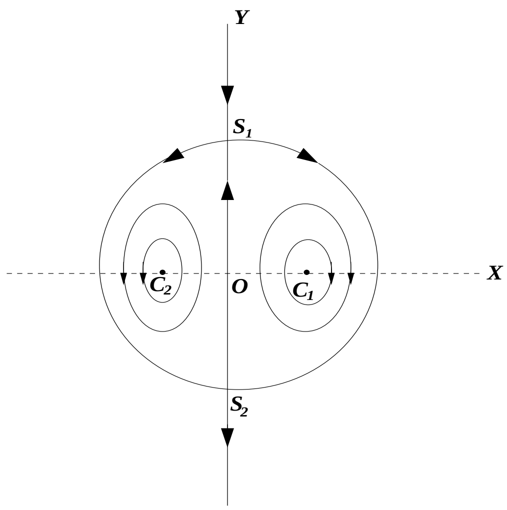


Figure 2 The case $A \in (0, 2)$

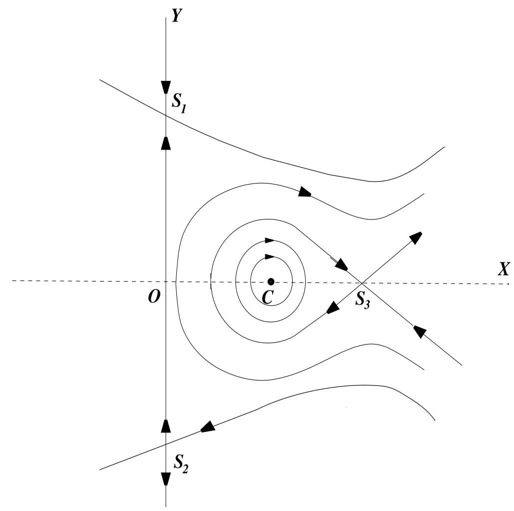


Figure 3 The case $A \in (-\infty, -1)$

We note that the elliptic segment case (6) is the only one in the above list which admits two period annuli.

The main results of the paper are stated as follows.

Theorem 1. *Consider the Hamiltonian H in* (1.4) *with $A \in (0, 2)$. Then $n_1 + n_2 \leq 2$, where n_1 and n_2 denote the cyclicities of the two period annuli of X_H under quadratic perturbations, respectively.*

Theorem 2. *Consider the Hamiltonian H in* (1.4) *with $A \in (-\infty, -1)$. Then the cyclicity of period annulus of X_H under quadratic perturbations equals* 2.

Combining results of [8],[10],[11],[18],[19] with Theorems 1, 2 above, we then conclude the following.

Theorem 3. *For any degenerate quadratic Hamiltonian system, the cyclicity of period annulus* (*or annuli*) *under quadratic perturbations is* 3 *for the triangle case and* 2 *for all other cases.*

Acknowledgment. This work is done when the second and third authors were visiting the National University of Singapore. The second author is partially supported by grants from NSFC and RFDP of China. The third author is partially supported by NSF grant DMS9803581. Also, the authors would like to thank the referee for valuable comments which lead to significant improvements of the paper.

2 Outline of proof

For the reader's convenience, we outline the proof of Theorem 1 in this section. As we will see in Section 6, the proof of Theorem 2 is very similar and in fact even simpler.

Let us first give the precise definition of cyclicity of period annulus. Suppose that h_c is the critical value of H corresponding to the center of X_H, and h_s is the value of H for which the period annulus terminates at a saddle loop (homoclinic or heteroclinic cycle). Without loss of generality we assume that $h_c < h_s$. For any $h \in (h_c, h_s)$, denote by $d(h, \epsilon)$ the displacement function of $X_H + \epsilon Y$, starting from a point on $H^{-1}(h) \cap \gamma$, where γ is a transversal segment to X_H, Y is any polynomial vector field in (x, y) of degree ≤ 2 and $0 < \epsilon \ll 1$. In [10],[11], by considering so-called essential perturbations, I.D. Iliev proved that if X_H is generic, then the displacement function has the form

$$d(h, \epsilon) = \epsilon M_1(h) + O(\epsilon^2),$$

where

$$M_1(h) = \int_{H=h} (\alpha + \beta x + \gamma y) y \, dx, \;\; h \in (h_c, h_s); \tag{2.1}$$

and, if X_H is degenerate, then the additional symmetry of X_H results in a decrease of the number of zeros of $M_1(h)$ and the displacement function (except the Hamiltonian triangle case) has the form

$$d(h, \epsilon) = \epsilon^2 M_2(h) + O(\epsilon^3),$$

where

$$M_2(h) = \int_{H=h} (\alpha + \beta x + \gamma x^{-1}) y \, dx, \;\; h \in (h_c, h_s), \tag{2.2}$$

and α, β and γ are constants.

We note that to obtain the maximal number of limit cycles for the perturbed systems using the expansion form of $d(h, \epsilon)$, one needs to apply the implicit function theorem and make use of the compactness of the interval of h.

Definition 2.1 *For any $\xi \in (h_c, h_s)$, let N_ξ be the maximal number of limit cycles in the compact region $\cup_{h \in [h_c, \xi]} H^{-1}(h)$ which can be bifurcated from X_H under all quadratic perturbations. The* cyclicity of period annulus *of X_H under quadratic perturbations is equal to* $\sup_{\xi \in (h_c, h_s)} N_\xi$.

Remark 2.1 (i) *If the period annulus is bounded by a homoclinic loop, then by a theory of R. Roussarie* ([15])*, the cyclicity of period annulus gives the total number of limit cycles including the ones bifurcated from the homoclinic loop, as pointed out in* [7]*. Hence the result in Theorem 2 can be extended to the boundary of the annulus. However, the result in Theorem 1 needs not be extended to the boundaries of annuli due to the unavailability of a similar theory in the heteroclinic case.*

(ii) *In Definition 2.1, the perturbations are made of all polynomials in (x, y) of degree less than or equal to 2, even if they are tangent to the Hamiltonian stratum.*

(iii) *As we will see in the present cases, for any $\xi \in (h_c, h_s)$, $N_\xi = 2$, hence the cyclicity equals 2.*

2.1 Main ideas

The proof of our results makes use of a result of I.D. Iliev ([12], Theorem 3, case v(i)), which, in term of Definition 2.1, can be rewritten as the following.

Lemma 2.1 *Consider a period annulus of a Hamiltonian H in* (1.4) *with $A \neq -1$ and let h_c and h_s denote the critical values of H as the periodic orbits in the annulus shrink to the center and expand to the separatrix loop, respectively. Then, the cyclicity of the period annulus under quadratic perturbations equals the maximal number of zeros in (h_c, h_s) or (h_s, h_c), counting multiplicity, of the function $M_2(h)$ defined by the Abelian integral* (2.2).

Thus, with this lemma, the proof of Theorems 1 and 2 amounts to a careful analysis of the Abelian integral (2.2). For this purpose, we rewrite (2.2) as

$$I(h) = \alpha I_0(h) + \beta I_1(h) + \gamma I_{-1}(h), \tag{2.3}$$

where $I_k(h) = \int_{\Gamma_h} x^k y dx$, $k = 0, 1, -1$, and, $\Gamma_h = \{(x, y) : H(x, y) = h\}$ for h lying in between h_c and h_s.

Note that if $h \neq h_c$, then $I_0(h)$ is just the area of the region bounded by Γ_h and $I_0'(h)$ is the period of the orbit Γ_h which is monotone ([4]). We thus have

$$I_0(h) I_0'(h) I_0''(h) \neq 0, \qquad h \neq h_c, \tag{2.4}$$

which allows us to define the C^∞ functions

$$P(h) = \frac{I_1(h)}{I_0(h)}, \quad Q(h) = \frac{I_{-1}(h)}{I_0(h)}. \tag{2.5}$$

After a rescaling, we assume that $\gamma = 1$ (see Section 3) and hence the formula (2.3) can be rewritten as

$$I(h) = I_0(h)(\alpha + \beta P(h) + Q(h)). \tag{2.6}$$

We now label the right and left period annulus of H by $i = 1, 2$ respectively (see Figure 2) and consider the respective curves

$$\Sigma_A^{(i)} = \left\{ (P, Q)(h) : h \in (h_c^{(i)}, h_s^{(i)}) \text{ or } (h_s^{(i)}, h_c^{(i)}) \right\}, \quad i = 1, 2, \tag{2.7}$$

in the (P, Q)-plane. It is easy to see that if $h \neq h_c^{(i)}$, then the number of zeros of (2.6) equals the number of intersections of the curve $\Sigma_A^{(i)}$ with the straight line

$$L_{\alpha,\beta} : \quad \alpha + \beta P + Q = 0, \tag{2.8}$$

counting multiplicity. Thus, if $\#(\Sigma_A^{(i)} \cap L_{\alpha,\beta}) = m_i$ denotes the smallest upper bound of the number of intersections of $\Sigma_A^{(i)}$ with $L_{\alpha,\beta}$ for all $\alpha, \beta \in \mathbf{R}$, then Theorem 1 is equivalent to the following.

Theorem 2.1 $m_1 + m_2 \leq 2$, $m_1, m_2 = 0, 1, 2$.

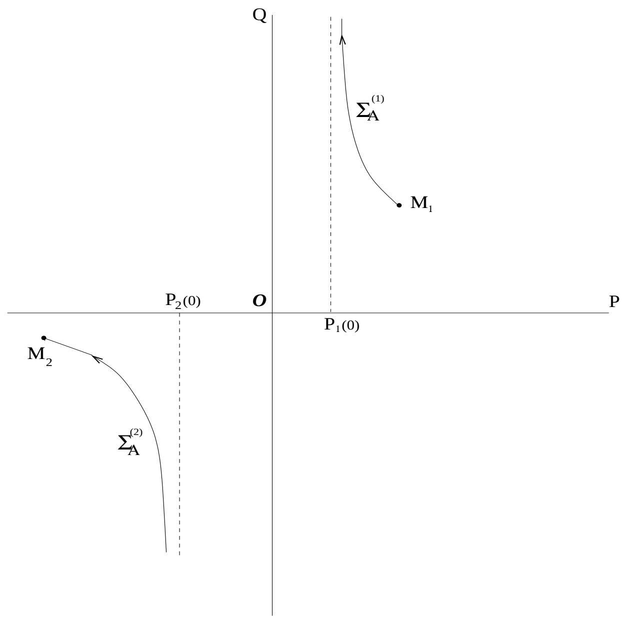


Figure 4 The behavior of $\Sigma_A^{(i)}, i = 1, 2$

This clearly follows from the following theorem.

Theorem 2.2 *The curve $\Sigma_A^{(1)}$ (resp. $\Sigma_A^{(2)}$) is located in the first (resp. third) quadrant, and it is strictly monotone with the vertical asymptote $\{P = P_1(0) > 0\}$ (resp. $\{P = P_2(0) < 0\}$). Moreover, $\frac{dQ}{dP}\Big|_{\Sigma_A^{(i)}} < 0$, $(-1)^{(i+1)} \frac{d^2Q}{dP^2}\Big|_{\Sigma_A^{(i)}} > 0$, for $i = 1, 2$, (see Figure 4).*

Thus, the most important step is to prove the convexity of the curve Σ_A, which will be carried out by showing the following:

- For a given (α, β) the zeros of $I(h)$ is given by the intersection of $L_{\alpha,\beta} \cap \Sigma_A$ (see (2.6)), where Σ_A is parameterized by $h \in (h_c, h_s)$.
- For the same (α, β), the zeros of $I''(h)$ is given by the the intersection of $L_{\alpha,\beta} \cap \Omega_A$, where the curve Ω_A is also parameterized by $h \in (h_c, h_s)$, see (2.11) and (2.12).
- If $L_{\alpha,\beta}$ is tangent to Σ_A at a point $(P, Q)(h_T)$, then $L_{\alpha,\beta}$ must cut Ω_A transversally at a unique point corresponding to $\tilde{h}$, see Figure 5.
- Since $I(h_c) = 0$, we must have $\tilde{h} < h_T$, hence the curvature of Σ_A is everywhere nonzero, see Figure 9.

2.2 Technical ingredients

We now summarize the key steps and technicalities involved in the proof of Theorem 2.2.

First of all, we show that both curves $\Sigma_A^{(i)}$ are regular in the sense that $P'(h) \neq 0$ for $h \neq h_c^{(i)}$, $i = 1, 2$ (see Section 3). The second, it is easy to see that

$$P(h) \to x_c^{(i)}, \quad Q(h) \to \frac{1}{x_c^{(i)}} \quad \text{as } h \to h_c^{(i)}, \tag{2.9}$$

where $x_c^{(i)} \neq 0$ is the x-coordinate of the center in the i-th period annulus for $i = 1, 2$ respectively. It is also easy to see from (1.4) that $h_s^{(1)} = h_s^{(2)} = 0$ and $x_c^{(1)} x_c^{(2)} < 0$ for all $A \in (0, 2)$. Hence, both curves $\Sigma_A^{(1)}$, $\Sigma_A^{(2)}$ are positioned as shown in Figure 4 with the end points $M_1 = (x_c^{(1)}, \frac{1}{x_c^{(1)}})$ and $M_2 = (x_c^{(2)}, \frac{1}{x_c^{(2)}})$ respectively, where the arrows in the figure correspond to the increasing order of the parameter $h \in (h_c^{(1)}, 0) \cup (0, h_c^{(2)})$.

The crucial part of the proof is the convexity of $\Sigma_A^{(i)}$. The study of the global behavior of $\Sigma_A^{(i)}$ is in general difficult since $P(h)$ and $Q(h)$ are given implicitly. As suggested by [6] in studying certain bifurcations for an integrable, non-Hamiltonian quadratic system under quadratic perturbations, one could derive a 3-dimensional system of ordinary differential equations for (h, P, Q), in which the curve Σ becomes the projection of a 1-dimensional stable (or unstable) manifold onto the (P, Q)-plane, and each of its inflection point corresponds to an intersection point of the manifold with the surface in the (h, P, Q)-space governed by the equation $Q''(h)P'(h) - Q'(h)P''(h) = F(h, P, Q) = 0$. Unfortunately, due to the dependence on parameter A, this method is not directly applicable to the present study.

After a rescaling, we can assume without loss of generality that $A \in (0, 1]$ (see Section 3). Since by (2.4) $I_0''(h) \neq 0$, the functions

$$\omega(h) = \frac{I_1''(h)}{I_0''(h)}, \quad \nu(h) = \frac{I_{-1}''(h)}{I_0''(h)} \tag{2.10}$$

are well defined, and hence

$$I''(h) = \alpha I_0''(h) + \beta I_1''(h) + I_{-1}''(h) = I_0''(h)(\alpha + \beta\omega(h) + \nu(h)). \tag{2.11}$$

We now consider the curves

$$\Omega_A^{(i)} = \left\{(\omega, \nu)(h) : h \in (h_c^{(i)}, h_s^{(i)}) \text{ or } (h_s^{(i)}, h_c^{(i)})\right\}, \quad i = 1, 2, \tag{2.12}$$

generated by the new functions ω, ν. If, for $i = 1$ or 2, $\Omega_A^{(i)}$ is strictly convex, which implies that $I''(h)$ has at most two zeros, one then has a control on the number of zeros of $I(h)$. However, difficulties arise in the present case due to the existence of a point of inflection on $\Omega_A^{(2)}$ for all $A \in (0, 1]$ (see Section 5).

As in [5], we will study the curves $\Sigma_A^{(i)}$ and $\Omega_A^{(i)}$ simultaneously by identifying the (P, Q)-plane with the (ω, ν)-plane. Consider

$$L_T^{(i)} = \left\{L_{\alpha,\beta} : \ L_{\alpha,\beta} \text{ is tangent to } \Sigma_A^{(i)} \text{ at a point on } \Sigma_A^{(i)}\right\}. \tag{2.13}$$

The key point is to show that any $L_{\alpha,\beta} \in L_T^{(i)}$ cuts $\Omega_A^{(i)}$ at a unique point, and the crossing is transversal. This will imply that the curvature of $\Sigma_A^{(i)}$ is everywhere non-zero. For otherwise, some $L_{\alpha,\beta} \in L_T^{(i)}$ would be tangent to $\Sigma_A^{(i)}$ at a point with tangency at least three. Then $I(h)$ must have at least four zeros, counting the multiplicity, since $I(h_c^{(i)}) = 0$. Hence $I''(h)$ admits at least two zeros for $h \neq h_c^{(i)}$, i.e., $L_{\alpha,\beta}$ cuts $\Omega_A^{(i)}$ at more than one point, a contradiction.

The analysis of course depends on the geometric nature of $\Omega_A^{(i)}$. At first glance, the study of $\Omega_A^{(i)}$ seems no easier than that of $\Sigma_A^{(i)}$. Nevertheless, $\nu(h)$ can be expressed as a function of h and $\omega(h)$, and $\omega(h)$ satisfies a Riccati equation. In this way, the study of $\Omega_A^{(i)}$ can be made within a 2-dimensional system. The upshot is that we can re-write (2.11) as

$$I''(h) = kI_0''(h)(h - h^*)(\omega(h) - U(h)), \tag{2.14}$$

where k, h^* are constants depending on A, α and β, and, $U(h)$ is a ratio of two linear functions of h which also depends on A, α and β. Let $C_\omega^{(i)}$ and C_U be curves in the (h, ω)-plane defined by $\omega = \omega(h)$ and $\omega = U(h)$ respectively. We further show that for $L_{\alpha,\beta} \in L_T^{(1)}$ and $A \in (0, 1]$, $C_\omega^{(1)}$ is always strictly increasing and it intersects only one branch of C_U, which is always strictly decreasing, and, they do not intersect at h^* (see Section 5). It follows that $I''(h)$ admits a unique zero, i.e., $L_{\alpha,\beta} \in L_T^{(1)}$ cuts $\Omega_A^{(1)}$ at a unique point, and the crossing is transversal. This shows the convexity of $\Sigma_A^{(1)}$.

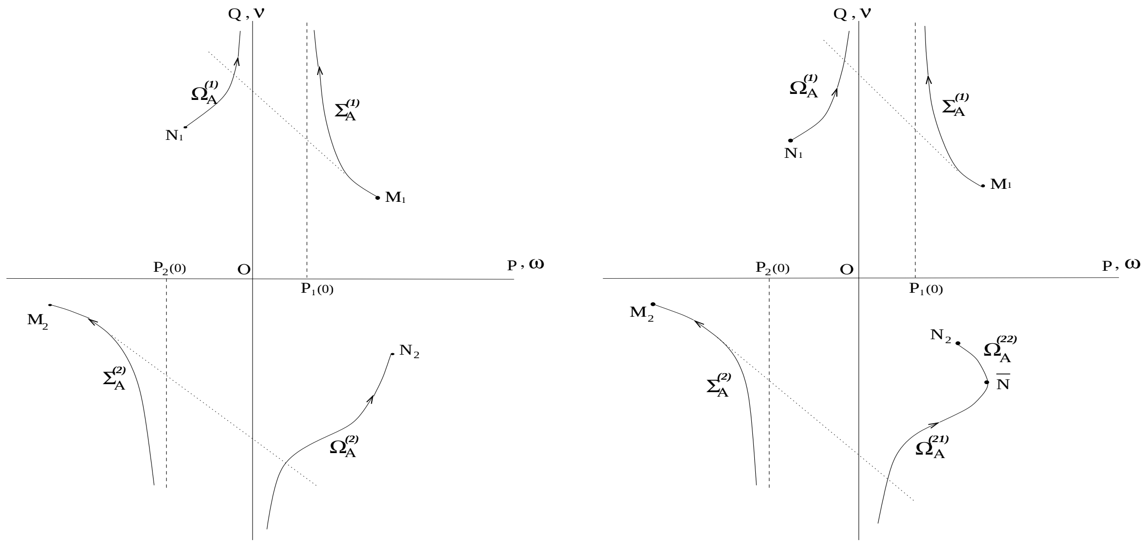
The study of $\Sigma_A^{(2)}$ varies with respect to two parameter ranges of A. It will be shown in Section 5 that there is a value $\bar{A} \approx 0.8177$, which is the unique root of the equation

$$7A^3 - 63A^2 + 141A - 77 = 0 \tag{2.15}$$

for $A \in (-\infty, -1) \cup (0, 1]$ such that the behavior of $\Omega_A^{(2)}$ (as well as $\Omega_A^{(1)}$) is as shown in Figures 5(a) and 5(b), depending on whether $A \geq \bar{A}$ or $A < \bar{A}$.

More precisely, if $A \in [\bar{A}, 1]$, then the analysis is completely similar to that of $\Sigma_A^{(1)}$. If $A \in (0, \bar{A})$, then $C_\omega^{(2)}$ has a unique maximum which corresponds to the turning point $\bar{N} \in \Omega_A^{(2)}$ and divides $\Omega_A^{(2)}$ into two parts $\Omega_A^{(21)} \cup \Omega_A^{(22)}$ with $\Omega_A^{(22)}$ being convex. Moreover, the upper end point N_2 of $\Omega_A^{(2)}$ precisely meets the tangent line of $\Sigma_A^{(2)}$ at M_2 and is always above all other $L_{\alpha,\beta} \in L_T^{(2)}$, while the lower end point of $\Omega_A^{(2)}$ is always below any $L_{\alpha,\beta} \in L_T^{(2)}$ since $(\omega, \nu)(h) \to (0, -\infty)$ as $h \to 0$. Thus, $\#(L_{\alpha,\beta} \cap \Omega_A^{(21)}) \leq 1$ and $\#(L_{\alpha,\beta} \cap \Omega_A^{(22)}) \leq 1$, which imply that $\#(L_{\alpha,\beta} \cap \Omega_A^{(2)}) = 1$ since this number must be odd.

The convexity of $\Omega_A^{(22)}$ will be shown using an argument of variation of the parameter A. The ideas are as follows. We first show the convexity of $\Omega_A^{(22)}$ near N_2 for all $A \in (0, \bar{A})$ (it is obvious that $\Omega_A^{(22)}$ is always convex near $\bar{N}$), and, for a particular A, say $A = \frac{1}{2}$, $\Omega_{1/2}^{(22)}$ is globally convex. If there is a A' such that $\Omega_{A'}^{(22)}$ is not convex, then it must admit at least two points of inflection. Varying A from $\frac{1}{2}$ to A', we then find a $\hat{A} \in (\frac{1}{2}, A')$ such that $\Omega_{\hat{A}}^{(22)}$ has a quadruple or higher order degenerate point. But as both X_H and the perturbation are quadratic, this is impossible (see Section 5 for more details).



(a) The case $A \in [\bar{A}, 1]$

(b) The case $A \in (0, \bar{A})$

Figure 5 The behavior of $\Omega_A^{(i)}$

3 Basic properties

3.1 The monotonicity of $P(h)$

Lemma 3.1 *For $H(x, y)$ in (1.4), we have that $P'(h) \neq 0$ for all $h \in (h_c^{(i)}, h_s^{(i)})$ or $(h_s^{(i)}, h_c^{(i)})$.*

Proof: Consider a perturbation of X_H as follows

$$\begin{cases} \dot{x} = 2xy, \\ \dot{y} = -3Ax^2 + 6(A-1)x - 3(A-2) - y^2 + \delta(\alpha + \beta x)y, \end{cases} \tag{3.1}$$

where α, β are constants, δ is a small parameter. Since the quadratic system (3.1) has an invariant line $\{x = 0\}$, it has at most one limit cycle, and, this limit cycle must be hyperbolic, if it exists ([3],[17]). Hence the corresponding Abelian integral

$$I(h) = \alpha I_0(h) + \beta I_1(h) = I_0(h)(\alpha + \beta P(h)) \tag{3.2}$$

has at most one zero, counting multiplicity, i.e., $P'(h) \neq 0$. ■

3.2 Rescaling of parameters

Using rescaling, we can make $\gamma = 1$ and $A \in (0, 1]$.

By using Lemma 3.1 and comparing (2.3) with (3.2), we see that if $\gamma = 0$, then the Abelian integral $I(h)$ has at most one zero, which is simple if it exists. So throughout the rest of the paper, we will assume that $\gamma \neq 0$ and re-scale it to 1. The fact that $\gamma \neq 0$ can be explained from a different point of view. If we add a perturbation term $\delta x^{-1}y$ to the second equation of (3.1), then the bifurcation diagram in (α, β, γ)-space has a cone-like structure (see [6], Section 2). Therefore, it is sufficient to consider the intersection of the bifurcation diagram with the half sphere $\{\alpha^2 + \beta^2 + \gamma^2 = 1, \gamma \geq 0\}$. Then the equator $\gamma = 0$ corresponds to the heteroclinic (and homoclinic) bifurcation – a case which we do not treat in this paper.

For $A \in (1, 2)$, we consider the following rescaling:

$$x = \frac{A-2}{A}u, \quad y = \sqrt{\frac{2-A}{A}}v, \quad dt = \sqrt{\frac{A}{2-A}}d\tau.$$

Then the form of the unperturbed part of (3.1) remains unchanged if we replace $A \in (1, 2)$ by $2 - A \in (0, 1)$, and (x, y, t) by (u, v, τ). Therefore, throughout the rest of paper, we will assume without loss of generality that $A \in (0, 1]$.

3.3 The Picard-Fuchs and Riccati equations

By using a standard procedure (see [10]), we have the following Picard-Fuchs equation for $(I_0(h), I_1(h), I_{-1}(h))$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4A & 0 \\ 0 & -3(A-1) & 2A \end{pmatrix} \begin{pmatrix} I_{-1} \\ I_0 \\ I_1 \end{pmatrix} = M \begin{pmatrix} I'_{-1} \\ I'_0 \\ I'_1 \end{pmatrix}, \tag{3.3}$$

where

$$M = \begin{pmatrix} 3h & -6(A-2) & 3(A-1) \\ -3(A-1)h & 6Ah & -3(A-3)(A+1) \\ 2(A-2)h & -4(A-1)h & 2Ah \end{pmatrix}.$$

A straightforward computation shows that (3.3) is equivalent to

$$G(h)\frac{d}{dh}\begin{pmatrix} I_0 \\ I_1 \\ I_{-1} \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} I_0 \\ I_1 \\ I_{-1} \end{pmatrix}, \tag{3.4}$$

where

$$\begin{aligned} G(h) &= 6h(h-(A-3))(A^2h-(A+1)(A-2)^2), \\ a_{00} &= 4A^2h^2 - 2(A-1)(2A^2-4A-9)h, \end{aligned}$$

$$
\begin{aligned}
a_{01} &= -12Ah, \\
a_{02} &= A(A-1)h^2 - (A+1)(A-2)(A-3)h, \\
a_{10} &= -A(A-1)h^2 + (A-2)(A^2-2A+9)h, \\
a_{11} &= 6A^2h^2 - 6A(A-1)(A-2)h, \\
a_{12} &= 2h^2, \\
a_{20} &= A(9A^2-18A+1)h - 9(A^2-1)(A-2)(A-3), \\
a_{21} &= -6A^2(A-1)h + 6A(A+1)(A-2)(A-3), \\
a_{22} &= 2A^2h^2 - 2(A^2-1)(A-3)h.
\end{aligned}
$$

Using (2.5) and (3.4), we then obtain the following differential equations for (h, P, Q):

$$
\begin{cases}
\dot{h} = G(h), \\
\dot{P} = a_{10} + a_{11}P + a_{12}Q - P(a_{00} + a_{01}P + a_{02}Q), \\
\dot{Q} = a_{20} + a_{21}P + a_{22}Q - Q(a_{00} + a_{01}P + a_{02}Q).
\end{cases}
\tag{3.5}
$$

Differentiating (3.3) with respect to h, we have

$$
\begin{pmatrix} -2 & 0 & 0 \\ 3(A-1) & -2A & 0 \\ -2(A-2) & (A-1) & 0 \end{pmatrix}
\begin{pmatrix} I'_{-1} \\ I'_0 \\ I'_1 \end{pmatrix}
= M \begin{pmatrix} I''_{-1} \\ I''_0 \\ I''_1 \end{pmatrix}
\tag{3.6}
$$

in which the variable I'_1 is already eliminated. By further removing I'_{-1} and I'_0 in (3.6), we then have

$$
I''_{-1} = \left(-\frac{4A(A-1)}{(A+1)(A-3)} + \frac{6(A-2)}{h}\right)I''_0 + \left(\frac{8A^2}{(A+1)(A-3)} - \frac{9(A-1)}{h}\right)I''_1,
\tag{3.7}
$$

which, when substituting into (2.10), yields

$$
\nu(h) = \frac{(8A^2h - 9(A^2-1)(A-3))\omega(h) - 4A(A-1)h + 6(A+1)(A-2)(A-3)}{(A+1)(A-3)h}.
\tag{3.8}
$$

By differentiating (3.6) with respect to h one more time and also substituting (3.7) back to (3.6), we obtain the following equation for (I''_0, I''_1):

$$
T(h)\frac{d}{dh}\begin{pmatrix} I''_0 \\ I''_1 \end{pmatrix} = \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix}\begin{pmatrix} I''_0 \\ I''_1 \end{pmatrix},
\tag{3.9}
$$

where

$$
\begin{aligned}
T(h) &= (A+1)(A-3)G(h), \\
b_{00} &= -4A^2(3A^2-6A-5)h^2 + 6(A^2-1)(A-3)(3A^2-6A-5)h \\
&\qquad - 6(A+1)^2(A-2)^2(A-3)^2, \\
b_{01} &= 8A^3(A-1)h^2 - A(A+1)(A-3)(17A^2-34A-3)h \\
&\qquad + 9(A-1)(A-2)(A+1)^2(A-3)^2, \\
b_{10} &= -A(A-1)(A^2-2A+5)h^2 + (A-2)(A+1)^2(A-3)^2h, \\
b_{11} &= -2A^2(3A^2-6A-17)h^2 + 6(A-1)(A+1)^2(A-3)^2h.
\end{aligned}
$$

Finally, from (2.10) and (3.9), we obtain the following Riccati equation for (h, ω):

$$\begin{cases} \dot{h} = T(h), \\ \dot{\omega} = -b_{01}\omega^2 + (b_{11} - b_{00})\omega + b_{10} = \phi(h, \omega). \end{cases} \tag{3.10}$$

4 Local behaviors of $\Sigma_A^{(i)}$ and $\Omega_A^{(i)}$

In the sequel, for simplicity, we sometimes suspend the up-script (i) on Σ_A, Ω_A, $P(h)$, $Q(h)$, etc. if there is no confusion.

Consider the following equation associated to the Hamiltonian (1.4):

$$\begin{cases} \dot{x} = 2xy, \\ \dot{y} = -3(A-2) + 6(A-1)x - 3Ax^2 - y^2. \end{cases} \tag{4.1}$$

For $A \in (0, 1]$, the system (4.1) clearly has two centers $C_1 = (1, 0)$ and $C_2 = (\frac{A-2}{A}, 0)$, two saddle points $S_1 = (0, \sqrt{3(2-A)})$ and $S_2 = (0, -\sqrt{3(2-A)})$ (see Figure 2). In this case, we have $h_{S_1} = h_{S_2} = 0$, and,

$$h_1 = h_{C_1} = A - 3 < 0, \quad h_2 = h_{C_2} = \frac{(A+1)(A-2)^2}{A^2} > 0. \tag{4.2}$$

It follows from (2.9) that

$$\begin{aligned} &(P(h), Q(h)) \to (1, 1), && \text{as } h \to h_1; \\ &(P(h), Q(h)) \to \left(\frac{A-2}{A}, \frac{A}{A-2}\right), && \text{as } h \to h_2. \end{aligned} \tag{4.3}$$

It is not difficult to see that the linearized matrix of (3.5) at the singularity $(h, P, Q) = (h_1, 1, 1)$ reads

$$B = 24(-A+3) \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{12}(A+1) & -1 & 0 \\ -\frac{1}{12}(A+3) & 0 & -1 \end{pmatrix}$$

which clearly has a simple eigenvalue $24(-A+3)$. We are interested in the unstable manifold $W^u = \{(h, P^{(1)}(h), Q^{(1)}(h))\}$ of (3.5) at the singularity $(h, P, Q) = (h_1, 1, 1)$ associated to the simple eigenvalue $24(-A+3)$, as $\Sigma_A^{(1)}$ is just the projection of W^u onto the (P, Q)-plane. Consider the expansions

$$\begin{aligned} h &= h_1 + t, \\ P &= 1 + a_1 t + \frac{1}{2} a_2 t^2 + \cdots, \\ Q &= 1 + b_1 t + \frac{1}{2} b_2 t^2 + \cdots \end{aligned}$$

and substitute them into (3.5). We then obtain $(a_1, b_1), (a_2, b_2)$ successively, which gives that

$$\begin{aligned} P'(h_1) &= a_1 = -\tfrac{1}{24}(A+1), \\ Q'(h_1) &= b_1 = \tfrac{1}{24}(A+3), \\ P''(h_1) &= a_2 = -\tfrac{1}{10368}(A+1)(55A^2 - 18A + 63), \\ Q''(h_1) &= b_2 = \tfrac{1}{10386}(55A^3 + 147A^2 + 369A + 549). \end{aligned} \tag{4.4}$$

Hence,

$$\begin{aligned} \left.\frac{dQ}{dP}\right|_{h_1} &= \frac{Q'(h_1)}{P'(h_1)} = -\frac{A+3}{A+1}, \\ \left.\frac{d^2Q}{dP^2}\right|_{h_1} &= \frac{Q''(h_1)P'(h_1) - Q'(h_1)P''(h_1)}{(P'(h_1))^3} = \frac{20}{A+1}. \end{aligned}$$

A similar analysis can be made at the singularity $(h_2, \frac{A-2}{A}, \frac{A}{A-2})$.

Summarizing up, we have the following result.

Lemma 4.1 *For $A \in (0, 1]$ and $(P, Q)(h) \in \Sigma_A^{(i)}$, $i = 1, 2$, we have*

$$\left.\frac{dQ}{dP}\right|_{h_1} = -\frac{A+3}{A+1}, \quad \left.\frac{d^2Q}{dP^2}\right|_{h_1} = \frac{20}{A+1},$$

$$\left.\frac{dQ}{dP}\right|_{h_2} = -\frac{A^2(A-5)}{(A-3)(A-2)^2}, \quad \left.\frac{d^2Q}{dP^2}\right|_{h_2} = -\frac{20A^3}{(A-3)(A-2)^3}.$$

We now turn to system (3.10). There are six distinct singularities: three saddles (h_1, ω_1), $(0, \omega_0)$, (h_2, ω_2) and three nodes $(h_1, 1)$, $(0, 0)$, (h_2, ω_2'), where

$$\begin{cases} \omega_1 = \dfrac{5A^2 - 6A - 3}{5A^2 + 6A + 9}, & \omega_0 = \dfrac{2(A-2)}{3(A-1)}, \\ \omega_2 = \dfrac{(A-2)(5A^2 - 14A + 5)}{A(5A^2 - 26A + 41)}, & \omega_2' = \dfrac{A-2}{A}. \end{cases} \tag{4.5}$$

Clearly, $\omega_1 < 0, \omega_0 > 0, \omega_2' < 0$ and $\omega_2 - \omega_2' > 0$ for all $A \in (0, 1]$. Since as $h \to h_i$ the level curve Γ_h shrinks to the center C_i, $I_0'''(h)$ is bounded for h near h_i. Let $h \to h_i$. It follows from the first equation of (3.9) that

$$\lim_{h \to h_i} \omega(h) = \lim_{h \to h_i} -\frac{b_{00}}{b_{01}} = \omega_i, \; i = 1, 2. \tag{4.6}$$

Hence, $C_\omega^{(1)} = \{(h, \omega(h)) : h \in (h_1, 0)\}$ (resp. $C_\omega^{(2)} = \{(h, \omega(h)) : h \in (0, h_2)\}$) is the stable manifold of (3.10) at the saddle point (h_1, ω_1) (resp. (h_2, ω_2)).

By using a similar argument as for Lemma 4.1, we have the following.

Lemma 4.2 *For all $A \in (0,1]$ and $(h,\omega)(h) \in \Omega_A^{(i)}$, $i = 1,2$,*

$$\begin{aligned}
\omega'(h_1) &= \frac{5(A+1)(A-3)p_{11}(A)}{24(5A^2+6A+9)^2} > 0,\\
\omega''(h_1) &= \frac{5(A+1)(A-3)p_{12}(A)}{10368(5A^2+6A+9)^3},\\
\omega'(h_2) &= \frac{5A(A+1)(A-3)p_{21}(A)}{24(A-2)(5A^2-26A+41)^2},\\
\omega''(h_2) &= -\frac{5A^3(A+1)(A-3)p_{22}(A)}{10368(A-2)^3(5A^2-26A+41)^3},
\end{aligned}$$

where $p_{21}(A)$ is the left hand side of (2.15), and

$$\begin{aligned}
p_{11}(A) &= 7A^3 + 21A^2 - 27A - 9,\\
p_{12}(A) &= 1925A^7 + 8967A^6 + 31689A^5 + 27027A^4 - 85617A^3 - 41067A^2\\
&\quad - 11421A - 5103,\\
p_{22}(A) &= 1925A^7 - 35917A^6 + 300993A^5 - 1420937A^4 + 3910879A^3 - 6074679A^2\\
&\quad + 4780955A - 1389619.
\end{aligned}$$

A straightforward computation using Lemma 4.2 and (3.8) yields the following.

Lemma 4.3 *For all $A \in (0,1]$ and $(\omega,\nu)(h) \in \Omega_A^{(i)}$, $i = 1,2$,*

$$\begin{aligned}
\left.\frac{d\nu}{d\omega}\right|_{h_1} &= -\frac{q_{11(A)}}{(A+1)(A-3)p_{11}(A)} > 0,\\
\left.\frac{d^2\nu}{d\omega^2}\right|_{h_1} &= -\frac{28(5A^2+6A+9)^3 q_{12}(A)}{5(A+1)(A-3)(p_{11}(A))^3}\\
\left.\frac{d\nu}{d\omega}\right|_{h_2} &= -\frac{A^2 q_{21}(A)}{(A+1)(A-3)(A-2)^2 p_{21}(A)},\\
\left.\frac{d^2\nu}{d\omega^2}\right|_{h_2} &= \frac{28A^3(5A^2-26A+41)^3 q_{22}(A)}{5(A+1)(A-3)(A-2)^3(p_{21}(A))^3},
\end{aligned}$$

where

$$\begin{aligned}
q_{11}(A) &= 7A^5 + 21A^4 - 450A^3 - 990A^2 - 837A - 567,\\
q_{12}(A) &= 55A^4 + 204A^3 + 162A^2 - 324A - 81,\\
q_{21}(A) &= 7A^5 - 91A^4 - 2A^3 + 2626A^2 - 8965A + 9241,\\
q_{22}(A) &= 55A^4 - 644A^3 + 2076A^2 - 4532A + 2431.
\end{aligned}$$

Remark 4.1 For $A \in (-\infty, -1) \cup (0, 1]$, the polynomials p_{ij}, q_{ij} have real roots approximately as follows, $p_{11}(A)$: $\widetilde{A} \approx -3.9037$ and -0.2785; $p_{12}(A)$: -2.563 and -0.4462; $p_{22}(A)$: 0.6422; $q_{11}(A)$: -8.6919 and -1.466; $q_{12}(A)$: -0.2306 and 0.9802; $q_{21}(A)$: -5.814; $q_{22}(A)$ has no real root in this region.

5 Global behaviors of $C_\omega^{(i)}$ and $\Sigma_A^{(i)}$

We first study some global properties of

$$C_\omega^{(1)} \cup C_\omega^{(2)} = \{(h, \omega(h)) : h \in (h_1, 0) \cup (0, h_2)\}.$$

Observe that the function ϕ defined in (3.10) is a polynomial of h and ω both of order 2, and moreover,

$$\begin{array}{l} \phi(h_1, \omega) = k_1(\omega - 1)(\omega - \omega_1), \\ \phi(0, \omega) = k_2\omega(\omega - \omega_0), \\ \phi(h_2, \omega) = k_3(\omega - \omega_2)(\omega - \omega_2'), \end{array} \tag{5.1}$$

where k_i, $i = 1, 2, 3$, are non-zero constants depending on A. Hence each branch of the 0-clines (i.e. the curves $(h, \omega_0(h))$ in (h, ω)-plane defined by $\phi(h, \omega) = 0$) of (3.10) meets the lines $\{h = h_1\}$, $\{h = 0\}$ and $\{h = h_2\}$ only at the singularities of (3.10), and the crossings are all transversal. From (3.10), we also see that, for $h \in (h_1, h_2)$,

$$\begin{array}{rcll} \phi(h, 0) & = & k_1^* h(h - h_1^*), & h_1^* > 0, \\ \phi(h, \omega_1) & = & k_2^*(h - h_1)(h - h_2^*), & h_2^* \notin (h_1, 0), \\ \phi(h, \omega_0) & = & k_3^* h(h - h_3^*), & h_3^* < 0, \\ \phi(h, \frac{\omega_2' h}{2h_2}) & = & k_4^* h\Phi(h, A), & \end{array} \tag{5.2}$$

where k_i^*, $i = 1, 2, 3, 4$, are non-zero constants, $\Phi(h, A) = m_3h^3 + m_2h^2 + m_1h + m_0$, and,

$$\begin{array}{rcl} m_3 & = & 8A^5(A - 1) < 0, \\ m_2 & = & -A^3(A + 1)(29A^3 - 133A^2 + 175A - 47), \\ m_1 & = & A(A - 1)(A - 2)(A + 1)^2(37A^3 - 190A^2 + 237A + 32) > 0, \\ m_0 & = & -4(4A + 1)(A - 3)^2(A + 1)^3(A - 2)^3 > 0. \end{array}$$

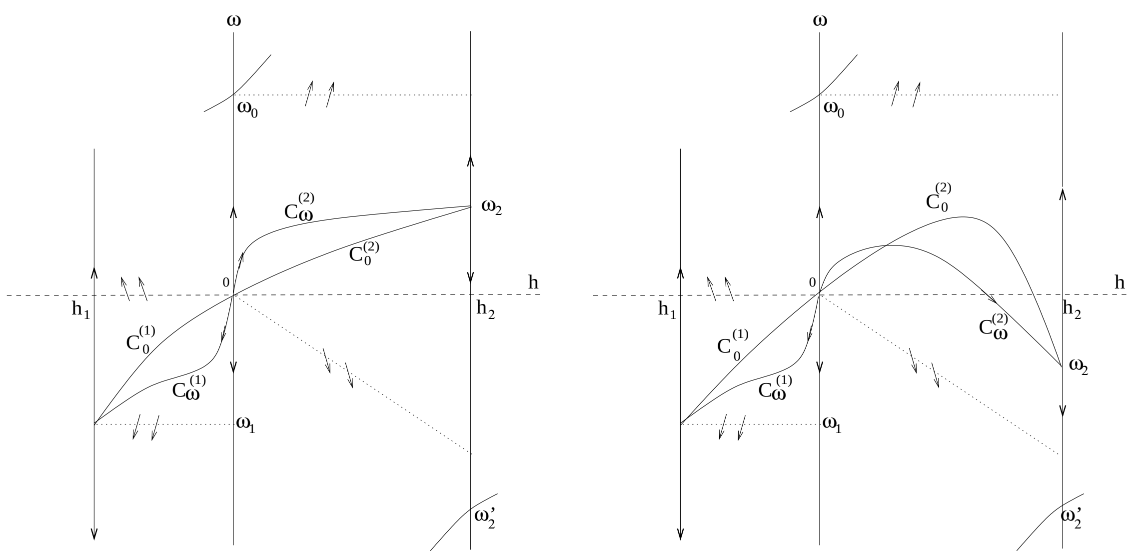
We claim that $\Phi(h, A) \neq 0$ for $h \in (0, h_2)$. Since $\Phi(0, A) = m_0 > 0$ and $\Phi(h_2, A) = \frac{2}{A}(A + 1)^3(A - 2)^3(5A^2 - 2A - 31) > 0$, $\Phi(h, A)$ must admit an even number of zeros for $h \in (0, h_2)$, and, since $\Phi(h_2, A) > 0$ and $m_3 < 0$ there is a zero of $\Phi(h, A)$ for $h \in (h_2, \infty)$. Thus, if $\Phi(h, A)$ has two zeros for $h \in (0, h_2)$, then $\Phi_h'(h, A)$ should have at least two zeros for $h \in (0, \infty)$, a contradiction to the fact that $m_1m_3 < 0$.

It is not difficult to find that the slopes of the 0-clines of (3.10) at the singularities (h_1, ω_1), $(0, 0)$ and (h_2, ω_2) are

$$\omega_{01}' = 2\omega'(h_1) > 0, \quad \omega_{00}' > 0, \quad \omega_{02}' = 2\omega'(h_2) \tag{5.3}$$

respectively, where $\omega'(h_1)$ and $\omega'(h_2)$ are as in Lemma 4.2.

The first two formulas in each of (5.1)–(5.3) together imply that there is a branch $C_0^{(1)}$ of the 0-clines of system (3.10) connecting the points (h_1, ω_1) and (0,0). Since $\omega_1 < 0$ and $\omega'_{01} > 0$, we claim that $C_0^{(1)}$ must be strictly increasing with respect to h for all $A \in (0, 1]$. For otherwise, we would find a horizontal line $\{\omega = \overline{\omega}\}$ which cuts $C_0^{(1)}$ for at least three times. This contradicts the fact that $\phi(h, \omega)$ is a polynomial of h of order 2. Similarly, the last two formulas in each of (5.1)–(5.3) together imply the existence of a branch $C_0^{(2)}$ of the 0-clines of (3.10) which connects the singularities $(0, 0)$ and (h_2, ω_2). From (4.5) we see that $\omega'_2 < 0$ and $\omega_2 - \frac{\omega'_2}{2} > 0$. By the same argument as above, we conclude that if $A \geq \bar{A}$, where $\bar{A} \approx 0.8177$ is the unique root of $p_{21}(A)$ for $A \in (0, 1]$, then $C_0^{(2)}$ is also strictly increasing; and, if $A < \bar{A}$, then $C_0^{(2)}$ has a unique extremum point, which must be a maximum point, see Figures 6(a) and 6(b).



(a) The case $A \in [\bar{A}, 1]$ (b) The case $A \in (0, \bar{A})$

Figure 6 The behavior of $C_0^{(i)}$ and $C_\omega^{(i)}$

Now, by using (5.2), the first and third formulas of (5.3), the monotone property of $C_0^{(i)}$, and the fact that $(0, 0)$ is an improper node of (3.10), we immediately obtain the following.

Lemma 5.1 $C_\omega^{(1)}$ *is the stable manifold of* (3.10) *from the saddle* (h_1, ω_1) *to the node* $(0, 0)$ *which stays in the region* $\{(h, \omega) : h_1 < h < 0,\ \omega_1 < \omega < 0\}$ *and is strictly increasing with respect to* h *for all* $A \in (0, 1]$*;* $C_\omega^{(2)}$ *is the stable manifold of* (3.10) *from the saddle* (h_2, ω_2) *to the node* $(0, 0)$ *which stays in the region* $\left\{(h, \omega) : 0 < h < h_2,\ \frac{\omega_2 h}{2h_2} < \omega < \omega_0\right\}$, *is strictly increasing with respect to* h *for all* $A \in [\bar{A}, 1]$, *and has a unique extremum (in* *fact, maximum) point for all* $A \in (0, \bar{A})$.

Next, substituting (3.8) into (2.11), we have that

$$I''(h) = \frac{I_0''(h)}{h}(\xi h + 6(A-2) + (\eta h - 9(A-1))\omega(h)), \tag{5.4}$$

where

$$\xi = \alpha - \frac{4A(A-1)}{(A+1)(A-3)}, \quad \eta = \beta + \frac{8A^2}{(A+1)(A-3)}. \tag{5.5}$$

If

$$L_0 = 3(A-1)\alpha + 2(A-2)\beta + 4A = 0, \tag{5.6}$$

then

$$\xi h + 6(A-2) = -\frac{2(A-2)}{3(A-1)}(\eta h - 9(A-1)),$$

and, (5.4) becomes

$$I''(h) = \frac{I_0''(h)}{h}(\eta h - 9(A-1))(\omega(h) - \omega_0), \tag{5.7}$$

where ω_0 is as in (4.5). By Lemma 5.1, $(C_\omega^{(1)} \cup C_\omega^{(2)}) \cap \{(h,\omega) : \omega = \omega_0\} = \emptyset$. Hence, by (5.7), if $\eta = 0$, then $I''(h) \neq 0$ for all $(h,\omega) \in C_\omega^{(1)} \cup C_\omega^{(2)}$; and, if $\eta \neq 0$ and $I''(h^*) = 0$, where

$$h^* = \frac{9(A-1)}{\eta} = \frac{9(A^2-1)(A-3)}{(A+1)(A-3)\beta + 8A^2}, \tag{5.8}$$

then $h = h^*$ is the unique zero of $I''(h)$ which is also simple.

If $L_0 \neq 0$ and $(\eta h - 9(A-1)) = 0$, then $\xi h + 6(A-2) \neq 0$, and $I''(h) \neq 0$. In the case that $(\eta h - 9(A-1)) \neq 0$, i.e. $h = h^*$ is not a zero of $I''(h)$ for $\eta \neq 0$, we rewrite (5.4) as

$$I''(h) = \frac{1}{h}I_0''(h)(\eta h - 9(A-1))(\omega(h) - U(h)), \tag{5.9}$$

where

$$U(h) = -\frac{\xi h + 6(A-2)}{\eta h - 9(A-1)}. \tag{5.10}$$

We note that

$$U'(h) = \frac{3(A+1)^2(A-3)^2 L_0}{(\eta h - 9(A-1))^2}, \tag{5.11}$$

where L_0 is as in (5.6). Let $C_U = \{(h,\omega) : \omega = U(h)\}$. If $\eta = 0$, then C_U is a straight line, and, if $\eta \neq 0$, then C_U consists of two strictly monotone branches which admit the same vertical asymptote $\{h = h^*\}$ and the same horizontal asymptote $\{\omega = \omega^*\}$, where

$$\omega^* = -\frac{\xi}{\eta} = -\frac{(A+1)(A-3)\alpha - 4A(A-1)}{(A+1)(A-3)\beta + 8A^2}. \tag{5.12}$$

It is clear that $\#(L_{\alpha,\beta} \cap \Omega_A^{(i)}) = \#(C_\omega^{(i)} \cap C_U)$ for $L_{\alpha,\beta} \in L_T^{(i)}$, and this number is controlled by the number of points on C_U at which the vector field (3.10) is tangent to C_U. Thus, we need to consider the zeros of $(-U'(h),1)\cdot(\dot{h},\dot{\omega})$ along C_U. Note that, by (3.10) and (5.10), we have

$$\dot{\omega}-U'(h)\dot{h}|_{\omega=U(h)}=\frac{h(n_3h^3+n_2h^2+n_1h+n_0)}{(\eta h-9(A-1))^2}, \tag{5.13}$$

where n_j, $j=0,1,2,3$, are constants depending on A,α and β.

Lemma 5.2 *For $A=\frac{1}{2}$, the curve $\Omega^{(2)}_{1/2}$ has a unique point of inflection.*

Proof: By taking $A=\frac{1}{2}$ in (4.2), (4.5) and (3.8), we have that $h_2=\frac{27}{2}$, $\omega_0=2$, $\omega_2=\frac{1}{13},\omega_2'=-3$, and

$$\nu(h)=V(h,\omega(h))=-\frac{8h+270+(16h-135)\omega(h)}{30h}. \tag{5.14}$$

Hence by (3.10), $(T(h))^2(\nu''(h)\omega'(h)-\omega''(h)\nu'(h))$ can be expressed as

$$TV_{hh}\phi+2V_{h\omega}\phi^2-V_h(T\phi_h+(\phi_\omega-T_h)\phi)=\frac{45\lambda(h,\omega(h))}{2048h^2},$$

where $\lambda(h,\omega)=l_3\omega^3+l_2\omega^2+l_1\omega+l_0$, and,

$$\begin{aligned}
l_3 &= 1024h^4+29916h^3+524880h^2-6506325h+7381125,\\
l_2 &= 4416h^4+122364h^3+1125900h^2+7709175h-29524500,\\
l_1 &= 48h^4+182052h^3+2157840h^2+13067325h+29524500,\\
l_0 &= -2h(476h^3-16056h^2+181035h+2460375).
\end{aligned}$$

Let

$$C_\lambda=\{(h,\mu(h)):\mu=\mu(h)\text{ is defined by }\lambda(h,\mu)=0\text{ for }h\in(0,h_2)\}.$$

Then the number of points of inflection of $\Omega^{(2)}_{1/2}$ equals $\#(C^{(2)}_\omega\cap C_\lambda)$. Since

$$\begin{aligned}
\lambda(0,\omega) &= 7381125\omega(\omega-2)^2,\\
\lambda(\frac{27}{2},\omega) &= 9447840(13\omega-1)(\omega+3)^2,\\
\lambda(h,2) &= 6250h^2(2h+45)^2,\\
\lambda(h,0) &= -2h(476h^3-16056h^2+181035h+2460375)\neq 0,\quad \text{for } h\in(0,\frac{27}{2}),
\end{aligned}$$

it follows from Lemma 5.1 and the same analysis as we did for the 0-clines of (3.10) that there is a unique branch of C_λ in the region where $C^{(2)}_\omega$ exists. Moreover, both C_λ and $C^{(2)}_\omega$ share the same end points $(0,0)$ and $(\frac{27}{2},\frac{1}{13})$. At $(0,0)$, $C^{(2)}_\omega$ and C_λ have slopes ∞ and $\frac{1}{6}$ respectively; while at $(\frac{27}{2},\frac{1}{13})$, they have the same slope $-\frac{475}{73008}$, but C_λ has a bigger second derivative. Hence $\#(C_\omega^{(2)} \cap C_\lambda) \geq 1$. A straightforward calculation shows that the equations

$$\begin{cases} \lambda(h,\omega) = 0, \\ \dfrac{\partial\lambda}{\partial h}\dot{h} + \dfrac{\partial\lambda}{\partial\omega}\dot{\omega}|_{(3.10)} = 0 \end{cases} \tag{5.15}$$

admit a unique solution for $h \in (0, \frac{27}{2})$, $\omega \in (0,2)$ at $(h,\omega) \approx (2.469, 0.180)$. By the saddle property of (3.10) at $(\frac{27}{2}, \frac{1}{13})$, if $\#(C_\omega^{(2)} \cap C_\lambda) \geq 2$, then there would be at least two points on C_λ at which the vector field (3.10) is tangent to C_λ, i.e., (5.15) would admit at least two solutions, a contradiction. Thus, $\#(C_\omega^{(2)} \cap C_\lambda) = 1$. ∎

Lemma 5.3 *Let $\Omega_A^{(22)}$ be the portion of $\Omega_A^{(2)}$ from N_2 to $\bar{N}$ (see Figure 5(b)). Then $\Omega_A^{(22)}$ is convex with negative curvature for all $A \in (0, \bar{A})$.*

Proof: By Lemma 5.1, $\omega(h) > 0$ for $0 < h \ll 1$ and $\lim_{h\to 0+0} \omega(h) = 0$. Hence by (3.8) $\lim_{h\to 0+0} \nu(h) = -\infty$, i.e., $\{\omega = 0\}$ is an asymptote of $\Omega_A^{(2)}$. On the other hand, $C_\omega^{(2)}$ has a maximum point for $A \in (0, \bar{A})$, which implies the existence of a turning point $\bar{N}$ on $\Omega_A^{(2)}$ (see Figure 5(b)). It follows that $\Omega_A^{(2)}$ has at least one point of inflection lying on $\Omega_A^{(2)} \setminus \Omega_A^{(22)}$. Thus, by Lemma 5.2, $\Omega_{1/2}^{(22)}$ has no point of inflection. By Lemma 4.3 we have $\frac{d^2\nu}{d\omega^2}\Big|_{\Omega_A^{(2)}} < 0$ for all $A \in (0, \bar{A})$ and h near h_2. If for some $A' \in (0, \bar{A})$, $\Omega_{A'}^{(22)}$ has a point of inflection, then it must have at least two, because the convexity near two end-points of $\Omega_A^{(22)}$ are the same and remain unchanged for all $A \in (0, \bar{A})$. Using the smoothness of the vector field, we let A vary from $\frac{1}{2}$ to A'. Then there must be a value $\hat{A}$ in between $\frac{1}{2}$ and A' such that $\Omega_{\hat{A}}^{(22)}$ has non-zero curvature except at a possible quadruple or higher order degenerate point $(\omega,\nu)(\hat{h}), \hat{h} \in (0, h_2)$. This means that if $L_{\hat{\alpha},\hat{\beta}}$ is the tangent line of $\Omega_{\hat{A}}^{(22)}$ at $(\omega,\nu)(\hat{h})$ (hence $\hat{\beta} = -\frac{\nu'(\hat{h})}{\omega'(\hat{h})}$), then $C_{\hat{U}}$ (with this $(\hat{\alpha},\hat{\beta})$) must meet $C_\omega^{(2)}$ at the point $(\hat{h}, \omega(\hat{h}))$ with tangency of order at least 4. Then by (5.13), for $h \neq 0$, $C_{\hat{U}}$ and $C_\omega^{(2)}$ have tangency at most 4. Hence along $C_{\hat{U}}$ there is no other point at which the vector field (3.10) is tangent to it. We now show that this is impossible. In fact, the convexity of $\Omega_{\hat{A}}^{(22)}$ and Lemma 4.3 imply that

$$\hat{\beta} = -\frac{\nu'(\hat{h})}{\omega'(\hat{h})} > -\frac{\nu'(h_2)}{\omega'(h_2)} \equiv \sigma_2 > 0.$$

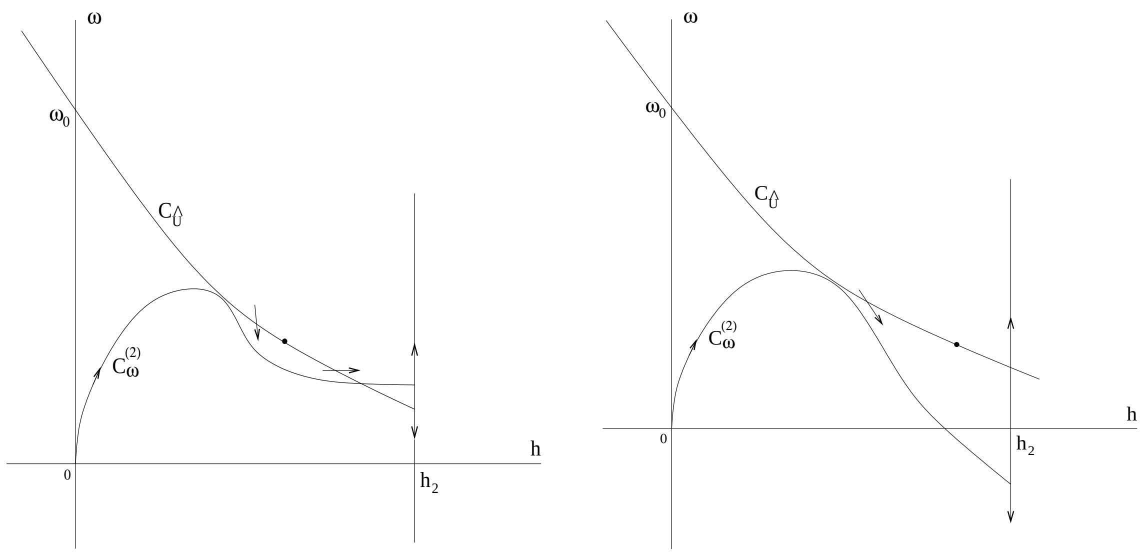
Using Lemma 4.3 again, we have

$$\begin{aligned} &(A+1)(A-3)\hat{\beta} + 8A^2 < (A+1)(A-3)\sigma_2 + 8A^2 \\ &= \frac{A^2(A+1)(A-3)(7A^3 - 77A^2 + 241A - 251)}{(A-2)^2(7A^3 - 63A^2 + 141A - 77)}, \end{aligned}$$

which is negative for $A \in (0, \bar{A})$. Hence, by (5.8), $\hat{h}^* < 0$, which implies that only the right branch of $C_{\hat{U}}$ can meet with $C_\omega^{(2)}$. On the other hand, we have

$$U(0) = \frac{2(A-2)}{3(A-1)} = \omega_0 > 0. \qquad (5.16)$$

Thus, if $C_\omega^{(2)}$ and $C_{\hat{U}}$ have a tangent point of order 4, then there are only two possibilities: a) $C_\omega^{(2)}$ and $C_{\hat{U}}$ have one more intersection point for $h \in (0, h_2)$ (see Figure 7(a)), which then gives one more "tangent point" on $C_{\hat{U}}$ with respect to the vector field (3.10), a contradiction to (5.13); b) $C_{\hat{U}}$ cuts $\{h = h_2\}$ above the saddle point (h_2, ω_2) (see Figure 7(b)), which also generates one more "tangent point" on $C_{\hat{U}}$ by the saddle property, a contradiction to (5.13) again. This completes the proof. ■



The case (a) The case (b)

Figure 7 Hypothetical relative positions of $C_\omega^{(2)}$ and $C_{\hat{U}}$

6 Proof of main results

6.1 Proof of Theorem 1

As explained in Section 2, it suffices to prove Theorem 2.2. By Lemmas 3.1, 4.1 and (2.5), we have that

$$\begin{aligned} P'(h) &< 0 \quad \text{for } h \in (h_1, 0) \cup (0, h_2), \\ P^{(1)}(h) &> P(0-0) > 0, \quad Q^{(1)}(h) > 0 \quad \text{for } h \in (h_1, 0), \end{aligned}$$

$$\begin{aligned} P^{(2)}(h) &< P(0+0)<0, \quad Q^{(2)}(h)<0 \quad \text{for } h\in(0,h_2), \\ \lim_{h\to 0-0} Q^{(1)}(h) &= \infty, \quad \lim_{h\to 0+0} Q^{(2)}(h) = -\infty; \\ \left.\frac{dQ}{dP}\right|_{\Sigma_A^{(i)}} &< 0, \quad (-1)^{i+1}\left.\frac{d^2Q}{dP^2}\right|_{\Sigma_A^{(i)}} > 0 \quad \text{for } 0<|h-h_i|\ll 1, i=1,2. \end{aligned}$$

Hence, if we can prove that the curves $\Sigma_A^{(i)}$, $i=1,2$, keep non-zero curvatures (i.e. $\frac{d^2Q}{dP^2}\big|_{\Sigma_A^{(i)}} \neq 0$) at all points, then Theorem 2.2 follows.

For this purpose, we will show that any straight line $L_{\alpha,\beta}\in L_T^{(i)}$ (see (2.13)) cuts $\Omega_A^{(i)}$ at a unique point, and the crossing is transversal. This is equivalent to show that, for all $A\in(0,1]$,

$$\#(C_\omega^{(i)}\cap C_U)=1, \;\; i=1,2. \tag{6.1}$$

Consider the multiple limit cycle bifurcation curve, defined in (α,β)-plane by

$$\widetilde{\Sigma}_A^{(i)} = \{(\alpha,\beta) : L_{\alpha,\beta}\in L_T^{(i)}\} = \{(\alpha,\beta)(h) : L_{\alpha,\beta} \text{ is tangent to } \Sigma_A^{(i)} \text{ at } (P,Q)(h)\in\Sigma_A^{(i)}\},$$

where $h\in(h_1,0)$ for $i=1$ and $h\in(0,h_2)$ for $i=2$. In other words, $(\alpha,\beta)(h)\in\widetilde{\Sigma}_A^{(i)}$ satisfies the equations

$$\begin{cases} \alpha+\beta P(h)+Q(h)=0, \\ \beta P'(h)+Q'(h)=0. \end{cases} \tag{6.2}$$

Applying the theory of Clairaut equation to the present case (see [6], Section 8, [5], (IV), Lemma 4.5), $\widetilde{\Sigma}_A^{(i)}$ is the envelope of a family of straight lines $\{\alpha+P(h)\beta+Q(h)=0\}$ parameterized by $h\in(h_1,0)$ for $i=1$ and $h\in(0,h_2)$ for $i=2$. If $\Sigma_A^{(i)}$ is convex in the (P,Q)-plane, then so is $\widetilde{\Sigma}_A^{(i)}$ in the (α,β)-plane; and, if $\Sigma_A^{(i)}$ has a point of inflection, then $\widetilde{\Sigma}_A^{(i)}$ has a cusp point for the corresponding value of h. Lemma 4.1, (6.2) and (4.3) now imply that the curve $\widetilde{\Sigma}_A^{(1)}$ (resp. $\widetilde{\Sigma}_A^{(2)}$) is tangent to the Hopf bifurcation line $L_1 = \alpha+P(h_1)\beta+Q(h_1)=\alpha+\beta+1=0$ (resp. $L_2=\alpha+P(h_2)\beta+Q(h_2)=\alpha+\frac{A-2}{A}\beta+\frac{A}{A-2}=0$) at the point $H_1=(\alpha_1,\beta_1)$ (resp. $H_2=(\alpha_2,\beta_2)$) of Hopf bifurcation of order 2, where

$$\begin{aligned} \alpha_1 &= -\tfrac{2(A+2)}{A+1}, & \beta_1 &= \tfrac{A+3}{A+1}, \\ \alpha_2 &= -\tfrac{2A(A-4)}{(A-2)(A-3)}, & \beta_2 &= \tfrac{A^2(A-5)}{(A-3)(A-2)^2}; \end{aligned} \tag{6.3}$$

and, as h slightly increases from h_1 (resp. slightly decreases from h_2), $\widetilde{\Sigma}_A^{(1)}$ (resp. $\widetilde{\Sigma}_A^{(2)}$) keeps its convexity and is located above the straight line $\{(\alpha,\beta): L_1=0\}$ (resp. $\{(\alpha,\beta): L_2=0\}$), again denoted by L_1 (resp. L_2), for (α,β) near H_1 (resp. H_2), since the absolute value of the slope of $L_{\alpha,\beta}\in L_T^{(i)}$, $\frac{1}{|P(h)|}$, is increasing. It is easy to check from (5.6) and (6.3) that $H_i\in L_i\cap L_0$, $i=1,2$, as shown in Figure 8, where the relative positions of L_0, L_1 and L_2 are qualitatively the same for all $A\in(0,1]$ (when $A=1$, L_1 and L_2 are symmetric with respect to $\beta-$axis).

Thus, $\widetilde{\Sigma}_A^{(i)}$ also stays above L_0 for $0<|h-h_i|\ll 1$, which implies that $U'(h)<0$ for $(\alpha,\beta)\in\Sigma_A^{(i)}$ but near H_i, $i=1,2$.

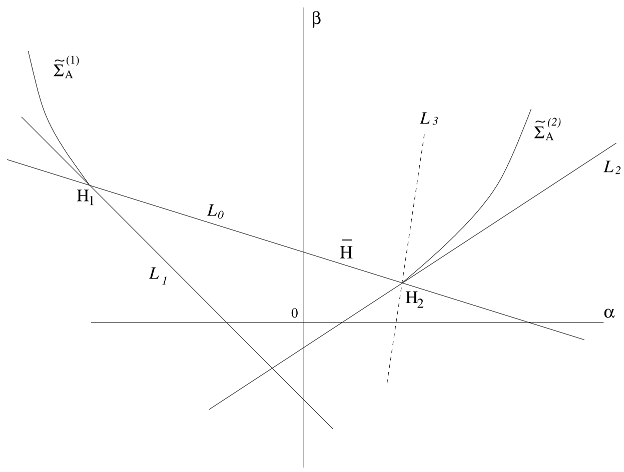


Figure 8 The relative positions of $L_0, L_1, L_2, L_3, \widetilde{\Sigma}_A^{(1)}$ and $\widetilde{\Sigma}_A^{(1)}$

On one hand, using (5.5) and (5.6), it is easy to check that the point

$$\bar{H} = (\frac{4A(A-1)}{(A+1)(A-3)}, \frac{-8A^2}{(A+1)(A-3)}),$$

at which $\xi = \eta = 0$, is also located on the line L_0 lying in between the points H_1 and H_2. Hence for $0 < h - h_1 \ll 1$ and $(\alpha, \beta) \in \widetilde{\Sigma}_A^{(1)}$ we have $\xi < 0$ and $\eta > 0$, which, together with (5.12), implies that $\omega^* > 0$. On the other hand, $C_\omega^{(1)}$ is increasing, and, by Lemma 5.1, stays below the ω axis. Hence, it can meet only one branch of C_U, which is decreasing since $U'(h) < 0$. This yields that $\#(C_\omega^{(1)} \cap C_U) \leq 1$. But since $I(h_1) = 0$ for all $A \in (0, 1]$, if $L_{\alpha,\beta} \in L_T^{(1)}$, then $I''(h)$ must have a zero for some $\tilde{h}_1 \in (h_1, h_T)$ (see Figure 9). It follows that $\#(C_\omega^{(1)} \cap C_U) = 1$ for $(\alpha, \beta) \in \widetilde{\Sigma}_A^{(1)}$ near H_1.

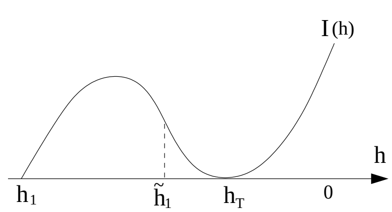


Figure 9 The behavior of $I(h)$ for $(\alpha, \beta) \in \widetilde{\Sigma}_A^{(1)}$

As h increases, we have that $(\alpha, \beta)(h) \in \widetilde{\Sigma}_A^{(1)}$ stays above L_0, while $\xi(h)$ and $\eta(h)$ keep negative and positive signs respectively. This implies (6.1) for all $h \in (h_1, 0)$ in the case of $i = 1$.

We have shown in the above that $(\alpha,\beta)(h) \in \widetilde{\Sigma}_A^{(2)}$ stays above L_2 (and L_0) and admits positive curvature for $0 < h_2 - h \ll 1$, hence $\alpha > \alpha_2 > 0$, $\beta > \beta_2 > 0, \xi > 0$. By (5.8), (5.5) and (4.2)

$$\begin{aligned} h^* - h_2 &= -\frac{(A+1)^2(\beta-\beta_2)}{A^2\eta} > 0, \qquad \text{if } \eta < 0; \\ h^* &= \frac{9(A-1)}{\eta} < 0, \qquad \text{if } \eta > 0. \end{aligned}$$

Thus, by (5.10), if $\eta = 0$, then C_U is a straight line with negative slope, and, if $\eta \neq 0$, then only one branch of C_U, which is strictly decreasing, can meet with $C_\omega^{(2)}$. In any case, by (5.16), this branch of C_U will pass through the point $(0, \omega_0)$.

By Lemma 5.1, if $A \in [\bar{A}, 1]$, then $C_\omega^{(2)}$ is strictly increasing and the same conclusion as for $\widetilde{\Sigma}_A^{(1)}$ holds. If $A \in (0, \bar{A})$, then $C_\omega^{(2)}$ is divided into two parts $C_\omega^{(21)} \cup C_\omega^{(22)}$ by the maximum point $(\tilde{h}, \omega(\tilde{h}))$, where $C_\omega^{(21)}$ (resp. $C_\omega^{(22)}$) is strictly increasing (resp. decreasing), hence

$$\#(C_\omega^{(21)} \cap C_U) \leq 1. \tag{6.4}$$

It remains to show that

$$\#(C_\omega^{(22)} \cap C_U) \leq 1. \tag{6.5}$$

Since $\#(C_\omega^{(22)} \cap C_U) = \#(L_{\alpha,\beta} \cap \Omega_A^{(22)})$ for $L_{\alpha,\beta} \in L_T^{(2)}$, and $\Omega_A^{(22)}$ is convex with negative curvature (Lemma 5.3), we only need to show that the endpoint $N_2 = (\omega(h_2), \nu(h_2))$ of $\Omega_A^{(22)}$ is always above $L_{\alpha,\beta} \in L_T^{(2)}$ as in Figure 5(b).

Consider

$$U(h_2) - \omega_2 = -\frac{L_3}{A(A-2)(5A^2-26A+41)(\beta-\beta_2)}, \tag{6.6}$$

where

$$L_3 = A(A-2)(5A^2-26A+41)\alpha + (A-2)^2(5A^2-14A+5)\beta + A^2(5A^2-38A+101).$$

A straightforward calculation shows that the straight line $\{(\alpha,\beta) : L_3 = 0\}$, again denoted by L_3, also passes through the point H_2, and, the slope of which, if positive, is bigger than the slope of L_2 (see Figure 8). If $L_{\alpha,\beta} \in L_T^{(2)}$, then the unique zero of $I''(h)$ cannot vanish at $h = h_2$ (see Figure 10). Hence $\widetilde{\Sigma}_A^{(2)}$ must stay to the right of L_3, which, by (6.6), implies that $U(h_2) - \omega_2 < 0$, i.e., in the (α,β)-plane, N_2 is above the straight line $L_{\alpha,\beta} \in L_T^{(2)}$ for $A \in (0, \bar{A})$ and $0 < h_2 - h \ll 1$.

Finally, since $\lim_{h\to 0+0}(\omega(h), \nu(h)) = (0, -\infty)$, we have that $\#(C_\omega^{(2)} \cap C_U)$ is odd, which, together with (6.4) and (6.5) implies (6.1) for $i = 2$ and $0 < h_2 - h \ll 1$. As h decreases, all conditions above remain unchanged and the same conclusion holds for all $h \in (0, h_2)$. ∎

6.2 Proof of Theorem 2

As the basic idea of the proof for Theorem 2 is the same as that for Theorem 1, we will mainly indicate the differences between these two.

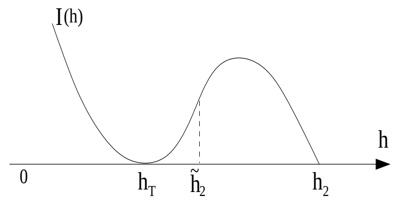


Figure 10 The behavior of $I(h)$ for $(\alpha, \beta) \in \widetilde{\Sigma}_A^{(2)}$

For $A \in (-\infty, -1)$, system (4.1) has a center at $(1, 0)$, corresponding to $h_1 = A - 3 < 0$, and three saddle points, two of which are on the invariant line $\{x = 0\}$ and the third one is at $(\frac{A-2}{A}, 0)$, corresponding to $h_2 = \frac{(A+1)(A-2)^2}{A^2} < 0$ (see Figure 3). In this case, we have

$$h_1 < h_2 < 0. \tag{6.7}$$

Using (2.5), Lemma 3.1 and (4.4), we obtain the following.

Lemma 6.1 *$P(h), Q(h), P'(h) > 0$ for all $A \in (-\infty, -1)$ and $h \in (h_1, h_2)$.*

Similarly to Lemma 4.1, we also have the following.

Lemma 6.2 *If $A \in (-\infty, -1)$ and $(P, Q)(h) \in \Sigma_A$, then*

$$\left.\frac{dQ}{dP}\right|_{h_1} = -\frac{A+3}{A+1}, \quad \left.\frac{d^2Q}{dP^2}\right|_{h_1} = \frac{20}{A+1} < 0.$$

For $A \in (-\infty, -1)$, system (3.10) has three saddle points at (h_1, ω_1), (h_2, ω_2) and $(0, \omega_0)$; and three nodes at $(h_1, 1)$, (h_2, ω_2') and $(0, 0)$, where ω_1, ω_2 and ω_2' are as in (4.5). But in this case, we have that

$$\omega_1 - 1 = \frac{-12(A+1)}{5A^2 + 6A + 9} > 0, \quad \omega_2' - \omega_2 = \frac{-12(A-2)(A-3)}{A(5A^2 - 26A + 41)} > 0.$$

Therefore, similar to Lemma 5.1, we have the following.

Lemma 6.3 *C_ω is the unstable manifold of* (3.10) *from the saddle (h_1, ω_1) connecting to the node (h_2, ω_2'). It is strictly decreasing for $A \in (-\infty, \widetilde{A}]$ and has a unique extremum (in fact, a maximum) point for $A \in (\widetilde{A}, -1)$, where $\widetilde{A} \approx -3.9037$ is as in* Remark 4.1.

Like the treatment for $\Omega_A^{(2)}$ in the proof of Theorem 1, when $A \in (\widetilde{A}, -1)$, we divide Ω_A into two parts $\Omega_A^{(+)} \cup \Omega_A^{(-)}$, corresponding to the increasing and decreasing part of C_ω respectively. Then similar to Lemma 5.3, we have the following.

Lemma 6.4 $\Omega_A^{(+)}$ *is convex with negative curvature for all* $A \in (\widetilde{A}, -1)$.

If we denote k_0 as the slope of L_0 (see (5.6)), then

$$k_0 + 1 = -\frac{(A+1)}{2(A-2)} < 0 \text{ for } A < -1.$$

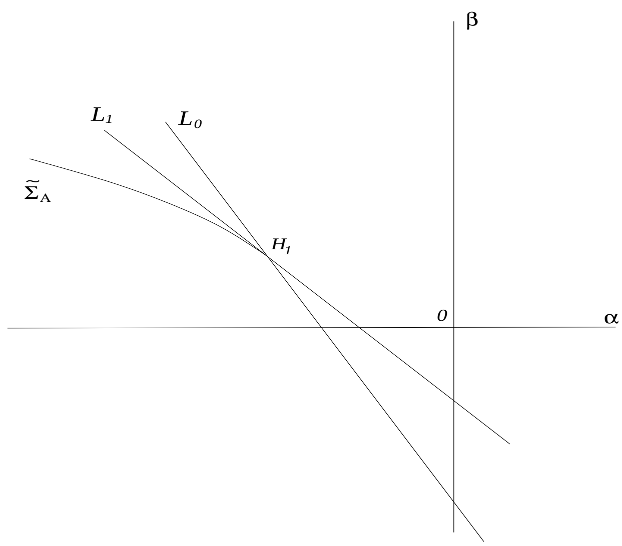
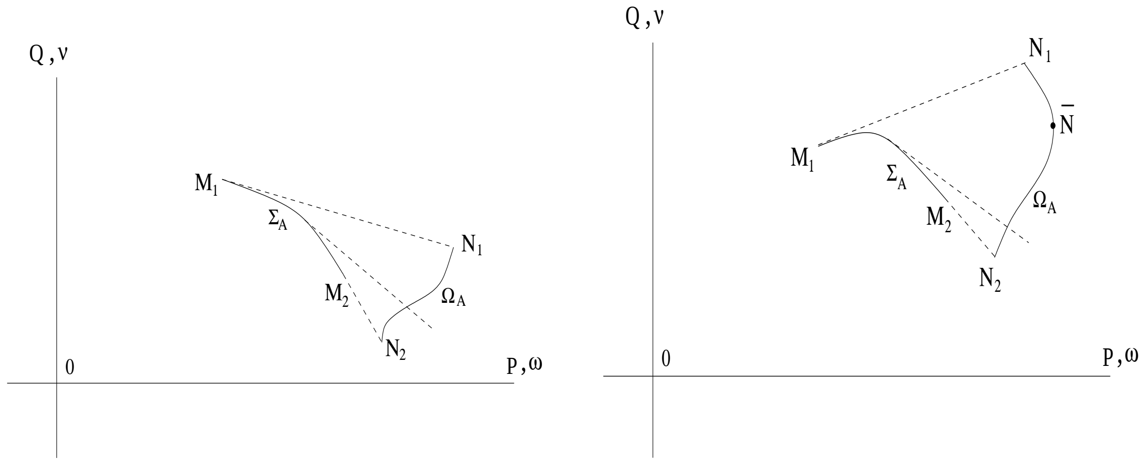


Figure 11 The relative positions of L_0, L_1 and $\widetilde{\Sigma}_A$ for $A \in (-\infty, -1)$

Instead of Figure 8, the relative positions of L_0 and L_1 are now shown in Figure 11. Since $\frac{1}{|P(h)|}$ is decreasing as h increases from h_1, $\widetilde{\Sigma}_A$ stays below L_1 and is tangent to L_1 at H_1 for $0 < h - h_1 \ll 1$. Hence $\widetilde{\Sigma}_A$ also stays below L_0, which implies that $U'(h) > 0$. The remaining proof of Theorem 2 is completely the same as for Theorem 1. The behaviors of Σ_A in the (P, Q)-plane and Ω_A in the (ω, ν)-plane are shown in Figure 12, where the two planes are identified.



(a) The case $A \in (-\infty, \widetilde{A}]$

(b) The case $A \in (\widetilde{A}, -1)$

Figure 12 The behavior of Σ_A and Ω_A for $A \in (-\infty, -1)$

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